Řešení cvičení 13: Určitý integrál II

Výpočet

Spočtěte následující integrály pro $k \in \mathbb{N}_0$

(a) $\int_0^2 \frac{1}{e^{\frac{x}{2}}+e^x}\,\mathrm{d}x$,

(b) $\int_0^9 x^3\sqrt[3]{1+x^2}\,\mathrm{d}x$,

(c) $\int_0^1 \frac{x^2}{(1-x)^{100}}\,\mathrm{d}x$,

(d) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\mathrm{d}x}{\cos^2(x)\sin^2(x)}$.

(a)

$$\int_0^2 \frac{1}{e^{\frac{x}{2}}+e^x}\,\mathrm{d}x = \int_0^2 \frac{e^{\frac{x}{2}}}{e^x+e^{\frac{3x}{2}}}\,\mathrm{d}x = \left\{\begin{array}{c} e^{\frac{x}{2}}=y \\ \frac{1}{2}e^{\frac{x}{2}}\mathrm{d}x=\mathrm{d}y \end{array}\right\} = \int_1^e \frac{2}{y^2+y^3}\,\mathrm{d}y = \int_1^e \frac{2}{y^2(1+y)}\,\mathrm{d}y =$$
$$\left\{\begin{array}{c} \frac{2}{y^2(1+y)} = \frac{a_1}{y}+\frac{b_1y+b_2}{y^2}+\frac{2c_1}{1+y} \\ 2 = b_2+y(a_1+b_1+b_2)+y^2(a_1+b_1+2c_1) \end{array}\right\} = \int_1^e -\frac{2}{y}+\frac{2}{y^2}+\frac{2}{1+y}\,\mathrm{d}y = \left[-2\ln(y)-\frac{2}{y}+2\ln(1+y)\right]_1^e$$

(b)

$$\int_0^9 x^3\sqrt[3]{1+x^2}\,\mathrm{d}x = \left\{\begin{array}{c} x^2=y \\ 2x\,\mathrm{d}x=\mathrm{d}y \end{array}\right\} = \frac{1}{2}\int_0^3 y\sqrt[3]{1+y}\,\mathrm{d}y \overset{pp.}{=} \left[y\frac{3(1+y)^{\frac{4}{3}}}{4}\right]_0^3 - \frac{1}{2}\int_0^3 \frac{3(1+y)^{\frac{4}{3}}}{4}\,\mathrm{d}y =$$
$$\left[y\frac{3(1+y)^{\frac{4}{3}}}{4}\right]_0^3 - \frac{3}{8}\left[\frac{3(1+y)^{\frac{7}{3}}}{7}\right]_0^3 = \frac{9}{4}4^{\frac{4}{3}} - \frac{3}{8}\left(\frac{3}{7}4^{\frac{7}{3}}-\frac{3}{7}\right).$$

(c)

$$\int_0^1 \frac{x^2}{(1+x)^{100}}\,\mathrm{d}x \overset{pp.}{=} \left[-\frac{x^2}{99(1+x)^{99}}\right]_0^1 - \int_0^1 -\frac{2x}{99(1+x)^{99}}\,\mathrm{d}x = -\frac{1}{99\cdot 2^{99}} + \int_0^1 \frac{2x}{99(1+x)^{99}}\,\mathrm{d}x \overset{pp.}{=}$$
$$-\frac{1}{99\cdot 2^{99}} + \left[-\frac{2x}{99\cdot 98(1+x)^{98}}\right]_0^1 + \int_0^1 \frac{2}{99\cdot 98(1+x)^{98}}\,\mathrm{d}x = -\frac{1}{99\cdot 2^{99}} - \frac{2}{99\cdot 98\cdot 2^{98}}$$
$$+\int_0^1 \frac{2}{99\cdot 98(1+x)^{98}}\,\mathrm{d}x = -\frac{1}{99\cdot 2^{99}} - \frac{2}{99\cdot 98\cdot 2^{98}} - \frac{2}{99\cdot 98\cdot 97\cdot 2^{97}} + \frac{2}{99\cdot 98\cdot 97}.$$

