

Řešení cvičení 6: Limity funkcí

Limity funkcí (snadné)

Spočtěte následující limity funkcí pro $m, n \in \mathbb{N}$

(a) $\lim_{x \rightarrow 0} \frac{\cos(x)+1}{\cos(x)-1},$

(e) $\lim_{x \rightarrow 1} [x] - x,$

(b) $\lim_{x \rightarrow 0} \frac{x^2-1}{2x^2-x-1},$

(f) $\lim_{x \rightarrow 0} \frac{(1+mx)^n - (1+nx)^m}{x^2},$

(c) $\lim_{x \rightarrow 1} \frac{x^2-1}{2x^2-x-1},$

(d) $\lim_{x \rightarrow 1} \frac{x^3-3x+2}{x^4-4x+3},$

(g) $\lim_{x \rightarrow 0} e^{\frac{\sqrt[3]{1-x^2}-1}{5x^2}}.$

(a) $\lim_{x \rightarrow 0} \frac{\cos(x)+1}{\cos(x)-1} = -\infty.$

(b) $\lim_{x \rightarrow 0} \frac{x^2-1}{2x^2-x-1} = 1.$

(c) $\lim_{x \rightarrow 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)(2x+1)} = \frac{2}{3}.$

(d) $\lim_{x \rightarrow 1} \frac{x^3-3x+2}{x^4-4x+3} = \lim_{x \rightarrow 1} \frac{(x+2)(x-1)^2}{(x^2+2x+3)(x-1)^2} = \frac{3}{6} = \frac{1}{2},$

(e) Tato limita neexistuje, protože $\lim_{x \rightarrow 1^-} [x] - x = 1$ a $\lim_{x \rightarrow 1^+} [x] - x = 0.$

(f)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{(1+mx)^n - (1+nx)^m}{x^2} &= \lim_{x \rightarrow 0} \frac{\sum_{i=0}^n \binom{n}{i} (mx)^i - \sum_{i=0}^m \binom{m}{i} (nx)^i}{x^2} = \\ &= \lim_{x \rightarrow 0} \frac{1 + mnmx + \binom{n}{2} (mx)^2 - (1 + mnmx + \binom{m}{2} (nx)^2) + o(x^3)}{x^2} = \\ &= \lim_{x \rightarrow 0} \frac{n(n+1)(mx)^2 - m(m+1)(nx)^2 + o(x^3)}{2x^2} = \\ &= \lim_{x \rightarrow 0} \frac{nm^2x^2 - mn^2x^2 + o(x^3)}{2x^2} = \frac{nm(n-m)}{2}. \end{aligned}$$

(g) $\lim_{x \rightarrow 0} e^{\frac{\sqrt[3]{1-x^2}-1}{5x^2}} = \lim_{x \rightarrow 0} e^{\frac{1-x^2-1}{5x^2 \cdot \left(\frac{2}{3} + \sqrt[3]{1-x^2} + 1\right)}} = e^{\lim_{x \rightarrow 0} \frac{-1}{5 \left(\frac{2}{3} + \sqrt[3]{1-x^2} + 1\right)}} = e^{-\frac{1}{15}}.$

Limity funkcí (obtížnější)

Spočtěte následující limity funkcí pro $a, b \in \mathbb{R}, b \neq 0$ a $m, n \in \mathbb{N}$

(a) $\lim_{x \rightarrow 0} \frac{1-\cos(x)}{x^2},$

(g) $\lim_{x \rightarrow 0} \left(\frac{1+x2^x}{1+x3^x} \right)^{\frac{1}{x^2}},$

(b) $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right),$

(h) $\lim_{x \rightarrow 0} \frac{\ln(a+x) + 2 \ln(a-x) - 2 \ln(a)}{x^2}, \quad a > 0,$

(c) $\lim_{x \rightarrow a} \frac{\tan(x) - \tan(a)}{x - a},$

(i) $\lim_{x \rightarrow 0} \frac{\ln(\cos(ax))}{\ln(\cos(bx))},$

(d) $\lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}},$

(j) $\lim_{x \rightarrow 0} \frac{\ln(\tan(\frac{\pi}{4} + ax))}{\sin(bx)},$

(e) $\lim_{x \rightarrow \pi} \frac{\sin(mx)}{\sin(nx)},$

(k) $\lim_{x \rightarrow 0^+} (\cos(\sqrt{x}))^{\frac{1}{x}},$

(f) $\lim_{x \rightarrow 1} (1-x) \log_x(2),$

(l) $\lim_{x \rightarrow 1} \frac{\sin(\pi x^a)}{\pi x^b}.$

(a) $\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \cos^2(x)}{x^2(1 + \cos(x))} = \lim_{x \rightarrow 0} \frac{\sin^2(x)}{x^2} \frac{1}{1 + \cos(x)} = \frac{1}{2}$.

