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Proof.

$E[X] \geq aP[X \geq a]$. □

Graphs with high girth and high chromatic number

Definition (Mostly reminder from earlier classes)

Let $G = (V, E)$ be a graph. A proper k-coloring is a function $c\colon V(G) \to \{1, \dots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$.
The chromatic number of G, denoted $\chi(G)$, is the minimal k such that G admits a proper k-coloring.

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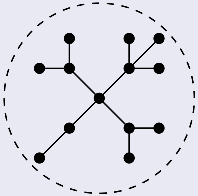
Problem

Are there graphs with arbitrarily large girth and arbitrarily large chromatic number?



Graphs with high girth and high chromatic number

Theorem (Erdős)

For every two integers $k, \ell > 0$ *there is a graph* G *with* $\chi(G) > k$ *and* $g(G) > \ell$.

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- $E[X] \leq \sum_{i=3}^{\ell} n^i p^i = \sum_{i=3}^{\ell} n^{\varepsilon i} \leq \ell n^{\frac{1}{2\ell}\ell} = o(n)$.
- If n is large enough, then $E[X] < \frac{n}{4}$.
- By Markov's inequality $P[X \geq n/2] < \frac{n/4}{n/2} = \frac{1}{2}$.

Proof of the Erdős theorem—continued

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- $\chi(G') \geq \frac{|V(G')|}{\alpha(G')} \geq \frac{n/2}{a-1} \geq \frac{n/2}{\frac{4}{p}\ln n} = \frac{1}{8}\frac{n^{\varepsilon}}{\ln n} \to \infty$.

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- So we could pick n so that $\chi(G') > k$. □

Bayes theorem

- Recall: $P[A|B] := \frac{P[A \cap B]}{P[B]}$ if $P[B] > 0$.

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Let $A, B_1, \ldots, B_n \subseteq \Omega$ be events such that $B_1, \ldots, B_n$ are pairwise disjoint, $B_1 \cup \cdots \cup B_n = \Omega$, and $P[A] > 0$, $P[B_j] > 0$ for $j \in [n]$. Then

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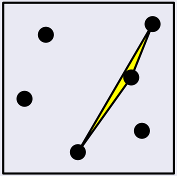
Proof of Bayes theorem.

$$P[B_i|A] = \frac{P[A \cap B_i]}{P[A]} = \frac{P[A|B_i]P[B_i]}{\sum_{j=1}^{n} P[A|B_j]P[B_j]}. \qquad \square$$

Avoiding triangles of small area

Problem



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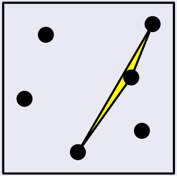

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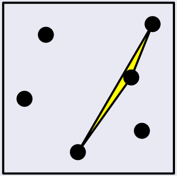

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- We will show:

Theorem

For every $n \in N$ there is $S \subseteq [0,1]^2$ with n points such that $T(S) \geq \frac{1}{101n^2}$.

Avoiding triangles of small area, proof

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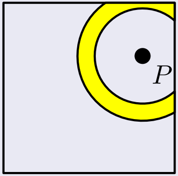
- Given $i \in \mathbb{N}$ and $\Delta > 0$ we get $P[w \in [(i-1)\Delta, i\Delta)] \leq \pi(i^2\Delta^2 - (i-1)^2\Delta^2) = \pi(2i-1)\Delta^2$.



Avoiding triangles of small area, proof

Proof, continued.

- $P[w \in [(i-1)\Delta, i\Delta)] \leq \pi(2i-1)\Delta^2$ where $w = \text{dist}(P, Q)$.

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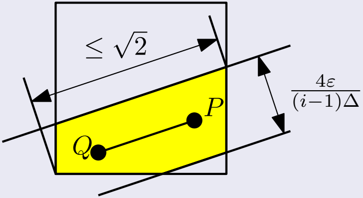

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$$\leq 1 \cdot \pi\Delta^2 + \sum_{\substack{i \geq 2 \\ (i-1)\Delta < \sqrt{2}}} \sqrt{2} \frac{4\varepsilon}{(i-1)\Delta} \pi(2i-1)\Delta^2 =$$

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$$\leq \pi\Delta^2 + \frac{\sqrt{2}}{\Delta} 12\sqrt{2}\varepsilon\pi\Delta = \pi\Delta^2 + 24\varepsilon\pi.$$

Avoiding triangles of small area, proof

Proof, continued.

