Cvičení 14: Aplikace integrálů

1 Délka křivky

(a)

$$\int_{-1}^{1} \sqrt{1+(2x)^2}\ \mathrm{d}x = \left\{ \begin{array}{c} 2x = y \\ 2\ \mathrm{d}x = \mathrm{d}y \end{array} \right\} = \frac{1}{2}\int_{-2}^{2} \sqrt{1+y^2}\ \mathrm{d}y =$$

Tento integrál silně připomíná integrál typu $\sqrt{1-x^2}$, ale zde máme +. Proto by se nám hodila funkce obdobná $\sin(x)$, ale taková, že $1-f^2(x)$ je nějaký čtverec. Takovou funkcí je tkz. hyperbolický sínus, definovaný pomocí

$$\sinh(x) = \frac{e^x - e^{-x}}{2},\ \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

Potom platí $\cosh^2(x) - \sinh^2(x) = 1$, což je zobecnění goniometrické jedničky. Pokud tedy zkusíme zde provést substituci

$$= \left\{ \begin{array}{c} y = \sinh(z) \\ \mathrm{d}y = \cosh(z)\mathrm{d}z \end{array} \right\} = \frac{1}{2}\int_{-\sinh(2)}^{\sinh(2)} \sqrt{1+\sinh^2(z)}\cosh(z)\ \mathrm{d}z =$$

$$\frac{1}{2}\int_{-\sinh(2)}^{\sinh(2)} \cosh^2(z)\ \mathrm{d}z = \frac{1}{2}\int_{-\sinh(2)}^{\sinh(2)} \frac{e^{2z}+2+e^{-2z}}{4}\ \mathrm{d}z = \frac{1}{2}\left[\frac{e^{2z}-e^{-2z}}{8} + 2z\right]_{-\sinh(2)}^{\sinh(2)}.$$

(b)

$$\int_0^5 \sqrt{1+e^{2x}}\ \mathrm{d}x \overset{2x=y}{=} \frac{1}{2}\int_0^{10} \sqrt{1+e^y}\ \mathrm{d}y = \left\{ \begin{array}{c} e^y = z \\ e^y\ \mathrm{d}y = \mathrm{d}z \end{array} \right\} = \frac{1}{2}\int_1^{e^{10}} \frac{\sqrt{1+z}}{z}\ \mathrm{d}z =$$

$$\left\{ \begin{array}{c} \sqrt{1+z} = t \\ \frac{1}{2\sqrt{1+z}}\ \mathrm{d}z = \mathrm{d}t \end{array} \right\} = \int_{\sqrt{2}}^{\sqrt{1+e^{10}}} \frac{t^2}{t^2-1}\ \mathrm{d}t = \int_{\sqrt{2}}^{\sqrt{1+e^{10}}} 1 + \frac{1}{t^2-1}\ \mathrm{d}t \overset{\text{parc. zl.}}{=}$$

$$\int_{\sqrt{2}}^{\sqrt{1+e^{10}}} 1 + \frac{1}{2(t-1)} + \frac{1}{2(t+1)}\ \mathrm{d}t = \left[t + \frac{1}{2}\ln\left(t^2-1\right)\right]_{\sqrt{2}}^{\sqrt{1+e^{10}}}.$$

(c)

$$\int_1^e \sqrt{1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2}\ \mathrm{d}x = \int_1^e \sqrt{\frac{x^4+2x^2+1}{4x^2}}\ \mathrm{d}x = \int_1^e \sqrt{\frac{(x^2+1)^2}{4x^2}}\ \mathrm{d}x = \int_1^e \frac{x^2+1}{2x}\ \mathrm{d}x =$$

$$\int_1^e \frac{x}{2} + \frac{1}{2x}\ \mathrm{d}x = \left[\frac{x^2}{4} + \frac{\ln(x)}{2}\right]_1^e = \frac{e^2}{4} + \frac{1}{2} - \frac{1}{4} = \frac{e^2+1}{4}.$$

