Probabilistic Techniques

Targets of the lecture:

- Show how the probability theory can be used for finding a solution to various problems in combinatorics, theoretical computer science, combinatorial geometry, etc...
- Overview basic notions in the probability theory. (But this lecture is not a replacement for the standard course in probability.)
- Mostly, we will focus to discrete probability. Occasionally, we will touch the continuous case.

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  - Put each edge into G randomly with probability $1/2$ independently of other edges.
  - Compute that there is a positive probability that G contains neither a clique nor an independent set on k vertices.

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- With high probability short cycles can be removed from G while keeping high chromatic number.

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- and $P \colon \Sigma \to [0, 1]$ is a probability measure:
  - $P[\emptyset] = 0$;
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Elements of Σ are called events (jevy). Elements of Ω are elementary events (elementární jevy). Given an event $A \in \Sigma$, $P[A]$ is the probability of A (pravděpodobnost jevu A).

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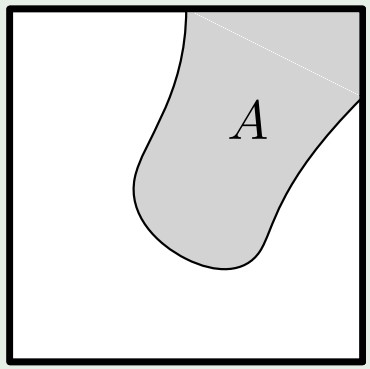
Example (Random graph)

- $G(n,p)$ is finite probability space of random graphs on n vertices with edge probability p;
- Elementary events: Graphs on fixed set of n vertices;
- Prob. of a graph G with m edges $p(G) = p^m(1-p)^{\binom{n}{2}-m}$.

Example of a continuous probability space

Example (Random point in a square)

- $\Omega = [0,1]^2$;
- $\Sigma =$ Lebesgue measurable subsets of $[0,1]^2$;
- $P[A] = \lambda(A)$, the Lebesgue measure of A (area of A).



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- For the last step $B \subseteq A \Rightarrow P[B] \leq P[A]$ because $P[A] = P[B \cup (A \setminus B)] = P[B] + P[A \setminus B] \geq P[B]$.

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- Limit behaviour (Stirling formula)

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \text{ i.e. } \lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1.$$

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- Even better for the middle binomial coefficient:

$$\frac{2^{2m}}{2\sqrt{m}} \leq \binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{2m}}.$$

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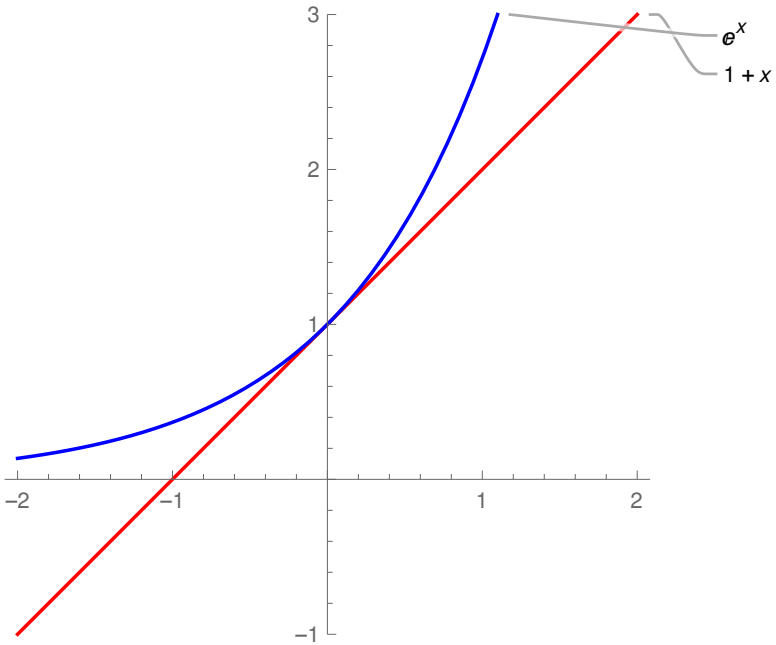

