

Řešení cvičení 12: Určitý integrál II

Výpočet

Spočítejte následující integrály pro $k \in \mathbb{N}_0$

(a)
$$\int_0^2 \frac{1}{e^{\frac{x}{2}} + e^x} dx,$$

(c)
$$\int_0^1 \frac{x^2}{(1-x)^{100}} dx,$$

(b)
$$\int_0^9 x^3 \sqrt[3]{1+x^2} dx,$$

(d)
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\cos^2(x) \sin^2(x)}.$$

(a)

$$\begin{aligned} \int_0^2 \frac{1}{e^{\frac{x}{2}} + e^x} dx &= \int_0^2 \frac{e^{\frac{x}{2}}}{e^x + e^{\frac{3x}{2}}} dx = \left\{ \frac{1}{2} e^{\frac{x}{2}} dx = dy \right\} = \int_1^e \frac{2}{y^2 + y^3} dy = \int_1^e \frac{2}{y^2(1+y)} dy = \\ \left\{ \frac{2}{y^2(1+y)} = \frac{a_1}{y} + \frac{b_1 y + b_2}{y^2} + \frac{2c_1}{1+y} \right\} &= \int_1^e \left(-\frac{2}{y} + \frac{2}{y^2} + \frac{2}{1+y} \right) dy = \left[-2 \ln(y) - \frac{2}{y} + 2 \ln(1+y) \right]_1^e \end{aligned}$$

(b)

$$\begin{aligned} \int_0^9 x^3 \sqrt[3]{1+x^2} dx &= \left\{ \begin{array}{l} x^2 = y \\ 2x dx = dy \end{array} \right\} = \frac{1}{2} \int_0^3 y \sqrt[3]{1+y} dy \stackrel{pp.}{=} \left[y \frac{3(1+y)^{\frac{4}{3}}}{4} \right]_0^3 - \frac{1}{2} \int_0^3 \frac{3(1+y)^{\frac{4}{3}}}{4} dy = \\ &= \left[y \frac{3(1+y)^{\frac{4}{3}}}{4} \right]_0^3 - \frac{3}{8} \left[\frac{3(1+y)^{\frac{7}{3}}}{7} \right]_0^3 = \frac{9}{4} 4^{\frac{4}{3}} - \frac{3}{8} \left(\frac{3}{7} 4^{\frac{7}{3}} - \frac{3}{7} \right). \end{aligned}$$

(c)

$$\begin{aligned} \int_0^1 \frac{x^2}{(1+x)^{100}} dx &\stackrel{pp.}{=} \left[-\frac{x^2}{99(1+x)^{99}} \right]_0^1 - \int_0^1 -\frac{2x}{99(1+x)^{99}} dx = -\frac{1}{99 \cdot 2^{99}} + \int_0^1 \frac{2x}{99(1+x)^{99}} dx \stackrel{pp.}{=} \\ &= -\frac{1}{99 \cdot 2^{99}} + \left[-\frac{2x}{99 \cdot 98(1+x)^{98}} \right]_0^1 + \int_0^1 \frac{2}{99 \cdot 98(1+x)^{98}} dx = -\frac{1}{99 \cdot 2^{99}} - \frac{2}{99 \cdot 98 \cdot 2^{98}} \\ &+ \int_0^1 \frac{2}{99 \cdot 98(1+x)^{98}} dx = -\frac{1}{99 \cdot 2^{99}} - \frac{2}{99 \cdot 98 \cdot 2^{98}} - \frac{2}{99 \cdot 98 \cdot 97 \cdot 2^{97}} + \frac{2}{99 \cdot 98 \cdot 97}. \end{aligned}$$

(d)

$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\cos^2(x) \sin^2(x)} &= \int_{\frac{\pi}{6}}^{\frac{2\pi}{6}} \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x) \sin^2(x)} dx = \int_{\frac{\pi}{6}}^{\frac{2\pi}{6}} \frac{1}{\sin^2(x)} + \frac{1}{\cos^2(x)} dx = \\ &= [\tan(x) + \cot(x)]_{\frac{\pi}{6}}^{\frac{2\pi}{6}} = \frac{4}{\sqrt{3}}. \end{aligned}$$

