Řešení cvičení 6: Limity funkcí

Limity funkcí (snadné)

Spočtěte následující limity funkcí pro $m, n \in \mathbb{N}$

(a) $\lim_{x\to 0} \frac{\cos(x)+1}{\cos(x)-1}$,

(b) $\lim_{x\to 0} \frac{x^2-1}{2x^2-x-1}$,

(c) $\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1}$,

(d) $\lim_{x\to 1} \frac{x^3-3x+2}{x^4-4x+3}$,

(e) $\lim_{x\to 1} \lfloor x\rfloor - x$,

(f) $\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2}$,

(g) $\lim_{x\to 0} e^{\frac{\sqrt[3]{1-x^2}-1}{5x^2}}$.

(a) $\lim_{x\to 0} \frac{\cos(x)+1}{\cos(x)-1} = -\infty$.

(b) $\lim_{x\to 0} \frac{x^2-1}{2x^2-x-1} = 1$.

(c) $\lim_{x\to 1} \frac{x^2-1}{2x^2-x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(2x+1)} = \frac{2}{3}$.

(d) $\lim_{x\to 1} \frac{x^3-3x+2}{x^4-4x+3} = \lim_{x\to 1} \frac{(x+2)(x-1)^2}{(x^2+2x+3)(x-1)^2} = \frac{3}{6} = \frac{1}{2}$,

(e) Tato limita neexistuje, protože $\lim_{x\to 1^-} \lfloor x\rfloor - x = 1$ a $\lim_{x\to 1^+} \lfloor x\rfloor - x = 0$.

(f)

$$\lim_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2} = \lim_{x\to 0} \frac{\sum_{i=0}^n \binom{n}{i}(mx)^i - \sum_{i=0}^m \binom{m}{i}(nx)^i}{x^2} =$$
$$\lim_{x\to 0} \frac{1+mnx+\binom{n}{2}(mx)^2-(1+mnx+\binom{m}{2}(nx)^2)+o(x^3)}{x^2} =$$
$$\lim_{x\to 0} \frac{n(n+1)(mx)^2-m(m+1)(nx)^2+o(x^3)}{2x^2} =$$
$$\lim_{x\to 0} \frac{nm^2x^2-mn^2x^2+o(x^3)}{2x^2} = \frac{nm(n-m)}{2}.$$

(g) $\lim_{x\to 0} e^{\frac{\sqrt[3]{1-x^2}-1}{5x^2}} = \lim_{x\to 0} e^{\frac{1-x^2-1}{5x^2((1-x^2)^{\frac{2}{3}}+\sqrt[3]{1-x^2}+1)}} = e^{\lim_{x\to 0} \frac{-1}{5((1-x^2)^{\frac{2}{3}}+\sqrt[3]{1-x^2}+1)}} = e^{-\frac{1}{15}}$.

Limity funkcí (obtížnější)

Spočtěte následující limity funkcí pro $a, b \in \mathbb{R}, b \neq 0$ a $m, n \in \mathbb{N}$

(a) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2}$,

(b) $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$,

(c) $\lim_{x\to a} \frac{\tan(x)-\tan(a)}{x-a}$,

(d) $\lim_{x\to 0^+} (1+x)^{\frac{1}{x}}$,

(e) $\lim_{x\to \pi} \frac{\sin(mx)}{\sin(nx)}$,

(f) $\lim_{x\to 1} (1-x)\log_x(2)$,

(g) $\lim_{x\to 0} \left(\frac{1+x2^x}{1+x3^x}\right)^{\frac{1}{x^2}}$,

(h) $\lim_{x\to 0} \frac{\ln(a+x)+2\ln(a-x)-2\ln(a)}{x^2}$, $\quad a > 0$,

(i) $\lim_{x\to 0} \frac{\ln(\cos(ax))}{\ln(\cos(bx))}$,

(j) $\lim_{x\to 0} \frac{\ln(\tan(\frac{\pi}{4}+ax))}{\sin(bx)}$,

(k) $\lim_{x\to 0^+} (\cos(\sqrt{x}))^{\frac{1}{x}}$,

(l) $\lim_{x\to 1} \frac{\sin(\pi x^a)}{\pi x^b}$.