(d)

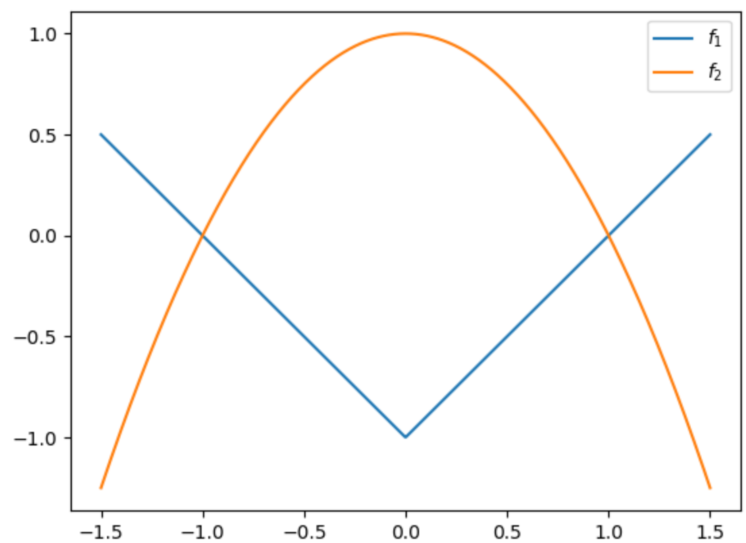
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\mathrm{d}x}{\cos^2(x)\sin^2(x)} = \int_{\frac{\pi}{6}}^{\frac{2\pi}{6}} \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)\sin^2(x)}\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{2\pi}{6}} \frac{1}{\sin^2(x)}+\frac{1}{\cos^2(x)}\mathrm{d}x =$$
$$[\tan(x)+\cot(x)]_{\frac{\pi}{6}}^{\frac{2\pi}{6}} = \frac{4}{\sqrt{3}}.$$

Oblasti mezi křivkami

Spočtšte obsah plochy ohraničené následujícími křivkami

(a) $f_1(x) = |x|-1$,
$f_2(x) = 1-x^2$

(b) $f_1(x) = x+1$,
$f_2(x) = -x+1$,
$f_3(x) = x-1$,
$f_4(x) = -x-1$,



(c) $f_1(x) = \frac{(x-1)^2}{6} - 1,$
$f_2(x) = \frac{x^2}{10} + 2,$

(d) $f_1(x) = \sqrt{1-x^2},$
$f_2(x) = -\sqrt{1-x^2}$

(a) $f_1(x) = |x| - 1,$
$f_2(x) = 1 - x^2$

$$S = \int_{-1}^{1} \left| |x| - 1 - (1-x^2) \right| \,\mathrm{d}x = \int_{-1}^{1} 1 - x^2 - |x| + 1 \,\mathrm{d}x = \int_{-1}^{0} 2 - x^2 + x \,\mathrm{d}x + \int_{0}^{1} 2 - x^2 - x \,\mathrm{d}x =$$

$$\left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{0} + \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{0}^{1} = 2 - \frac{1}{3} - \frac{1}{2} + 2 - \frac{1}{3} - \frac{1}{2} = 3 - \frac{2}{3}.$$

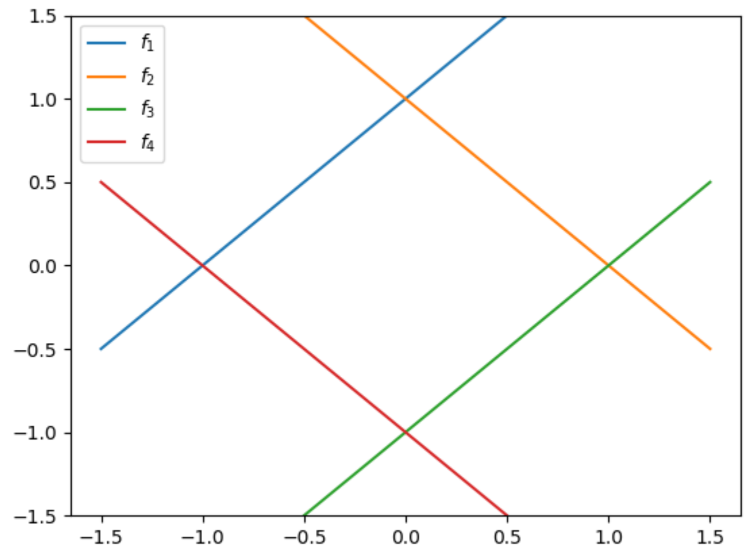
(b) $f_1(x) = x + 1,$
$f_2(x) = -x + 1,$
$f_3(x) = x - 1,$
$f_4(x) = -x - 1,$