Oblasti mezi křivkami

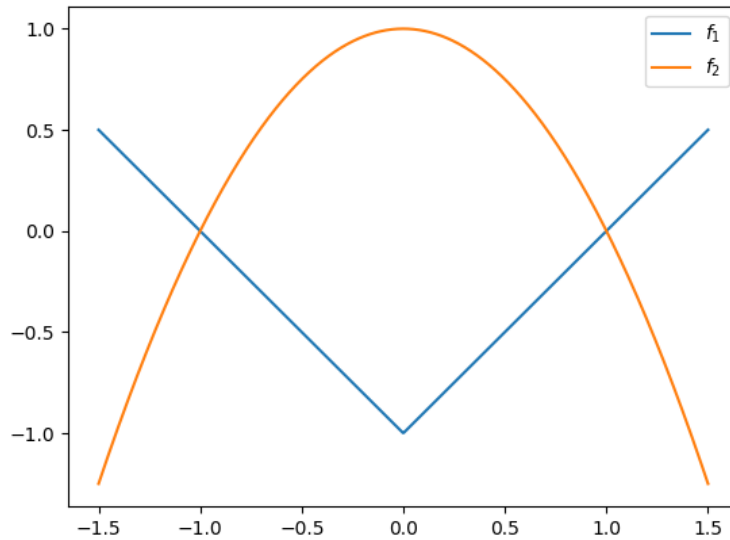
Spočítejte obsah plochy ohraničené následujícími křivkami

(a)
$$\begin{aligned} f_1(x) &= |x| - 1, \\ f_2(x) &= 1 - x^2 \end{aligned}$$

$$\begin{aligned} f_2(x) &= -x + 1, \\ f_3(x) &= x - 1, \end{aligned}$$

(b)
$$f_1(x) = x + 1,$$

$$f_4(x) = -x - 1,$$



$$(c) \quad f_1(x) = \frac{(x-1)^2}{6} - 1, \\ f_2(x) = \frac{x^2}{10} + 2,$$

$$(d) \quad f_1(x) = \sqrt{1-x^2}, \\ f_2(x) = -\sqrt{1-x^2}$$

$$(a) \quad f_1(x) = |x| - 1, \\ f_2(x) = 1 - x^2$$

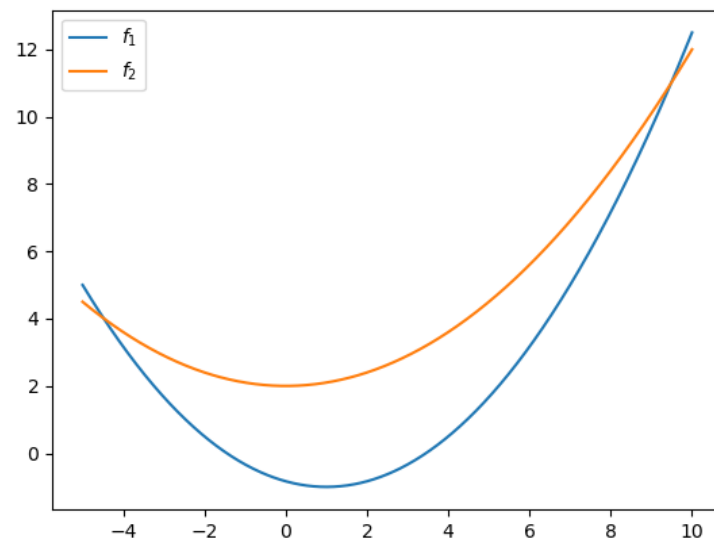
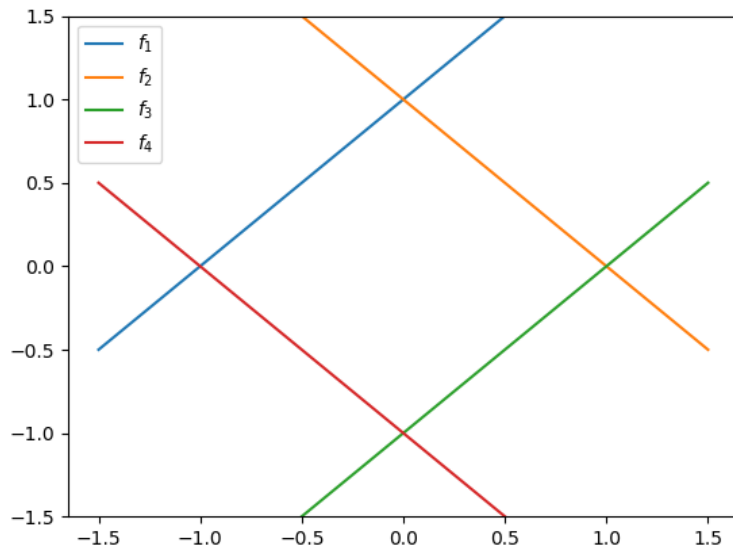
$$S = \int_{-1}^1 ||x| - 1 - (1 - x^2)| \, dx = \int_{-1}^1 1 - x^2 - |x| + 1 \, dx = \int_{-1}^0 2 - x^2 + x \, dx + \int_0^1 2 - x^2 - x \, dx = \\ \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^0 + \left[2x - \frac{x^3}{3} - \frac{x^2}{2} \right]_0^1 = 2 - \frac{1}{3} - \frac{1}{2} + 2 - \frac{1}{3} - \frac{1}{2} = 3 - \frac{2}{3}.$$