- $P[w \in [(i-1)\Delta, i\Delta)] \leq \pi(2i-1)\Delta^2$ where $w = \mathrm{dist}(P, Q)$.
- Let B_i be the event corresponding to $w \in [(i-1)\Delta, i\Delta)$.
- $P[\lambda \leq \varepsilon] = \sum_{\substack{i \in \mathbb{N} \\ (i-1)\Delta < \sqrt{2}}} P[\lambda \leq \varepsilon | B_i] P[B_i] =$

$$= P[\lambda \leq \varepsilon | B_1] P[B_1] + \sum_{\substack{i \geq 2 \\ (i-1)\Delta < \sqrt{2}}} P[\lambda \leq \varepsilon | B_i] P[B_i] \leq$$

$$\leq 1 \cdot \pi\Delta^2 + \sum_{\substack{i \geq 2 \\ (i-1)\Delta < \sqrt{2}}} \sqrt{2} \frac{4\varepsilon}{(i-1)\Delta} \pi(2i-1)\Delta^2 =$$

$$= \pi\Delta^2 + \sum_{i=2}^{\lfloor \frac{\sqrt{2}}{\Delta} \rfloor + 1} 4\sqrt{2}\pi\varepsilon \frac{2i-1}{i-1} \Delta \leq \pi\Delta^2 + \sum_{i=2}^{\lfloor \frac{\sqrt{2}}{\Delta} \rfloor + 1} 4\sqrt{2}\pi\varepsilon 3\Delta \leq$$

$$\leq \pi\Delta^2 + \frac{\sqrt{2}}{\Delta} 12\sqrt{2}\varepsilon\pi\Delta = \pi\Delta^2 + 24\varepsilon\pi.$$

- This gives $P[\lambda \leq \varepsilon] \leq 24\varepsilon\pi$ considering $\Delta \to 0$.

Avoiding triangles of small area, proof

Proof, continued.

- At this moment we have $P[\lambda \leq \varepsilon] \leq 24\varepsilon\pi$ where λ is the area of a triangle PQR formed by three random points PQR.

Avoiding triangles of small area, proof

Proof, continued.

- At this moment we have $P[\lambda \leq \varepsilon] \leq 24\varepsilon\pi$ where λ is the area of a triangle PQR formed by three random points PQR.
- Consider random points $P_1, \ldots, P_{2n}$ in $[0,1]^2$.

Avoiding triangles of small area, proof

Proof, continued.

- At this moment we have $P[\lambda \leq \varepsilon] \leq 24\varepsilon\pi$ where λ is the area of a triangle PQR formed by three random points PQR.
- Consider random points $P_1, \ldots, P_{2n}$ in $[0,1]^2$.
- Set X to be the number of triangles with area $\leq \frac{1}{101n^2}$.

Avoiding triangles of small area, proof

Proof, continued.

- At this moment we have $P[\lambda \leq \varepsilon] \leq 24\varepsilon\pi$ where λ is the area of a triangle PQR formed by three random points PQR.
- Consider random points $P_1, \ldots, P_{2n}$ in $[0,1]^2$.
- Set X to be the number of triangles with area $\leq \frac{1}{101n^2}$.
- $E[X] \leq \binom{2n}{3}\frac{24\pi}{101n^2} \leq \frac{8n^3}{6}\frac{24\pi}{101n^2} < n$.

Avoiding triangles of small area, proof

Proof, continued.

- At this moment we have $P[\lambda \leq \varepsilon] \leq 24\varepsilon\pi$ where λ is the area of a triangle PQR formed by three random points PQR.
- Consider random points $P_1, \ldots, P_{2n}$ in $[0,1]^2$.
- Set X to be the number of triangles with area $\leq \frac{1}{101n^2}$.
- $E[X] \leq \binom{2n}{3}\frac{24\pi}{101n^2} \leq \frac{8n^3}{6}\frac{24\pi}{101n^2} < n$.
- Find $P_1, \ldots, P_{2n}$ so that $X < n$. After removal at most n points we get required S without triangles of small area. □