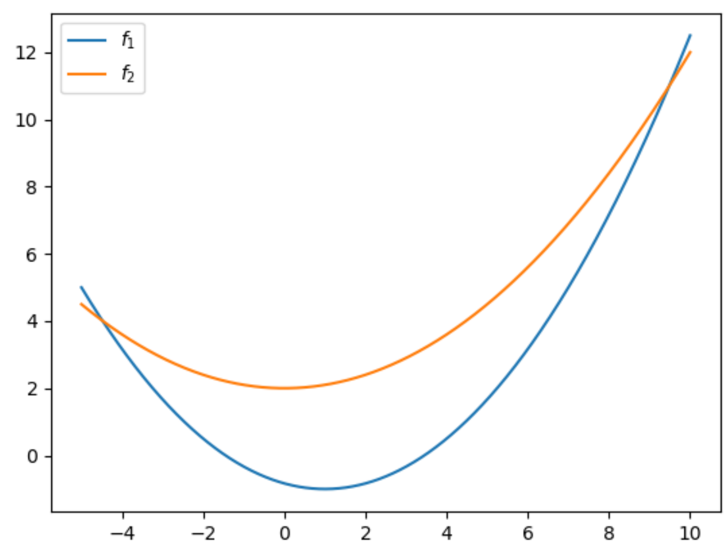
$$S = \int_{-1}^{0} |(x+1) - (-x-1)| \,\mathrm{d}x + \int_{0}^{1} |(-x+1) - (x-1)| \,\mathrm{d}x = \int_{-1}^{0} |2+2x| \,\mathrm{d}x + \int_{0}^{1} |2-2x| \,\mathrm{d}x =$$

$$\int_{-1}^{0} 2 + 2x \,\mathrm{d}x + \int_{0}^{1} 2 - 2x \,\mathrm{d}x = [2x + x^2]_{-1}^{0} + [2x - x^2]_{0}^{1} = 2.$$

(c) $f_1(x) = \frac{(x-1)^2}{6} - 1,$
$f_2(x) = \frac{x^2}{10} + 2,$ Průsečíky spočteme pomocí

$$\frac{(x-1)^2}{6} - 1 = \frac{x^2}{10} + 2 \overset{\text{Wolfram}}{\Rightarrow} x = \frac{5}{2} \pm \frac{\sqrt{195}}{2}.$$





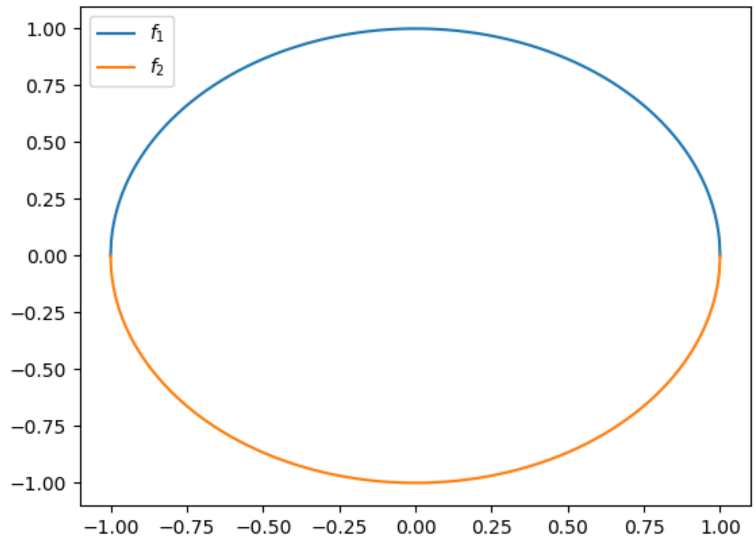
Pak (označme $x_\pm := \frac{5}{2} \pm \frac{\sqrt{195}}{2}$)