(d)

$$\int_{-1}^{1} \sqrt{1+\frac{x^2}{1-x^2}}\ \mathrm{d}x = \int_{-1}^{1} \sqrt{\frac{1}{1-x^2}}\ \mathrm{d}x = \left\{ \begin{array}{c} x = \cos(y) \\ \mathrm{d}x = -\sin(y)\ \mathrm{d}y \end{array} \right\} =$$

$$-\int_{\pi}^{0} \sqrt{\frac{1}{\sin^2(y)}}\sin(y)\ \mathrm{d}y = -\int_{\pi}^{0}\ \mathrm{d}y = \int_0^{\pi}\ \mathrm{d}y = \pi.$$

2 Objem tělesa

(a)

$$\int_{-1}^{1} \pi(1-x^2)\ \mathrm{d}x = \left\{ \begin{array}{c} x=\cos(y) \\ \mathrm{d}x = -\sin(y)\ \mathrm{d}y \end{array} \right\} = -\pi \int_{\pi}^{0} (1-\cos^2(y))\sin(y)\ \mathrm{d}y = -\pi \int_{\pi}^{0} \sin^3(y)\ \mathrm{d}y,$$

kde

$$I = \int_{\pi}^{0} \sin^3(y)\ \mathrm{d}y \overset{pp.}{=} \bigg/ \begin{array}{c} F=-\cos(y)\ G=\sin^2(y) \\ f=\sin(y)\ g=2\sin(y)\cos(y) \end{array} \bigg/ = \left[-\cos(y)\sin^2(y)\right]_{\pi}^{0} + 2\int_{\pi}^{0} \cos^2(y)\sin(y)\ \mathrm{d}y =$$

$$2\int_{\pi}^{0} (1-\sin^2(y))\sin(y)\ \mathrm{d}y = 2\int_{\pi}^{0} \sin(y)\ \mathrm{d}y - 2\pi I = 2[-\cos(y)]_{\pi}^{0} - 2I = -4 - 2I,$$

neboli $I = -\frac{4}{3}$ a

$$\int_{-1}^{1} \pi(1-x^2)\ \mathrm{d}x = -\pi \int_{\pi}^{0} \sin^3(y)\ \mathrm{d}y = \frac{4\pi}{3}.$$

(b)

$$\int_{0}^{h} \pi\frac{r^2}{h^2}x^2\ \mathrm{d}x = \pi\frac{r^2}{h^2}\int_{0}^{h} x^2\ \mathrm{d}x = \pi\frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_{0}^{h} = \pi\frac{r^2 h}{3}.$$

Tohle je opravdu objem kužele o výšce h a poloměru r.

(c)

$$\int_{0}^{\infty} \pi\frac{1}{x^2}\ \mathrm{d}x = \pi\int_{0}^{\infty} \frac{1}{x^2}\ \mathrm{d}x = \pi\left[-\frac{1}{x}\right]_{0}^{\infty} = \infty.$$

(d)

$$\int_{0}^{h} \pi r^2\ \mathrm{d}x = \pi r^2 \int_{0}^{h}\ \mathrm{d}x = \pi r^2 h,$$

což je objem válce.

3 Povrch tělesa

(a)

$$\int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\ \mathrm{d}x = 2\pi\int_{-1}^{1} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\ \mathrm{d}x = 2\pi\int_{-1}^{1}\ \mathrm{d}x = 4\pi,$$

což je opravdu povrch koule o poloměru 1.

(b)

$$\int_{0}^{h} 2\pi\frac{r}{h}x\sqrt{1+\frac{r^2}{h^2}}\ \mathrm{d}x = 2\pi\frac{r}{h}\sqrt{1+\frac{r^2}{h^2}}\int_{0}^{h} x\ \mathrm{d}x = 2\pi r\sqrt{1+\frac{r^2}{h^2}}.$$

(c)

$$\int_{0}^{h} 2\pi r\sqrt{1+0}\ \mathrm{d}x = 2\pi r h.$$

(d)

$$\int_{0}^{5} 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\ \mathrm{d}x = \pi\int_{0}^{5} \sqrt{1+4x}\ \mathrm{d}x = \pi\left[\frac{2(1+4x)^{\frac{3}{2}}}{4\cdot 3}\right]_{0}^{5} = \frac{\pi}{6}(21^{\frac{3}{2}}-1).$$