

Řešení cvičení 12: Určitý integrál

Snadné integrály

Spočtěte následující integrály

(a) $\int_0^5 x^3 + 2x^2 + \frac{x}{3} dx,$

(e) $\int_5^5 \frac{\arctan(x^{0.75}+3)}{e^{x^3+x+2}} dx,$

(b) $\int_0^2 \frac{x}{(1+2x^2)^2} dx,$

(f) $\int_0^{\frac{\pi}{2}} \frac{\sin(x) \cos^2(x)}{1+\cos^2(x)} dx,$

(c) $\int_4^1 \sqrt{x} e^{1-\sqrt{x^3}} dx,$

(g) $\int_0^1 \frac{x^3}{3+x} dx,$

(d) $\int_0^\infty \frac{3}{5+2x} dx,$

(h) $\int_0^1 x^2(2-3x^2)^2 dx.$

(a)

$$\int_0^5 x^3 + 2x^2 + \frac{x}{3} dx = \int_0^5 x^3 dx + \int_0^5 2x^2 dx + \int_0^5 \frac{x}{3} dx = \left[\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{6} \right]_0^5 = \frac{5^4}{4} + \frac{2 \cdot 5^3}{3} + \frac{5^2}{6} - 0.$$

(b)

$$\int_0^2 \frac{x}{(1+2x^2)^2} dx = \left\{ \begin{array}{l} 2x^2 = y \\ 4x dx = dy \end{array} \right\} = \frac{1}{4} \int_0^4 \frac{1}{(1+y)^2} dy = \frac{1}{4} \left[-\frac{1}{1+y} \right]_0^4 = -\frac{1}{4} \left[\frac{1}{1+4} - 1 \right] = \frac{1}{5}.$$

(c)

$$\int_4^1 \sqrt{x} e^{1-\sqrt{x^3}} dx = - \int_1^4 \sqrt{x} e^{1-\sqrt{x^3}} dx = \left\{ \begin{array}{l} x^{\frac{3}{2}} = y \\ \frac{3}{2} \sqrt{x} dx = dy \end{array} \right\} = -\frac{2}{3} \int_1^{4^{\frac{3}{2}}} e^{1-y} dy = -\frac{2}{3} [-e^{1-y}]_1^{4^{\frac{3}{2}}} = -\frac{2}{3} [1 - e^{1-4^{\frac{3}{2}}}] .$$

(d)

$$\int_0^\infty \frac{3}{5+2x} dx = \frac{3}{5} \int_0^\infty \frac{1}{1+\frac{2x}{5}} dx = \left\{ \begin{array}{l} \frac{2x}{5} = y \\ \frac{2}{5} dx = dy \end{array} \right\} = \frac{3}{2} \int_0^\infty \frac{1}{1+y} dy = \frac{3}{2} [\ln(1+y)]_0^\infty = \infty.$$

(e)

$$\int_5^5 \frac{\arctan(x^{0.75}+3)}{e^{x^3+x+2}} dx = 0,$$

protože integrujeme přes interval nulové délky.

(f)

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x) \cos^2(x)}{1+\cos^2(x)} dx = \left\{ \begin{array}{l} \cos(x) = y \\ \sin(x) dx = dy \end{array} \right\} = \int_1^0 \frac{y^2}{1+y^2} dy = \int_1^0 1 - \frac{1}{1+y^2} dy = [y - \arctan(y)]_1^0 = \arctan(1) - 1.$$

(g)

$$\int_0^1 \frac{x^3}{3+x} dx = \left\{ \frac{x^3}{3+x} = x^2 - 3x + 9 - \frac{27}{3+x} \right\} = \int_0^1 x^2 - 3x + 9 - \frac{27}{3+x} dx = \left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - 27 \ln(3+x) \right]_0^1 = \frac{1}{3} - \frac{3}{2} + 9 - 27 \ln\left(\frac{4}{3}\right).$$

(h)

$$\int_0^1 x^2(2-3x^2)^2 dx = \int_0^1 x^2(4-12x^2+9x^4) dx = \int_0^1 4x^2-12x^4+9x^6 dx = \left[\frac{4}{3}x^3 - \frac{12}{5}x^5 + \frac{9}{7}x^7 \right]_0^1 = \frac{4}{3} - \frac{12}{5} + \frac{9}{7}.$$

Složitější integrály

Spočítejte následující integrály

(a) $\int_0^{\ln(2)} \sqrt{e^x-1} dx,$

(c) $\int_0^\infty e^{-ax} \cos(bx) dx,$

(b) $\int_0^1 x \ln(x) dx,$

(d) $\int_0^\infty x^3 e^{-\frac{x^2}{2}} dx.$

(a)

$$\begin{aligned} \int_0^{\ln(2)} \sqrt{e^x-1} dx &= \int_0^{\ln(2)} \frac{e^x}{e^x} \sqrt{e^x-1} dx = \left\{ \begin{array}{l} e^x = y \\ e^x dx = dy \end{array} \right\} = \int_1^2 \frac{\sqrt{y-1}}{y} dy = \\ &= \int_1^2 \frac{\sqrt{y-1}^2}{\sqrt{y-1}^2 + 1} \frac{1}{\sqrt{y-1}} dy = \left\{ \begin{array}{l} y-1 = z^2 \\ dy = 2z dz \end{array} \right\} = 2 \int_0^1 \frac{z^2}{z^2+1} dz = \\ &= 2 \int_0^1 1 - \frac{1}{z^2+1} dz = 2[z - \arctan(z)]_0^1 = 2(1 - \arctan(1)). \end{aligned}$$