(a) $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{1-\cos^2(x)}{x^2(1+\cos(x))} = \lim_{x\to 0} \frac{\sin^2(x)}{x^2} \frac{1}{1+\cos(x)} = \frac{1}{2}$.

(b) Limita neexistuje, protože máme Heineho větu, která pro posloupnosti $a_n = \frac{1}{\pi n}$ a $a_n = \frac{1}{\pi(2n+1)}$ dává odlišné výsledky.

(c) Použijeme sčítací vzorec pro tan

$$\lim_{x\to a} \frac{\tan(x)-\tan(a)}{x-a} = \lim_{x\to a} \frac{\tan(x-a)(1+\tan(a)\tan(x))}{x-a} = \lim_{x\to a} \frac{\sin(x-a)(1+\tan(a)\tan(x))}{(x-a)\cos(x-a)} =$$

$$\lim_{x\to a} \frac{\sin(x-a)}{x-a} \frac{(1+\tan(a)\tan(x))}{\cos(x-a)} \overset{x-a=h}{=} \lim_{h\to 0} \underbrace{\frac{\sin(h)}{h}}_{\to 1} \frac{(1+\tan(a)\tan(a+h))}{\cos(h)} = 1+\tan^2(a).$$

(d) $\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\frac{\ln(1+x)}{x}} = e$.

(e) $\lim_{x\to\pi} \frac{\sin(mx)}{\sin(nx)} = \lim_{x\to 0} \frac{-\sin(mx)}{-\sin(nx)} = \lim_{x\to 0} \frac{\sin(mx)}{mnx} \frac{mnx}{\sin(nx)} = \frac{m}{n}$.

(f) $\lim_{x\to 1} (1-x)\log_x(2) = \lim_{x\to 1} -\frac{x-1}{\log_2(x)} \overset{y=x-1}{=} \lim_{y\to 0} -\frac{y}{\log_2(1+y)} = \lim_{y\to 0} -\ln(2)\frac{y}{\ln(1+y)} = -\ln(2)$.

(g)

$$\lim_{x\to 0} \left(\frac{1+x2^x}{1+x3^x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0} e^{\frac{1}{x^2}\ln\left(\frac{1+x2^x}{1+x3^x}\right)} = \lim_{x\to 0} e^{\frac{1}{x^2}\ln\left(1+\frac{x(2^x-3^x)}{1+x3^x}\right)} =$$

$$e^{\lim_{x\to 0} \frac{\frac{x(2^x-3^x)}{1+x3^x}}{x^2} \frac{\ln\left(1+\frac{x(2^x-3^x)}{1+x3^x}\right)}{\frac{x(2^x-3^x)}{1+x3^x}}} = e^{\lim_{x\to 0} \frac{2^x-3^x}{x(1+x3^x)} \frac{\ln\left(1+\frac{x(2^x-3^x)}{1+x3^x}\right)}{\frac{x(2^x-3^x)}{1+x3^x}}},$$

kde je problém pouze v limitě

$$\lim_{x\to 0} \frac{2^x-3^x}{x} = \lim_{x\to 0} 2^x \frac{1-\left(\frac{3}{2}\right)^x}{x} = \lim_{x\to 0} 2^x \frac{1-e^{x\ln\left(\frac{3}{2}\right)}}{x} = \lim_{x\to 0} 2^x \ln\left(\frac{3}{2}\right) \frac{1-e^{x\ln\left(\frac{3}{2}\right)}}{x\ln\left(\frac{3}{2}\right)} = \ln\left(\frac{3}{2}\right),$$

tedy

$$\lim_{x\to 0} \left(\frac{1+x2^x}{1+x3^x}\right)^{\frac{1}{x^2}} = e^{\ln\left(\frac{3}{2}\right)} = \frac{3}{2}.$$