(b) Limita neexistuje, protože máme Heineho větu, která pro posloupnosti $a_n = \frac{1}{\pi n}$ a $a_n = \frac{1}{\pi(2n+1)}$ dává odlišné výsledky.

(c) Použijeme sčítací vzorec pro \tan

$$\begin{aligned} \lim_{x \rightarrow a} \frac{\tan(x) - \tan(a)}{x - a} &= \lim_{x \rightarrow a} \frac{\tan(x - a)(1 + \tan(a)\tan(x))}{x - a} = \lim_{x \rightarrow a} \frac{\sin(x - a)(1 + \tan(a)\tan(x))}{(x - a)\cos(x - a)} = \\ &= \lim_{x \rightarrow a} \frac{\sin(x - a)}{x - a} \frac{(1 + \tan(a)\tan(x))}{\cos(x - a)} \stackrel{x-a=h}{=} \lim_{h \rightarrow 0} \underbrace{\frac{\sin(h)}{h}}_{\rightarrow 1} \frac{(1 + \tan(a)\tan(a+h))}{\cos(h)} = 1 + \tan^2(a). \end{aligned}$$

(d) $\lim_{x \rightarrow 0^+} (1 + x)^{\frac{1}{x}} = \lim_{x \rightarrow 0^+} e^{\frac{\ln(1+x)}{x}} = e$.

(e) $\lim_{x \rightarrow \pi} \frac{\sin(mx)}{\sin(nx)} = \lim_{x \rightarrow 0} \frac{-\sin(mx)}{-\sin(nx)} = \lim_{x \rightarrow 0} \frac{\sin(mx)}{mnx} \frac{mnx}{\sin(nx)} = \frac{m}{n}$.

(f) $\lim_{x \rightarrow 1} (1 - x) \log_x(2) = \lim_{x \rightarrow 1} -\frac{x-1}{\log_2(x)} \stackrel{y=x-1}{=} \lim_{y \rightarrow 0} -\frac{y}{\log_2(1+y)} = \lim_{y \rightarrow 0} -\ln(2) \frac{y}{\ln(1+y)} = -\ln(2)$.

(g)

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1 + x2^x}{1 + x3^x} \right)^{\frac{1}{x^2}} &= \lim_{x \rightarrow 0} e^{\frac{1}{x^2} \ln\left(\frac{1 + x2^x}{1 + x3^x}\right)} = \lim_{x \rightarrow 0} e^{\frac{1}{x^2} \ln\left(1 + \frac{x(2^x - 3^x)}{1 + x3^x}\right)} = \\ &= e^{\lim_{x \rightarrow 0} \frac{\frac{x(2^x - 3^x)}{1 + x3^x} \ln\left(1 + \frac{x(2^x - 3^x)}{1 + x3^x}\right)}{x^2} \frac{x(2^x - 3^x)}{1 + x3^x}} = e^{\lim_{x \rightarrow 0} \frac{2^x - 3^x}{x(1 + x3^x)} \frac{\ln\left(1 + \frac{x(2^x - 3^x)}{1 + x3^x}\right)}{\frac{x(2^x - 3^x)}{1 + x3^x}}}, \end{aligned}$$

kde je problém pouze v limitě

$$\lim_{x \rightarrow 0} \frac{2^x - 3^x}{x} = \lim_{x \rightarrow 0} 2^x \frac{1 - \left(\frac{3}{2}\right)^x}{x} = \lim_{x \rightarrow 0} 2^x \frac{1 - e^{x \ln\left(\frac{3}{2}\right)}}{x} = \lim_{x \rightarrow 0} 2^x \ln\left(\frac{3}{2}\right) \frac{1 - e^{x \ln\left(\frac{3}{2}\right)}}{x \ln\left(\frac{3}{2}\right)} = \ln\left(\frac{3}{2}\right),$$

tedy

$$\lim_{x \rightarrow 0} \left(\frac{1 + x2^x}{1 + x3^x} \right)^{\frac{1}{x^2}} = e^{\ln\left(\frac{3}{2}\right)} = \frac{3}{2}.$$

