Cvičení 13: Určité integrály II

1 Snadné integrály

(a)

$$\int_5^5 \frac{\arctan(x^{0.75}+3)}{e^{x^3}+x+2}\,\mathrm{d}x = 0,$$

protože integrujeme přes interval nulové délky.

(b)

$$\int_0^{\frac{\pi}{2}} \frac{\sin(x)\cos^2(x)}{1+\cos^2(x)}\,\mathrm{d}x = \left\{ \begin{array}{c} \cos(x) = y \\ \sin(x)\,\mathrm{d}x = \mathrm{d}y \end{array} \right\} = \int_1^0 \frac{y^2}{1+y^2}\,\mathrm{d}y = \int_1^0 1 - \frac{1}{1+y^2}\,\mathrm{d}y =$$
$$[y - \arctan(y)]_1^0 = \arctan(1) - 1.$$

(c)

$$\int_0^1 \frac{x^3}{3+x}\,\mathrm{d}x = \left\{ \frac{x^3}{3+x} = x^2 - 3x + 9 - \frac{27}{3+x} \right\} = \int_0^1 x^2 - 3x + 9 - \frac{27}{3+x}\,\mathrm{d}x =$$
$$\left[\frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - 27\ln(3+x)\right]_0^1 = \frac{1}{3} - \frac{3}{2} + 9 - 27\ln\left(\frac{4}{3}\right).$$

(d)

$$\int_0^1 x^2(2-3x^2)^2\,\mathrm{d}x = \int_0^1 x^2(4-12x^2+9x^4)\,\mathrm{d}x = \int_0^1 4x^2-12x^4+9x^6\,\mathrm{d}x = \left[\frac{4}{3}x^3 - \frac{12}{5}x^5 + \frac{9}{7}x^7\right]_0^1 = \frac{4}{3} - \frac{12}{5} + \frac{9}{7}.$$

2 Složitější integrály

(a)

$$\int_0^2 \frac{1}{e^{\frac{x}{2}}+e^x}\,\mathrm{d}x = \int_0^2 \frac{e^{\frac{x}{2}}}{e^x + e^{\frac{3x}{2}}}\,\mathrm{d}x = \left\{ \begin{array}{c} e^{\frac{x}{2}} = y \\ \frac{1}{2}e^{\frac{x}{2}}\mathrm{d}x = \mathrm{d}y \end{array} \right\} = \int_1^e \frac{2}{y^2+y^3}\,\mathrm{d}y = \int_1^e \frac{2}{y^2(1+y)}\,\mathrm{d}y =$$
$$\left\{ \begin{array}{c} \frac{2}{y^2(1+y)} = \frac{a_1}{y} + \frac{b_1 y + b_2}{y^2} + \frac{2c_1}{1+y} \\ 2 = b_2 + y(a_1+b_1+b_2) + y^2(a_1+b_1+2c_1) \end{array} \right\} = \int_1^e -\frac{2}{y} + \frac{2}{y^2} + \frac{2}{1+y}\,\mathrm{d}y = \left[-2\ln(y) - \frac{2}{y} + 2\ln(1+y)\right]_1^e$$

(b)

$$\int_0^9 x^3\sqrt[3]{1+x^2}\,\mathrm{d}x = \left\{ \begin{array}{c} x^2 = y \\ 2x\,\mathrm{d}x = \mathrm{d}y \end{array} \right\} = \frac{1}{2}\int_0^3 y\sqrt[3]{1+y}\,\mathrm{d}y \overset{pp.}{=} \left[y\frac{3(1+y)^{\frac{4}{3}}}{4}\right]_0^3 - \frac{1}{2}\int_0^3 \frac{3(1+y)^{\frac{4}{3}}}{4}\,\mathrm{d}y =$$
$$\left[y\frac{3(1+y)^{\frac{4}{3}}}{4}\right]_0^3 - \frac{3}{8}\left[\frac{3(1+y)^{\frac{7}{3}}}{7}\right]_0^3 = \frac{9}{4}4^{\frac{4}{3}} - \frac{3}{8}\left(\frac{3}{7}4^{\frac{7}{3}} - \frac{3}{7}\right).$$

