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- This is less then 1 as $m < 2^k$, thus a satisfying assignment exists. □

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- Question to the audience: How big is the largest intersecting family if $n < 2k$?

Lemma for Erdős-Ko-Rado theorem

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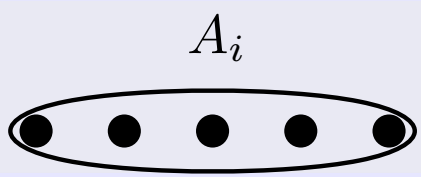

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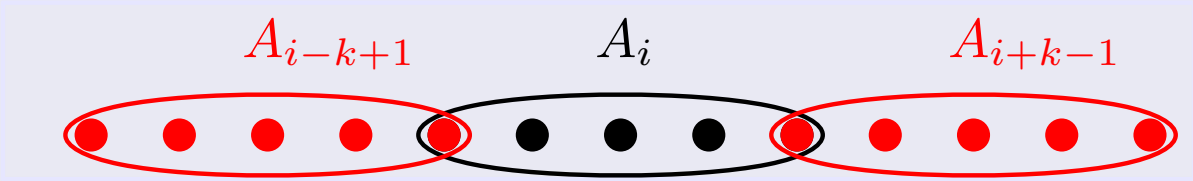

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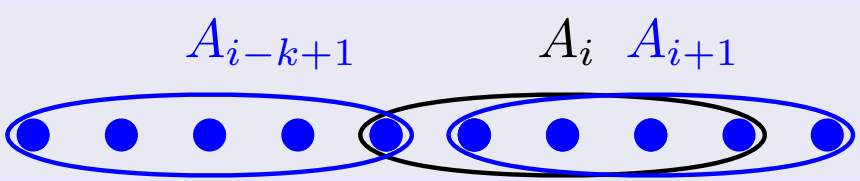

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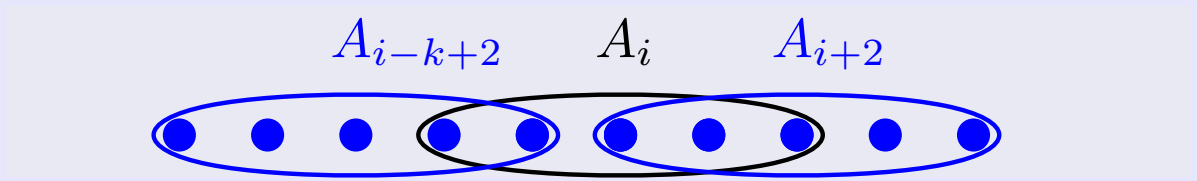

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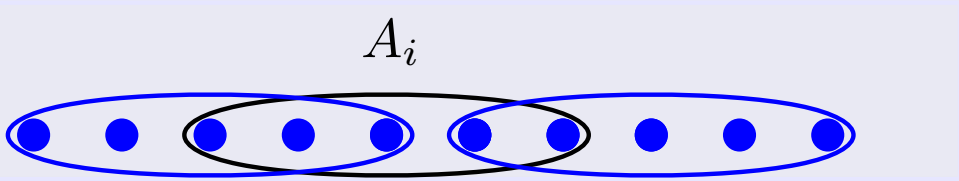

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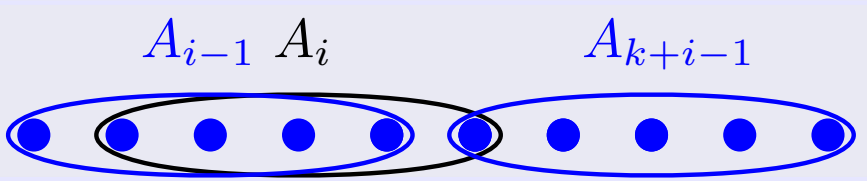

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- Altogether:

$$|\mathcal{F}| \leq \frac{k}{n}\binom{n}{k} = \frac{k}{n}\frac{n!}{k!(n-k)!} = \binom{n-1}{k-1}. \quad \square$$

Erdös-Ko-Rado, additional explanation

- This slide explains in more detail the following item from the previous slide.
  - $P[\sigma(A_s) \in \mathcal{F}] \leq \frac{k}{n}$ by the lemma (first we pick σ, then s)
- Note that the formula $P[\sigma(A_s) \in \mathcal{F}]$ regards the case where both σ and s are taken uniformly in random.
- First consider a fixed σ and let p_σ be the probability that $\sigma(A_s)$ belongs to $\mathcal{F}$ for $s \in \{0, \ldots, n-1\}$ chosen uniformly at random.
- By lemma, we have $p_\sigma \leq \frac{k}{n}$ because at most k sets among the sets A_s may be intersecting which is equivalent with saying that at most k sets among $\sigma(A_s)$ may be intersecting.
- Now we get $P[\sigma(A_s) \in \mathcal{F}] = \sum_\sigma \frac{1}{n!} p_\sigma$ because we have probability $\frac{1}{n!}$ to pick an individual σ and we have to consider all possible σ. (This is in principle the conditional probability that I mentioned during the lecture but for somebody it maybe easier to see it directly.)

Random variables and the expectation

Definition

Random variable (náhodná veličina) on probability space (Ω, Σ, P) is a P-measurable function $X : \Omega \to R$. (P-measurable: $\forall a \in \mathbb{R}$: $\{\omega \in \Omega : X(\omega) \leq a\} \in \Sigma$).

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Lemma (Linearity of expectation)

Let X, Y be random variables and $\alpha, \beta \in \mathbb{R}$. Then $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$.

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Proof (only for finite prob. spaces).

$$\begin{aligned} E[\alpha X + \beta Y] &= \sum_{\omega \in \Omega} p(\omega)\big(\alpha X(\omega) + \beta Y(\omega)\big) \\ &= \alpha \sum_{\omega \in \Omega} p(\omega) X(\omega) + \beta \sum_{\omega \in \Omega} p(\omega) Y(\omega) \\ &= \alpha E[X] + \beta E[Y]. \end{aligned}$$

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&= \sum_{\substack{a \in X(\Omega) \\ b \in Y(\Omega)}} abP[X = a \wedge Y = b] \\
&= \sum_{\substack{a \in X(\Omega) \\ b \in Y(\Omega)}} abP[X = a]P[X = b] \\
&= \left(\sum_{a \in X(\Omega)} aP[X = a] \right) \left(\sum_{b \in Y(\Omega)} bP[Y = b] \right) = E[X]E[Y] \square
\end{aligned}
$$

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Given an event A, the indicator (indikátor) of A is a random variable defined as

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- $E[I_{A_j}] = P[A_j] = \frac{1}{n}$.
- Then the expectation of the number of fixed points in a random permutation equals

$$\sum_{j=1}^{n} E[I_{A_j}] = n \cdot \frac{1}{n} = 1.$$