$$S = \int_{x_-}^{x_+} \left| \left(\frac{(x-1)^2}{6} - 1 \right) - \left(\frac{x^2}{10} + 2 \right) \right| \, \mathrm{d}x = \int_{x_-}^{x_+} \frac{x^2}{10} + 2 - \frac{(x-1)^2}{6} + 1 \, \mathrm{d}x =$$

$$\int_{x_-}^{x_+} \left(\frac{1}{10} - \frac{1}{6} \right) x^2 + \frac{x}{3} + 3 - \frac{1}{6} \, \mathrm{d}x = \left[\left(\frac{1}{10} - \frac{1}{6} \right) \frac{x^3}{3} + \frac{x}{6} + \left(+3 - \frac{1}{6} \right) x \right]_{x_-}^{x_+} \overset{\text{Wolfram}}{=}$$

$$\frac{13}{2} \sqrt{\frac{65}{3}} \approx 30.26.$$

(d) $f_1(x) = \sqrt{1 - x^2}$,



$f_2(x) = -\sqrt{1-x^2}$,

$$S = \int_{-1}^{1} |\sqrt{1-x^2} - (-\sqrt{1-x^2})| \, \mathrm{d}x = 2\int_{-1}^{1} \sqrt{1-x^2} \, \mathrm{d}x = \left\{ \begin{array}{c} x = \sin(y) \\ \mathrm{d}x = \cos(y)\mathrm{d}y \end{array} \right\} =$$

$$2\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\sin^2(x)}\cos(x) \, \mathrm{d}x = 2\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} |\cos(x)|\cos(x) \, \mathrm{d}x = -2\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos^2(x) \, \mathrm{d}x,$$

kde

$$I = \int \cos^2(x) \, \mathrm{d}x \overset{pp.}{=} \cos(x)\sin(x) + \int \sin^2(x) \, \mathrm{d}x = \cos(x)\sin(x) + \int 1 - \cos^2(x) \, \mathrm{d}x = \cos(x)\sin(x) + x - I + c,$$

tedy $\int \cos^2(x) \, \mathrm{d}x = \frac{1}{2}(\cos(x)\sin(x) + x) + c$ a

$$S = -2\int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos^2(x) \, \mathrm{d}x = -2\frac{1}{2}\left[\cos(x)\sin(x) + x\right]_{\frac{3\pi}{2}}^{\frac{\pi}{2}} = -\left(0 + \frac{\pi}{2} - 0 - \frac{3\pi}{2}\right) = \pi.$$

Délka křivky

Spočtěte délky následujících křivek mezi a a b

(a) $f(x) = x^2, \; a = -1, \; b = 1,$

(b) $f(x) = e^x, \; a = 0, \; b = 5,$

(c) $f(x) = \frac{x^2}{4} - \frac{\ln(x)}{2}, \; a = 1, \; b = e,$

(d) $f(x) = \sqrt{1-x^2}, \; a = -1, \; b = 1.$

(a)

$$\int_{-1}^{1} \sqrt{1+(2x)^2} \, \mathrm{d}x = \left\{ \begin{array}{c} 2x = y \\ 2 \, \mathrm{d}x = \mathrm{d}y \end{array} \right\} = \frac{1}{2}\int_{-2}^{2} \sqrt{1+y^2} \, \mathrm{d}y =$$

Tento integrál silně připomíná integrál typu $\sqrt{1-x^2}$, ale zde máme +. Proto by se nám hodila funkce obdobná $\sin(x)$, ale taková, že $1 - f^2(x)$ je nějaký čtverec. Takovou funkcí je tkz. hyperbolický sínus, definovaný pomocí