(c)

$$\int_0^1 \frac{x^2}{(1+x)^{100}}\ \mathrm{d}x \overset{pp.}{=} \left[-\frac{x^2}{99(1+x)^{99}}\right]_0^1 - \int_0^1 -\frac{2x}{99(1+x)^{99}}\ \mathrm{d}x = -\frac{1}{99\cdot 2^{99}} + \int_0^1 \frac{2x}{99(1+x)^{99}}\ \mathrm{d}x \overset{pp.}{=}$$
$$-\frac{1}{99\cdot 2^{99}} + \left[-\frac{2x}{99\cdot 98(1+x)^{98}}\right]_0^1 + \int_0^1 \frac{2}{99\cdot 98(1+x)^{98}}\ \mathrm{d}x = -\frac{1}{99\cdot 2^{99}} - \frac{2}{99\cdot 98\cdot 2^{98}}$$
$$+\int_0^1 \frac{2}{99\cdot 98(1+x)^{98}}\ \mathrm{d}x = -\frac{1}{99\cdot 2^{99}} - \frac{2}{99\cdot 98\cdot 2^{98}} - \frac{2}{99\cdot 98\cdot 97\cdot 2^{97}} + \frac{2}{99\cdot 98\cdot 97}.$$

(d)

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\mathrm{d}x}{\cos^2(x)\sin^2(x)} = \int_{\frac{\pi}{6}}^{\frac{2\pi}{6}} \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)\sin^2(x)}\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{2\pi}{6}} \frac{1}{\sin^2(x)} + \frac{1}{\cos^2(x)}\mathrm{d}x =$$
$$[\tan(x)+\cot(x)]_{\frac{\pi}{6}}^{\frac{2\pi}{6}} = \frac{4}{\sqrt{3}}.$$

3 Integrální kritérium konvergence

(a)

$$\sum_{i=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x}\ \mathrm{d}x = [\ln(x)]_1^{\infty} = \infty.$$

(b)

$$\sum_{i=2}^{\infty} \frac{1}{n\ln(n)} \geq \int_2^{\infty} \frac{1}{x\ln(x)}\ \mathrm{d}x = \left\{ \begin{array}{c} \ln(x) = y \\ \frac{1}{x}\ \mathrm{d}x = \mathrm{d}y \end{array} \right\} = \int_{\ln(2)}^{\infty} \frac{1}{y}\ \mathrm{d}y = [\ln(y)]_{\ln(2)}^{\infty} = \infty.$$

(c)

$$\sum_{i=1}^{\infty} \frac{n}{(n^2+1)\ln(n^2+1)} \geq \int_1^{\infty} \frac{x}{(x^2+1)\ln(x^2+1)}\ \mathrm{d}x = \left\{ \begin{array}{c} x^2+1 = y \\ 2x\ \mathrm{d}x = \mathrm{d}y \end{array} \right\} = \int_2^{\infty} \frac{1}{2y\ln(y)}\ \mathrm{d}y =$$
$$\left\{ \begin{array}{c} \ln(y) = z \\ \frac{1}{y}\ \mathrm{d}y = \mathrm{d}z \end{array} \right\} = \int_{\ln(2)}^{\infty} \frac{1}{2z}\ \mathrm{d}z = \infty.$$

(d)

$$0 \leq \sum_{i=1}^{\infty} \frac{e^n}{1+e^{2n}} \leq \int_1^{\infty} \frac{e^{x+1}}{1+e^{2x+2}}\ \mathrm{d}x \overset{x+1=y}{=} \int_2^{\infty} \frac{e^y}{1+e^{2y}}\ \mathrm{d}y = \left\{ \begin{array}{c} e^y = z \\ e^y\ \mathrm{d}y = \mathrm{d}z \end{array} \right\} =$$
$$\int_{e^2}^{\infty} \frac{1}{1+z^2}\ \mathrm{d}z = [\arctan(z)]_{e^2}^{\infty} = \frac{\pi}{2} - \arctan(e^2).$$