$$(b) \quad f_1(x) = x + 1, \\ f_2(x) = -x + 1, \\ f_3(x) = x - 1, \\ f_4(x) = -x - 1,$$

$$S = \int_{-1}^0 |(x+1) - (-x-1)| \, dx + \int_0^1 |(-x+1) - (x-1)| \, dx = \int_{-1}^0 |2+2x| \, dx + \int_0^1 |2-2x| \, dx = \\ \int_{-1}^0 2+2x \, dx + \int_0^1 2-2x \, dx = [2x+x^2]_{-1}^0 + [2x-x^2]_0^1 = 2.$$

$$(c) \quad f_1(x) = \frac{(x-1)^2}{6} - 1, \\ f_2(x) = \frac{x^2}{10} + 2, \text{ Průsečíky spočteme pomocí}$$

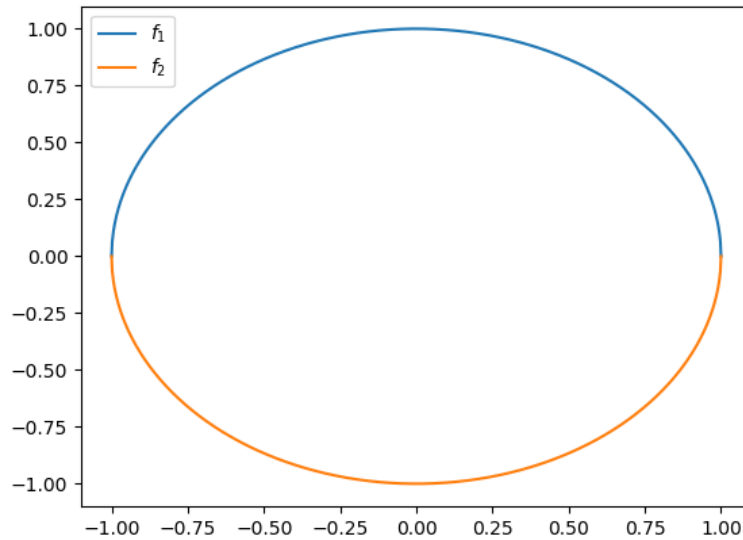
$$\frac{(x-1)^2}{6} - 1 = \frac{x^2}{10} + 2 \xrightarrow{\text{Wolfram}} x = \frac{5}{2} \pm \frac{\sqrt{195}}{2}.$$



Pak (označme $x_{\pm} := \frac{5}{2} \pm \frac{\sqrt{195}}{2}$)

$$\begin{aligned}
 S &= \int_{x_-}^{x_+} \left| \left(\frac{(x-1)^2}{6} - 1 \right) - \left(\frac{x^2}{10} + 2 \right) \right| dx = \int_{x_-}^{x_+} \frac{x^2}{10} + 2 - \frac{(x-1)^2}{6} + 1 dx = \\
 &= \int_{x_-}^{x_+} \left(\frac{1}{10} - \frac{1}{6} \right) x^2 + \frac{x}{3} + 3 - \frac{1}{6} dx = \left[\left(\frac{1}{10} - \frac{1}{6} \right) \frac{x^3}{3} + \frac{x}{6} + \left(+3 - \frac{1}{6} \right) x \right]_{x_-}^{x_+} \stackrel{\text{Wolfram}}{=} \\
 &= \frac{13}{2} \sqrt{\frac{65}{3}} \approx 30.26.
 \end{aligned}$$

