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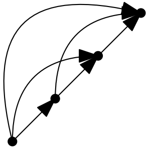

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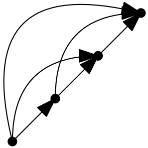

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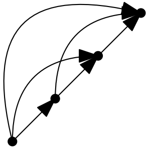

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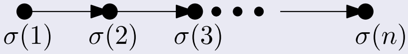

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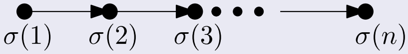

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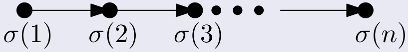

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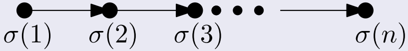

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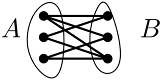

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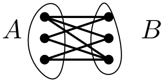

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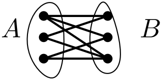

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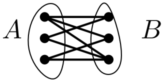

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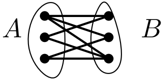

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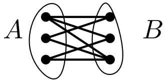

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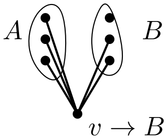

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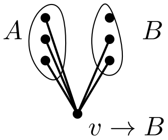

- In each step at least half of the considered edges enters the cut.

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- Can be maintained above $\frac{2^k - 1}{2^k} m$ based on $E[I_A] = \frac{1}{2}(E[I_{A|B}] + E[I_{A|B^c}])$ if $P[B] = P[B^c] = \frac{1}{2}$.

Applications of linearity of exp. value: Balancing vectors

Proposition

Let $v_1, \ldots, v_n$ be unit vectors in $\mathbb{R}^n$.

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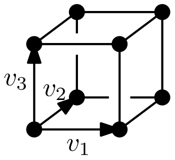
- Proposition cannot be improved.



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- Therefore, there is a choice of ε_i as needed. $\square$

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- May be sometimes difficult (almost impossible).
- Find, using a probabilistic method, an object with 'almost' desired property and modify it to an object with the desired property.

Weak Turán theorem

Proposition (Weak Turán theorem)

Let $G = (V, E)$ be an arbitrary graph with n vertices and m edges, then $\alpha(G) \geq \frac{n}{2d}$ where $\alpha(G)$ is the size of the largest independent set in G and $d := \frac{2m}{n}$ is the average degree of G.

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- Let S be a random subset of V, each vertex belongs to S with probability p (which will be determined later).

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Let $G = (V, E)$ be an arbitrary graph with n vertices and m edges, then $\alpha(G) \geq \frac{n}{2d}$ where $\alpha(G)$ is the size of the largest independent set in G and $d := \frac{2m}{n}$ is the average degree of G.

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- Let S be a random subset of V, each vertex belongs to S with probability p (which will be determined later).
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- We want to maximize this expression—we set $p = \frac{1}{d}$.
- $E[X - Y] = \frac{n}{2d}$. Take S with $X - Y \geq \frac{n}{2d}$.
- Obtain $S' \subseteq S$ by removing a vertex from each edge of $G[S]$. Then S' is independent and $|S'| \geq \frac{n}{2d}$. $\square$

Relation to the Turán theorem

- Turán's theorem: G without $K_{r+1} \Rightarrow m \leq (1 - \frac{1}{r})\frac{n^2}{2}$ where $n := |V(G)|$ and $m := |E(G)|$.

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- Thus, Turán's theorem is equivalent to: $\alpha(G) \leq r \Rightarrow r \geq \frac{n}{d+1}$.
- This is same as $\alpha(G) \geq \frac{n}{d+1}$. $\square$