(b)

$$\int_0^1 x \ln(x) dx \stackrel{pp.}{=} \left[\frac{x^2}{2} \ln(x) \right]_0^1 - \int_0^1 \frac{x^2}{2} \frac{1}{x} dx = \left[\frac{x^2}{2} \ln(x) \right]_0^1 - \int_0^1 \frac{x}{2} dx = \left[\frac{x^2}{2} \ln(x) - \frac{x^2}{4} \right]_0^1 = -\frac{1}{4},$$

kde jsme použili

$$\lim_{x \rightarrow 0^+} \frac{x^2}{2} \ln(x) = \frac{1}{2} \lim_{x \rightarrow 0^+} \frac{\ln(x)}{\frac{1}{x^2}} \stackrel{l'H}{=} \frac{1}{2} \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = -\frac{1}{2} \lim_{x \rightarrow 0^+} \frac{x^2}{2} = 0.$$

(c)

$$\begin{aligned} I &= \int_0^\infty e^{-ax} \cos(bx) dx \stackrel{pp.}{=} - \left[\frac{e^{-ax}}{a} \cos(bx) \right]_0^\infty + \int_0^\infty \frac{e^{-ax}}{a} b \sin(bx) dx = \\ &= \frac{1}{a} + \frac{b}{a} \int_0^\infty e^{-ax} \sin(bx) dx \stackrel{pp.}{=} \frac{1}{a} - \frac{b}{a} \left[\frac{e^{-ax}}{a} \sin(bx) \right]_0^\infty - \frac{b^2}{a^2} \int_0^\infty \frac{e^{-ax}}{a} \cos(bx) dx = \frac{1}{a} - \frac{b}{a^2} - \frac{b^2}{a^2} I, \end{aligned}$$

tedy

$$I = \frac{\frac{1}{a} - \frac{b}{a^2}}{1 + \frac{b^2}{a^2}} = \frac{\frac{a-b}{a^2}}{\frac{a^2-b^2}{a^2}} = \frac{1}{a+b}.$$

(d)

$$\begin{aligned} \int_0^\infty x^3 e^{-\frac{x^2}{2}} dx &= \left\{ \begin{array}{l} \frac{x^2}{2} = y \\ x dx = dy \end{array} \right\} = 2 \int_0^\infty y e^{-y} dy \stackrel{pp.}{=} 2 [-y e^{-y}]_0^\infty + 2 \int_0^\infty e^{-y} dy = \\ &= 2 [-y e^{-y} - e^{-y}]_0^\infty = 2. \end{aligned}$$

Integrační kritérium konvergence

Vyšetřete konvergenci následujících řad

(a) $\sum_{i=1}^{\infty} \frac{1}{n}$,

(c) $\sum_{i=1}^{\infty} \frac{n}{(n^2+1)\ln(n^2+1)}$,

(b) $\sum_{i=2}^{\infty} \frac{1}{n \ln(n)}$,

(d) $\sum_{i=1}^{\infty} \frac{e^n}{1+e^{2n}}$.

(a)

$$\sum_{i=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x} dx = [\ln(x)]_1^{\infty} = \infty.$$

(b)

$$\sum_{i=2}^{\infty} \frac{1}{n \ln(n)} \geq \int_2^{\infty} \frac{1}{x \ln(x)} dx = \left\{ \begin{array}{l} \ln(x) = y \\ \frac{1}{x} dx = dy \end{array} \right\} = \int_{\ln(2)}^{\infty} \frac{1}{y} dy = [\ln(y)]_{\ln(2)}^{\infty} = \infty.$$

(c)

$$\begin{aligned} \sum_{i=1}^{\infty} \frac{n}{(n^2+1)\ln(n^2+1)} &\geq \int_1^{\infty} \frac{x}{(x^2+1)\ln(x^2+1)} dx = \left\{ \begin{array}{l} x^2+1 = y \\ 2x dx = dy \end{array} \right\} = \int_2^{\infty} \frac{1}{2y \ln(y)} dy = \\ &\left\{ \begin{array}{l} \ln(y) = z \\ \frac{1}{y} dy = dz \end{array} \right\} = \int_{\ln(2)}^{\infty} \frac{1}{2z} dz = \infty. \end{aligned}$$

(d)

$$\begin{aligned} 0 \leq \sum_{i=1}^{\infty} \frac{e^n}{1+e^{2n}} &\leq \int_1^{\infty} \frac{e^{x-1}}{1+e^{2x-2}} dx \stackrel{x-1=y}{=} \int_2^{\infty} \frac{e^y}{1+e^{2y}} dy = \left\{ \begin{array}{l} e^y = z \\ e^y dy = dz \end{array} \right\} = \\ &\int_{e^2}^{\infty} \frac{1}{1+z^2} dz = [\arctan(z)]_{e^2}^{\infty} = \frac{\pi}{2} - \arctan(e^2). \end{aligned}$$