(d) $f_1(x) = \sqrt{1-x^2}$,



$$f_2(x) = -\sqrt{1-x^2},$$

$$S = \int_{-1}^1 |\sqrt{1-x^2} - (-\sqrt{1-x^2})| dx = 2 \int_{-1}^1 \sqrt{1-x^2} dx = \left\{ \begin{array}{l} x = \sin(y) \\ dx = \cos(y) dy \end{array} \right\} =$$

$$2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2(x)} \cos(x) dx = 2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} |\cos(x)| \cos(x) dx = -2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos^2(x) dx,$$

kde

$$I = \int \cos^2(x) dx \stackrel{pp.}{=} \cos(x) \sin(x) + \int \sin^2(x) dx = \cos(x) \sin(x) + \int 1 - \cos^2(x) dx = \cos(x) \sin(x) + x - I + c,$$

$$\text{tedy } \int \cos^2(x) dx = \frac{1}{2}(\cos(x) \sin(x) + x) + c \text{ a}$$

$$S = -2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos^2(x) dx = -2 \frac{1}{2} [\cos(x) \sin(x) + x]_{\frac{3\pi}{2}}^{\frac{\pi}{2}} = - \left(0 + \frac{\pi}{2} - 0 - \frac{3\pi}{2} \right) = \pi.$$

Délka křivky

Spočtete délky následujících křivek mezi a a b

(a) $f(x) = x^2$, $a = -1$, $b = 1$,

(c) $f(x) = \frac{x^2}{4} - \frac{\ln(x)}{2}$, $a = 1$, $b = e$,

(b) $f(x) = e^x$, $a = 0$, $b = 5$,

(d) $f(x) = \sqrt{1-x^2}$, $a = -1$, $b = 1$.

(a)

$$\int_{-1}^1 \sqrt{1+(2x)^2} dx = \left\{ \begin{array}{l} 2x = y \\ 2 dx = dy \end{array} \right\} = \frac{1}{2} \int_{-2}^2 \sqrt{1+y^2} dy =$$

Tento integrál silně připomíná integrál typu $\sqrt{1-x^2}$, ale zde máme $+$. Proto by se nám hodila funkce obdobná $\sin(x)$, ale taková, že $1-f^2(x)$ je nějaký čtverec. Takovou funkcí je tkz. hyperbolický sinus, definovaný pomocí

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Potom platí $\cosh^2(x) - \sinh^2(x) = 1$, což je zobecnění goniometrické jedničky. Pokud tedy zkusíme zde provést substituci

$$= \left\{ \begin{array}{l} y = \sinh(z) \\ dy = \cosh(z) dz \end{array} \right\} = \frac{1}{2} \int_{-\sinh(2)}^{\sinh(2)} \sqrt{1 + \sinh^2(z)} \cosh(z) dz =$$

$$\frac{1}{2} \int_{-\sinh(2)}^{\sinh(2)} \cosh^2(z) dz = \frac{1}{2} \int_{-\sinh(2)}^{\sinh(2)} \frac{e^{2z} + 2 + e^{-2z}}{4} dz = \frac{1}{2} \left[\frac{e^{2z} - e^{-2z}}{8} + 2z \right]_{-\sinh(2)}^{\sinh(2)}.$$

(b)

$$\int_0^5 \sqrt{1 + e^{2x}} dx \stackrel{2x=y}{=} \frac{1}{2} \int_0^{10} \sqrt{1 + e^y} dy = \left\{ \begin{array}{l} e^y = z \\ e^y dy = dz \end{array} \right\} = \frac{1}{2} \int_1^{e^{10}} \frac{\sqrt{1+z}}{z} dz =$$

$$\left\{ \begin{array}{l} \sqrt{1+z} = t \\ \frac{1}{2\sqrt{1+z}} dz = dt \end{array} \right\} = \int_{\sqrt{2}}^{\sqrt{1+e^{10}}} \frac{t^2}{t^2-1} dt = \int_{\sqrt{2}}^{\sqrt{1+e^{10}}} 1 + \frac{1}{t^2-1} dt \stackrel{\text{parc. zl.}}{=} \\ \int_{\sqrt{2}}^{\sqrt{1+e^{10}}} 1 + \frac{1}{2(t-1)} + \frac{1}{2(t+1)} dt = \left[t + \frac{1}{2} \ln(t^2-1) \right]_{\sqrt{2}}^{\sqrt{1+e^{10}}}.$$

(c)

$$\int_1^e \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2} dx = \int_1^e \sqrt{\frac{x^4 + 2x^2 + 1}{4x^2}} dx = \int_1^e \sqrt{\frac{(x^2+1)^2}{4x^2}} dx = \int_1^e \frac{x^2+1}{2x} dx =$$

$$\int_1^e \frac{x}{2} + \frac{1}{2x} dx = \left[\frac{x^2}{4} + \frac{\ln(x)}{2} \right]_1^e = \frac{e^2}{4} + \frac{1}{2} - \frac{1}{4} = \frac{e^2+1}{4}.$$