(h)

$$\lim_{x\to 0} \frac{\ln(a+x)+2\ln(a-x)-2\ln(a)}{x^2} = \lim_{x\to 0} \frac{\ln\left(\frac{(a+x)(a-x)^2}{a^2}\right)}{x^2} = \lim_{x\to 0} \frac{\ln\left(1+\frac{(a+x)(a-x)^2-a^2}{a^2}\right)}{x^2} =$$

$$\lim_{x\to 0} \frac{\ln\left(1+\frac{(a+x)(a-x)^2-a^2}{a^2}\right)}{\frac{(a+x)(a-x)^2-a^2}{a^2}} \frac{\frac{(a+x)(a-x)^2-a^2}{a^2}}{x^2} =$$

$$\lim_{x\to 0} \frac{\ln\left(1+\frac{(a+x)(a-x)^2-a^2}{\frac{(a+x)(a-x)^2-a^2}{a^2}}\right)}{x^2} \frac{a^3+a^2x-2a^2x-2ax^2+ax^2+x^3-a^2}{a^2x^2} =$$

$$\lim_{x\to 0} \underbrace{\frac{\ln\left(1+\frac{(a+x)(a-x)^2-a^2}{a^2}\right)}{\frac{(a+x)(a-x)^2-a^2}{a^2}}}_{\to 1} \frac{a^3-a^2x-ax^2+x^3-a^2}{a^2x^2} = \begin{cases} -\infty & a\in(0,1] \\ \infty & a>1 \end{cases}.$$

(i)

$$\lim_{x\to 0}\frac{\ln(\cos(ax))}{\ln(\cos(bx))}=\lim_{x\to 0}\frac{\ln(\cos^2(ax))}{\ln(\cos^2(bx))}=\lim_{x\to 0}\frac{\ln(1-\sin^2(ax))}{\ln(1-\sin^2(bx))}=$$
$$\lim_{x\to 0}\frac{\ln(1-\sin^2(ax))}{-\sin^2(ax)}\frac{-\sin^2(bx)}{\ln(1-\sin^2(bx))}\frac{-\sin^2(ax)}{-\sin^2(bx)}=$$
$$\lim_{x\to 0}\underbrace{\frac{\ln(1-\sin^2(ax))}{-\sin^2(ax)}}_{\to 1}\underbrace{\frac{-\sin^2(bx)}{\ln(1-\sin^2(bx))}}_{\to 1}\underbrace{\frac{\sin^2(ax)}{(ax)^2}}_{\to 1}\underbrace{\frac{(bx)^2}{\sin^2(bx)}}_{\to 1}\frac{(ax)^2}{(bx)^2}=\frac{a^2}{b^2}.$$

(j)

$$\lim_{x\to 0}\frac{\ln(\tan(\frac{\pi}{4}+ax))}{\sin(bx)}=\lim_{x\to 0}\underbrace{\frac{\ln(1+(\tan(\frac{\pi}{4}+ax)-1))}{\tan(\frac{\pi}{4}+ax)-1}}_{\to 1}\underbrace{\frac{bx}{\sin(bx)}}_{\to 1}\frac{\tan(\frac{\pi}{4}+ax)-1}{bx}=\lim_{x\to 0}\frac{\tan(\frac{\pi}{4}+ax)-1}{bx},$$

kde použijeme sčítací vzorec $\tan(A+B)=\frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$ a dostáváme

$$\lim_{x\to 0}\frac{\tan(\frac{\pi}{4}+ax)-1}{bx}=\lim_{x\to 0}\frac{\frac{\tan(ax)+1}{1-\tan(ax)}-1}{bx}=\lim_{x\to 0}\frac{2\tan(ax)}{bx(1-\tan(ax))}=$$
$$\lim_{x\to 0}\frac{2\sin(ax)}{bx\cos(ax)(1-\tan(ax))}\overset{\frac{\sin(x)}{x}\to 0}{=}\lim_{x\to 0}\frac{2a}{b\cos(ax)(1-\tan(ax))}=\frac{2a}{b}.$$