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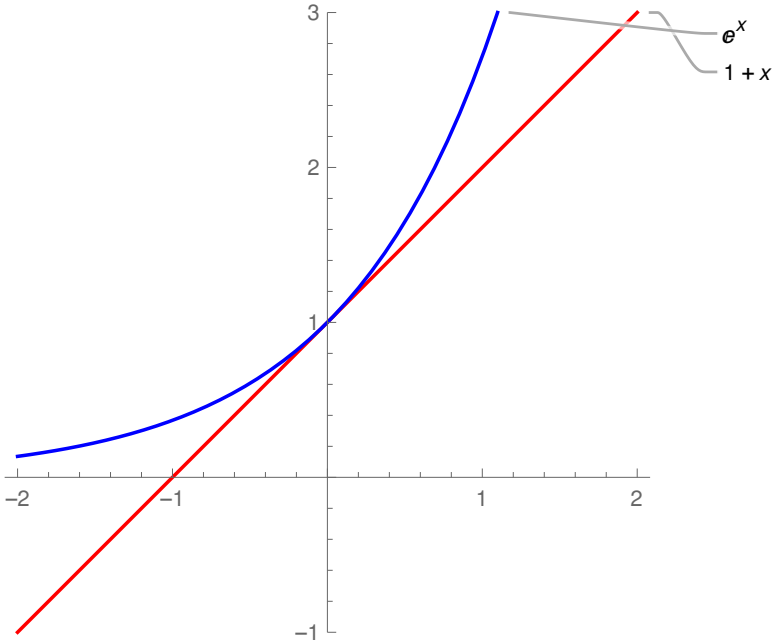

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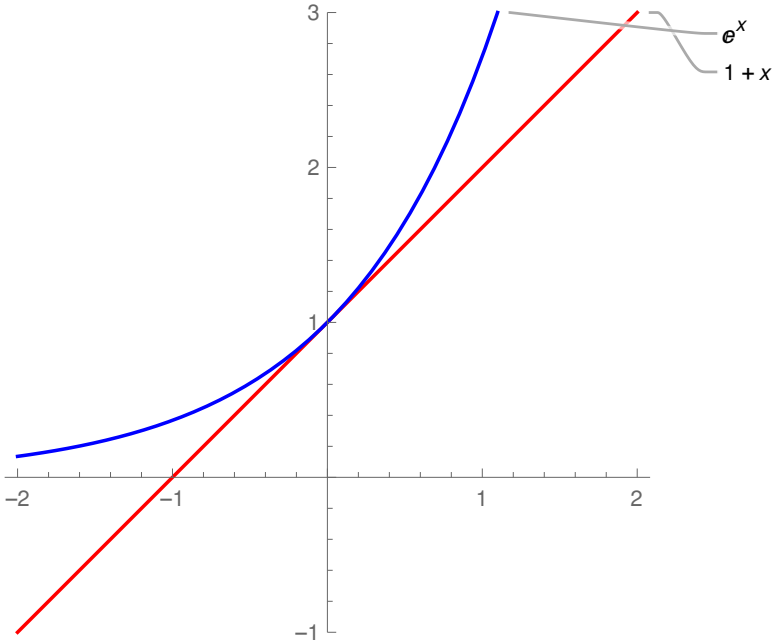