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \; \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Potom platí $\cosh^2(x) - \sinh^2(x) = 1$, což je zobecnění goniometrické jedničky. Pokud tedy zkusíme zde provést substituci

$$= \left\{ \begin{array}{c} y = \sinh(z) \\ \mathrm{d}y = \cosh(z)\mathrm{d}z \end{array} \right\} = \frac{1}{2} \int_{-\sinh(2)}^{\sinh(2)} \sqrt{1 + \sinh^2(z)} \cosh(z)\ \mathrm{d}z =$$

$$\frac{1}{2} \int_{-\sinh(2)}^{\sinh(2)} \cosh^2(z)\ \mathrm{d}z = \frac{1}{2} \int_{-\sinh(2)}^{\sinh(2)} \frac{e^{2z} + 2 + e^{-2z}}{4}\ \mathrm{d}z = \frac{1}{2} \left[\frac{e^{2z} - e^{-2z}}{8} + 2z \right]_{-\sinh(2)}^{\sinh(2)}.$$

(b)

$$\int_0^5 \sqrt{1 + e^{2x}}\ \mathrm{d}x \overset{2x=y}{=} \frac{1}{2} \int_0^{10} \sqrt{1 + e^y}\ \mathrm{d}y = \left\{ \begin{array}{c} e^y = z \\ e^y\ \mathrm{d}y = \mathrm{d}z \end{array} \right\} = \frac{1}{2} \int_1^{e^{10}} \frac{\sqrt{1+z}}{z}\ \mathrm{d}z =$$

$$\left\{ \begin{array}{c} \sqrt{1+z} = t \\ \frac{1}{2\sqrt{1+z}}\ \mathrm{d}z = \mathrm{d}t \end{array} \right\} = \int_{\sqrt{2}}^{\sqrt{1+e^{10}}} \frac{t^2}{t^2 - 1}\ \mathrm{d}t = \int_{\sqrt{2}}^{\sqrt{1+e^{10}}} 1 + \frac{1}{t^2-1}\ \mathrm{d}t \overset{\text{parc. zl.}}{=}$$

$$\int_{\sqrt{2}}^{\sqrt{1+e^{10}}} 1 + \frac{1}{2(t-1)} + \frac{1}{2(t+1)}\ \mathrm{d}t = \left[t + \frac{1}{2} \ln\left(t^2 - 1\right) \right]_{\sqrt{2}}^{\sqrt{1+e^{10}}}.$$

(c)

$$\int_1^e \sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x} \right)^2}\ \mathrm{d}x = \int_1^e \sqrt{\frac{x^4 + 2x^2 + 1}{4x^2}}\ \mathrm{d}x = \int_1^e \sqrt{\frac{(x^2+1)^2}{4x^2}}\ \mathrm{d}x = \int_1^e \frac{x^2+1}{2x}\ \mathrm{d}x =$$

$$\int_1^e \frac{x}{2} + \frac{1}{2x}\ \mathrm{d}x = \left[\frac{x^2}{4} + \frac{\ln(x)}{2} \right]_1^e = \frac{e^2}{4} + \frac{1}{2} - \frac{1}{4} = \frac{e^2+1}{4}.$$

(d)

$$\int_{-1}^1 \sqrt{1 + \frac{x^2}{1-x^2}}\ \mathrm{d}x = \int_{-1}^1 \sqrt{\frac{1}{1-x^2}}\ \mathrm{d}x = \left\{ \begin{array}{c} x = \cos(y) \\ \mathrm{d}x = -\sin(y)\ \mathrm{d}y \end{array} \right\} =$$

$$-\int_\pi^0 \sqrt{\frac{1}{\sin^2(y)}} \sin(y)\ \mathrm{d}y = -\int_\pi^0\ \mathrm{d}y = \int_0^\pi\ \mathrm{d}y = \pi.$$