(k)

$$\lim_{x\to 0^+}(\cos(\sqrt{x}))^{\frac{1}{x}}=\lim_{x\to 0^+}e^{\frac{\ln(\cos(\sqrt{x}))}{x}}=\lim_{x\to 0^+}e^{\frac{\ln(1-\sin^2(\sqrt{x}))}{2x}}=\lim_{x\to 0^+}e^{\frac{\ln(1-\sin^2(\sqrt{x}))}{-\sin^2(\sqrt{x})}\frac{-\sin^2(\sqrt{x})}{2x}}=$$
$$\lim_{x\to 0^+}e^{-\frac{1}{2}\underbrace{\frac{\ln(1-\sin^2(\sqrt{x}))}{-\sin^2(\sqrt{x})}}_{\to 1}\left(\underbrace{\frac{\sin(\sqrt{x})}{\sqrt{x}}}_{\to 1}\right)^2}=e^{-\frac{1}{2}}.$$

(l) $\lim_{x\to 1}\frac{\sin(\pi x^a)}{\pi x^b}=0.$

Derivace

Spočtěte

$$\mathcal{F}[f(x)]=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

pro

(a) $f(x)=x^\alpha,\quad \alpha\in\mathbb{R}$,

(b) $f(x)=\sin(x)$,

(c) $f(x)=\cos(x)$,

(d) $f(x)=e^x$,

(e) $f(x)=\ln(x)$,

(f) $f(x)=\arctan(x)$.

(a)

$$\lim_{h\to 0}\frac{(x+h)^\alpha - x^\alpha}{h} = \lim_{h\to 0} x^\alpha \frac{\left(1+\frac{h}{x}\right)^\alpha - 1}{h} =$$
$$\lim_{h\to 0} x^\alpha \frac{e^{\alpha\ln\left(1+\frac{h}{x}\right)} - 1}{\alpha\ln\left(1+\frac{h}{x}\right)}\frac{\alpha\ln\left(1+\frac{h}{x}\right)}{h} = \lim_{h\to 0} x^{\alpha-1}\frac{\alpha\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = \alpha x^{\alpha-1}.$$

(b)

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} =$$
$$\lim_{h\to 0} h\underbrace{\frac{\sin(x)(\cos(h)-1)}{h^2}}_{\to\frac{\sin(x)}{2}} + \frac{\cos(x)\sin(h)}{h} = 0+\cos(x).$$

(c)

$$\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} = \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h} =$$
$$\lim_{h\to 0} h\cos(x)\underbrace{\frac{\cos(h)-1}{h^2}}_{\to-\frac{1}{2}} - \frac{\sin(x)\sin(h)}{h} = 0-\sin(x).$$

(d)

$$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} e^x\frac{e^h-1}{h} = e^x.$$

(e)

$$\lim_{h\to 0}\frac{\ln(x+h)-\ln(x)}{h} = \lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0}\frac{\ln\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\frac{1}{x} = \frac{1}{x}.$$

(f)

$$\lim_{h\to 0}\frac{\arctan(x+h)-\arctan(x)}{h} = \lim_{h\to 0}\frac{\arctan(x+h)-\arctan(x)}{x+h-x},$$

kde zavedeme nové proměnné $x+h=\tan(u)$ a $x=\tan(v)$, což dává

$$\lim_{u\to v}\frac{u-v}{\tan(u)-\tan(v)} = \frac{1}{1+\tan^2(v)} = \frac{1}{1+x^2}.$$

což víme z příkladu 2(c).