(h)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln(a+x) + 2\ln(a-x) - 2\ln(a)}{x^2} &= \lim_{x \rightarrow 0} \frac{\ln\left(\frac{(a+x)(a-x)^2}{a^2}\right)}{x^2} = \lim_{x \rightarrow 0} \frac{\ln\left(1 + \frac{(a+x)(a-x)^2 - a^2}{a^2}\right)}{x^2} = \\ &= \lim_{x \rightarrow 0} \frac{\ln\left(1 + \frac{(a+x)(a-x)^2 - a^2}{a^2}\right)}{\frac{(a+x)(a-x)^2 - a^2}{a^2}} \frac{(a+x)(a-x)^2 - a^2}{x^2} = \\ &= \lim_{x \rightarrow 0} \frac{\ln\left(1 + \frac{(a+x)(a-x)^2 - a^2}{\frac{(a+x)(a-x)^2 - a^2}{a^2}}\right)}{x^2} \frac{a^3 + a^2x - 2a^2x - 2ax^2 + ax^2 + x^3 - a^2}{a^2x^2} = \\ &= \lim_{x \rightarrow 0} \underbrace{\frac{\ln\left(1 + \frac{(a+x)(a-x)^2 - a^2}{\frac{(a+x)(a-x)^2 - a^2}{a^2}}\right)}{\frac{(a+x)(a-x)^2 - a^2}{a^2}}}_{\rightarrow 1} \frac{a^3 - a^2x - ax^2 + x^3 - a^2}{a^2x^2} = \begin{cases} -\infty & a \in (0, 1] \\ \infty & a > 1 \end{cases}. \end{aligned}$$

(i)

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln(\cos(ax))}{\ln(\cos(bx))} &= \lim_{x \rightarrow 0} \frac{\ln(\cos^2(ax))}{\ln(\cos^2(bx))} = \lim_{x \rightarrow 0} \frac{\ln(1 - \sin^2(ax))}{\ln(1 - \sin^2(bx))} = \\ &= \lim_{x \rightarrow 0} \frac{\ln(1 - \sin^2(ax))}{-\sin^2(ax)} \frac{-\sin^2(bx)}{\ln(1 - \sin^2(bx))} \frac{-\sin^2(ax)}{-\sin^2(bx)} = \\ &= \lim_{x \rightarrow 0} \underbrace{\frac{\ln(1 - \sin^2(ax))}{-\sin^2(ax)}}_{\rightarrow 1} \underbrace{\frac{-\sin^2(bx)}{\ln(1 - \sin^2(bx))}}_{\rightarrow 1} \underbrace{\frac{\sin^2(ax)}{(ax)^2}}_{\rightarrow 1} \underbrace{\frac{(bx)^2}{\sin^2(bx)}}_{\rightarrow 1} \frac{(ax)^2}{(bx)^2} = \frac{a^2}{b^2}. \end{aligned}$$

(j)

$$\lim_{x \rightarrow 0} \frac{\ln(\tan(\frac{\pi}{4} + ax))}{\sin(bx)} = \lim_{x \rightarrow 0} \frac{\ln(1 + (\tan(\frac{\pi}{4} + ax) - 1))}{\tan(\frac{\pi}{4} + ax) - 1} \underbrace{\frac{bx}{\sin(bx)}}_{\rightarrow 1} \frac{\tan(\frac{\pi}{4} + ax) - 1}{bx} = \lim_{x \rightarrow 0} \frac{\tan(\frac{\pi}{4} + ax) - 1}{bx},$$

kde použijeme sčítací vzorec $\tan(A + B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}$ a dostáváme

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan(\frac{\pi}{4} + ax) - 1}{bx} &= \lim_{x \rightarrow 0} \frac{\frac{\tan(ax) + 1}{1 - \tan(ax)} - 1}{bx} = \lim_{x \rightarrow 0} \frac{2 \tan(ax)}{bx(1 - \tan(ax))} = \\ &= \lim_{x \rightarrow 0} \frac{2 \sin(ax)}{bx \cos(ax)(1 - \tan(ax))} \stackrel{\frac{\sin(x)}{x} \rightarrow 1}{=} \lim_{x \rightarrow 0} \frac{2a}{b \cos(ax)(1 - \tan(ax))} = \frac{2a}{b}. \end{aligned}$$