(d)

$$\int_{-1}^1 \sqrt{1 + \frac{x^2}{1-x^2}} dx = \int_{-1}^1 \sqrt{\frac{1}{1-x^2}} dx = \left\{ \begin{array}{l} x = \cos(y) \\ dx = -\sin(y) dy \end{array} \right\} =$$

$$- \int_{\pi}^0 \sqrt{\frac{1}{\sin^2(y)}} \sin(y) dy = - \int_{\pi}^0 dy = \int_0^{\pi} dy = \pi.$$

Objem tělesa

Spočítejte objem těles vzniklých rotací následujících křivek od a do b

(a) $f(x) = \sqrt{1-x^2}$, $a = -1$, $b = 1$,

(c) $f(x) = \frac{1}{x}$, $a = 0$, $b = \infty$,

(b) $f(x) = \frac{r}{h}x$, $a = 0$, $b = h$,

(d) $f(x) = r$, $a = 0$, $b = h$.

(a)

$$\int_{-1}^1 \pi(1-x^2) dx = \left\{ \begin{array}{l} x = \cos(y) \\ dx = -\sin(y) dy \end{array} \right\} = -\pi \int_{\pi}^0 (1 - \cos^2(y)) \sin(y) dy = -\pi \int_{\pi}^0 \sin^3(y) dy,$$

kde

$$I = \int_{\pi}^0 \sin^3(y) dy \stackrel{pp.}{=} \int_{\pi}^0 \frac{F = -\cos(y) \quad G = \sin^2(y)}{f = \sin(y) \quad g = 2 \sin(y) \cos(y)} = [-\cos(y) \sin^2(y)]_{\pi}^0 + 2 \int_{\pi}^0 \cos^2(y) \sin(y) dy =$$

$$2 \int_{\pi}^0 (1 - \sin^2(y)) \sin(y) dy = 2 \int_{\pi}^0 \sin(y) dy - 2\pi I = 2[-\cos(y)]_{\pi}^0 - 2I = -4 - 2I,$$

neboli $I = -\frac{4}{3}$ a

$$\int_{-1}^1 \pi(1-x^2) dx = -\pi \int_{\pi}^0 \sin^3(y) dy = \frac{4\pi}{3}.$$

(b)

$$\int_0^h \pi \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2} \int_0^h x^2 dx = \pi \frac{r^2}{h^2} \left[\frac{x^3}{3} \right]_0^h = \pi \frac{r^2 h}{3}.$$

Tohle je opravdu objem kužele o výšce h a poloměru r .

(c)

$$\int_0^{\infty} \pi \frac{1}{x^2} dx = \pi \int_0^{\infty} \frac{1}{x^2} dx = \pi \left[-\frac{1}{x} \right]_0^{\infty} = \infty.$$

(d)

$$\int_0^h \pi r^2 dx = \pi r^2 \int_0^h dx = \pi r^2 h,$$

což je objem válce.

Povrch tělesa

Spočítejte povrch těles vzniklých rotací následujících křivek od a do b

(a) $f(x) = \sqrt{1-x^2}$, $a = -1$, $b = 1$,

(c) $f(x) = r$, $a = 0$, $b = h$.

(b) $f(x) = \frac{r}{h}x$, $a = 0$, $b = h$,

(d) $f(x) = \sqrt{x}$, $a = 0$, $b = 5$.

(a)

$$\int_{-1}^1 2\pi \sqrt{1-x^2} \sqrt{1+\frac{x^2}{1-x^2}} dx = 2\pi \int_{-1}^1 \sqrt{1-x^2} \sqrt{\frac{1}{1-x^2}} dx = 2\pi \int_{-1}^1 dx = 4\pi,$$

což je opravdu povrch koule o poloměru 1.

(b)

$$\int_0^h 2\pi \frac{r}{h} x \sqrt{1+\frac{r^2}{h^2}} dx = 2\pi \frac{r}{h} \sqrt{1+\frac{r^2}{h^2}} \int_0^h x dx = 2\pi r \sqrt{1+\frac{r^2}{h^2}}.$$

(c)

$$\int_0^h 2\pi r \sqrt{1+0} dx = 2\pi r h.$$

(d)

$$\int_0^5 2\pi \sqrt{x} \sqrt{1+\frac{1}{4x}} dx = \pi \int_0^5 \sqrt{1+4x} dx = \pi \left[\frac{2(1+4x)^{\frac{3}{2}}}{4 \cdot 3} \right]_0^5 = \frac{\pi}{6} (21^{\frac{3}{2}} - 1).$$