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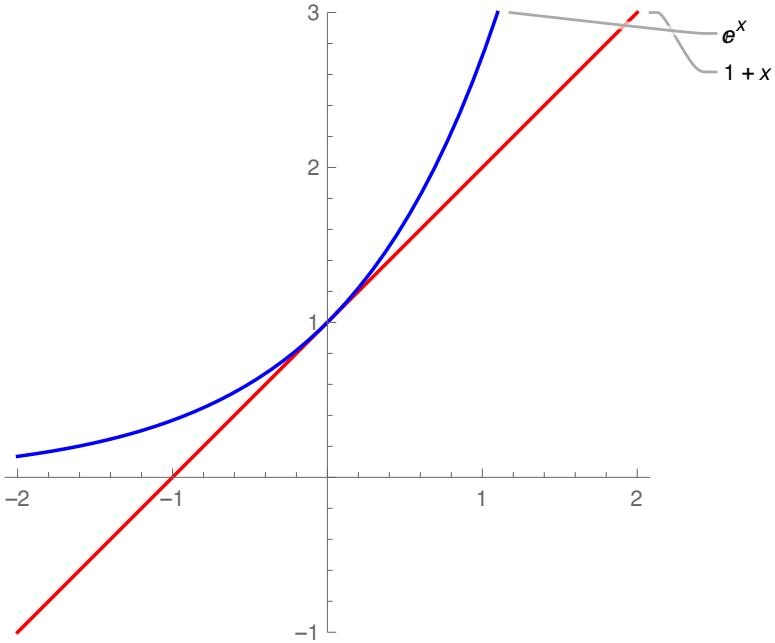

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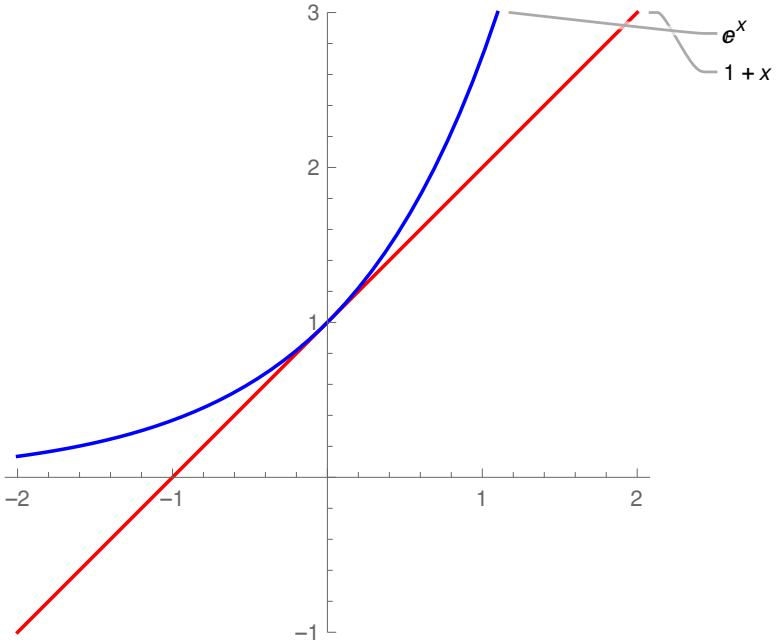

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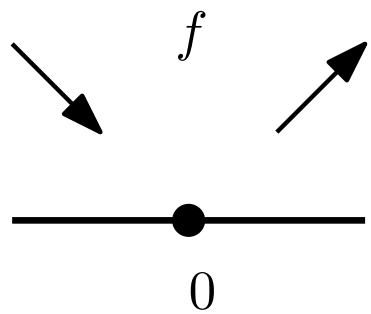

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- Take the random graph $G(n, 1/2)$.
- For a fixed subset of k vertices the probability that it forms a clique or independent set is $2 \cdot 2^{-\binom{k}{2}}$.
- By the union bound: Probability that $G(n, 1/2)$ contains a clique or an independent set on k vertices is at most $\binom{n}{k} \cdot 2 \cdot 2^{-\binom{k}{2}}$.

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$$\binom{n}{k} \cdot 2 \cdot 2^{-\binom{k}{2}} \leq \frac{n^k}{k!} 2^{-(\frac{k(k-1)}{2}-1)} \leq \frac{2^{k \cdot k/2}}{k!} 2^{-(\frac{k(k-1)}{2}-1)} =$$

$$= \frac{2^{k/2+1}}{k!} < 1.$$

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- If A and B are independent, then $P[A|B] = P[A]$.