(k)

$$\begin{aligned} \lim_{x \rightarrow 0^+} (\cos(\sqrt{x}))^{\frac{1}{x}} &= \lim_{x \rightarrow 0^+} e^{\frac{\ln(\cos(\sqrt{x}))}{x}} = \lim_{x \rightarrow 0^+} e^{\frac{\ln(1 - \sin^2(\sqrt{x}))}{2x}} = \lim_{x \rightarrow 0^+} e^{\frac{\ln(1 - \sin^2(\sqrt{x})) - \sin^2(\sqrt{x})}{- \sin^2(\sqrt{x})} \frac{-\sin^2(\sqrt{x})}{2x}} = \\ &= \lim_{x \rightarrow 0^+} e^{-\frac{1}{2} \underbrace{\frac{\ln(1 - \sin^2(\sqrt{x}))}{-\sin^2(\sqrt{x})}}_{\rightarrow 1} \underbrace{\left(\frac{\sin(\sqrt{x})}{\sqrt{x}}\right)^2}_{\rightarrow 1}} = e^{-\frac{1}{2}}. \end{aligned}$$

(l) $\lim_{x \rightarrow 1} \frac{\sin(\pi x^a)}{\pi x^b} = 0.$

Derivace

Spočtěte

$$\mathcal{F}[f(x)] = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

pro

(a) $f(x) = x^\alpha, \quad \alpha \in \mathbb{R},$

(d) $f(x) = e^x,$

(b) $f(x) = \sin(x),$

(e) $f(x) = \ln(x),$

(c) $f(x) = \cos(x),$

(f) $f(x) = \arctan(x).$

(a)

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(x+h)^\alpha - x^\alpha}{h} &= \lim_{h \rightarrow 0} x^\alpha \frac{(1 + \frac{h}{x})^\alpha - 1}{h} = \\ &= \lim_{h \rightarrow 0} x^\alpha \frac{e^{\alpha \ln(1 + \frac{h}{x})} - 1}{\alpha \ln(1 + \frac{h}{x})} \frac{\alpha \ln(1 + \frac{h}{x})}{h} = \lim_{h \rightarrow 0} x^{\alpha-1} \frac{\alpha \ln(1 + \frac{h}{x})}{\frac{h}{x}} = \alpha x^{\alpha-1}. \end{aligned}$$

(b)

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} = \\ &= \lim_{h \rightarrow 0} h \underbrace{\frac{\sin(x)(\cos(h) - 1)}{h^2}}_{\rightarrow \frac{\sin(x)}{2}} + \frac{\cos(x)\sin(h)}{h} = 0 + \cos(x). \end{aligned}$$

(c)

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos(x)}{h} &= \lim_{h \rightarrow 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} = \\ &= \lim_{h \rightarrow 0} h \cos(x) \underbrace{\frac{\cos(h) - 1}{h^2}}_{\rightarrow -\frac{1}{2}} - \frac{\sin(x)\sin(h)}{h} = 0 - \sin(x). \end{aligned}$$

(d)

$$\lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \rightarrow 0} e^x \frac{e^h - 1}{h} = e^x.$$

(e)

$$\lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h \rightarrow 0} \frac{\ln(\frac{x+h}{x})}{h} = \lim_{h \rightarrow 0} \frac{\ln(1 + \frac{h}{x})}{\frac{h}{x}} \frac{1}{x} = \frac{1}{x}.$$

(f)

$$\lim_{h \rightarrow 0} \frac{\arctan(x+h) - \arctan(x)}{h} = \lim_{h \rightarrow 0} \frac{\arctan(x+h) - \arctan(x)}{x+h-x},$$

kde zavedeme nové proměnné $x+h = \tan(u)$ a $x = \tan(v)$, což dává

$$\lim_{u \rightarrow v} \frac{u - v}{\tan(u) - \tan(v)} = \frac{1}{1 + \tan^2(v)} = \frac{1}{1 + x^2}.$$

což víme z příkladu 2(c).