Objem tělesa

Spočtěte objem těles vzniklých rotací následujících křivek od a do b

(a) $f(x) = \sqrt{1-x^2},\ a = -1,\ b = 1,$

(b) $f(x) = \frac{r}{h}x,\ a = 0,\ b = h,$

(c) $f(x) = \frac{1}{x},\ a = 0,\ b = \infty,$

(d) $f(x) = r,\ a = 0,\ b = h.$

(a)

$$\int_{-1}^1 \pi(1 - x^2)\ \mathrm{d}x = \left\{ \begin{array}{c} x = \cos(y) \\ \mathrm{d}x = -\sin(y)\ \mathrm{d}y \end{array} \right\} = -\pi \int_\pi^0 (1 - \cos^2(y)) \sin(y)\ \mathrm{d}y = -\pi \int_\pi^0 \sin^3(y)\ \mathrm{d}y,$$

kde

$$I = \int_\pi^0 \sin^3(y)\ \mathrm{d}y \overset{pp.}{=} \Bigg/ \begin{array}{c} F = -\cos(y)\ \ G = \sin^2(y) \\ f = \sin(y)\ \ g = 2\sin(y)\cos(y) \end{array} \Bigg/ = \left[-\cos(y)\sin^2(y) \right]_\pi^0 + 2\int_\pi^0 \cos^2(y)\sin(y)\ \mathrm{d}y =$$

$$2\int_\pi^0 (1 - \sin^2(y)) \sin(y)\ \mathrm{d}y = 2\int_\pi^0 \sin(y)\ \mathrm{d}y - 2\pi I = 2[-\cos(y)]_\pi^0 - 2I = -4 - 2I,$$

neboli $I = -\frac{4}{3}$ a

$$\int_{-1}^{1} \pi(1-x^2)\ \mathrm{d}x = -\pi \int_{\pi}^{0} \sin^3(y)\ \mathrm{d}y = \frac{4\pi}{3}.$$

(b)

$$\int_0^h \pi \frac{r^2}{h^2} x^2\ \mathrm{d}x = \pi \frac{r^2}{h^2} \int_0^h x^2\ \mathrm{d}x = \pi \frac{r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \pi \frac{r^2 h}{3}.$$

Tohle je opravdu objem kužele o výšce h a poloměru r.

(c)

$$\int_0^\infty \pi \frac{1}{x^2}\ \mathrm{d}x = \pi \int_0^\infty \frac{1}{x^2}\ \mathrm{d}x = \pi \left[-\frac{1}{x}\right]_0^\infty = \infty.$$

(d)

$$\int_0^h \pi r^2\ \mathrm{d}x = \pi r^2 \int_0^h\ \mathrm{d}x = \pi r^2 h,$$

což je objem válce.

Povrch tělesa

Spočtěte povrch těles vzniklých rotací následujících křivek od a do b

(a) $f(x) = \sqrt{1-x^2},\ a = -1,\ b = 1,$

(b) $f(x) = \frac{r}{h}x,\ a = 0,\ b = h,$

(c) $f(x) = r,\ a = 0,\ b = h.$

(d) $f(x) = \sqrt{x},\ a = 0,\ b = 5.$

(a)

$$\int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\ \mathrm{d}x = 2\pi \int_{-1}^{1} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\ \mathrm{d}x = 2\pi \int_{-1}^{1}\ \mathrm{d}x = 4\pi,$$

což je opravdu povrch koule o poloměru 1.

(b)

$$\int_0^h 2\pi \frac{r}{h} x \sqrt{1+\frac{r^2}{h^2}}\ \mathrm{d}x = 2\pi \frac{r}{h}\sqrt{1+\frac{r^2}{h^2}} \int_0^h x\ \mathrm{d}x = 2\pi r \sqrt{1+\frac{r^2}{h^2}}.$$

(c)

$$\int_0^h 2\pi r\sqrt{1+0}\ \mathrm{d}x = 2\pi r h.$$

(d)

$$\int_0^5 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\ \mathrm{d}x = \pi \int_0^5 \sqrt{1+4x}\ \mathrm{d}x = \pi \left[\frac{2(1+4x)^{\frac{3}{2}}}{4\cdot 3}\right]_0^5 = \frac{\pi}{6}(21^{\frac{3}{2}}-1).$$