Cvičení 12: Určité integrály

1 Snadné integrály

(a)

$$\int_0^5 x^3+2x^2+\frac{x}{3}\ \mathrm{d}x=\int_0^5 x^3\ \mathrm{d}x+\int_0^5 2x^2\ \mathrm{d}x+\int_0^5 \frac{x}{3}\ \mathrm{d}x=\left[\frac{x^4}{4}+\frac{2x^3}{3}+\frac{x^2}{6}\right]_0^5=\frac{5^4}{4}+\frac{2\cdot 5^3}{3}+\frac{5^2}{6}-0.$$

(b)

$$\int_0^2 \frac{x}{(1+2x^2)^2}\ \mathrm{d}x=\left\{\begin{array}{c}2x^2=y\\ 4x\mathrm{d}x=\mathrm{d}y\end{array}\right\}=\frac{1}{4}\int_0^4\frac{1}{(1+y)^2}\ \mathrm{d}y=\frac{1}{4}\left[-\frac{1}{1+y}\right]_0^4=-\frac{1}{4}\left[\frac{1}{1+4}-1\right]=\frac{1}{5}.$$

(c)

$$\int_4^1 \sqrt{x}e^{1-\sqrt{x^3}}\ \mathrm{d}x=-\int_1^4 \sqrt{x}e^{1-\sqrt{x^3}}\ \mathrm{d}x=\left\{\begin{array}{c}x^{\frac{3}{2}}=y\\ \frac{3}{2}\sqrt{x}\mathrm{d}x=\mathrm{d}y\end{array}\right\}=-\frac{2}{3}\int_1^{4^{\frac{3}{2}}} e^{1-y}\ \mathrm{d}y=$$
$$-\frac{2}{3}\left[-e^{1-y}\right]_1^{4^{\frac{3}{2}}}=-\frac{2}{3}\left[1-e^{1-4^{\frac{3}{2}}}\right].$$

(d)

$$\int_0^\infty \frac{3}{5+2x}\ \mathrm{d}x=\frac{3}{5}\int_0^\infty\frac{1}{1+\frac{2x}{5}}\ \mathrm{d}x=\left\{\begin{array}{c}\frac{2x}{5}=y\\ \frac{2}{5}\mathrm{d}x=\mathrm{d}y\end{array}\right\}=\frac{3}{2}\int_0^\infty\frac{1}{1+y}\ \mathrm{d}y=\frac{3}{2}\left[\ln(1+y)\right]_0^\infty=\infty.$$

2 Složitější integrály

(a)

$$\int_0^{\ln(2)}\sqrt{e^x-1}\ \mathrm{d}x=\int_0^{\ln(2)}\frac{e^x}{e^x}\sqrt{e^x-1}\ \mathrm{d}x=\left\{\begin{array}{c}e^x=y\\ e^x\mathrm{d}x=\mathrm{d}y\end{array}\right\}=\int_1^2\frac{\sqrt{y-1}}{y}\ \mathrm{d}y=$$
$$\int_1^2\frac{\sqrt{y-1}^2}{\sqrt{y-1}^2+1}\frac{1}{\sqrt{y-1}}\ \mathrm{d}y\int_0^{4\pi}\frac{1}{1+\sin^2(x)}\ \mathrm{d}x=2\int_0^1\frac{z^2}{z^2+1}\ \mathrm{d}z=$$
$$2\int_0^1 1-\frac{1}{z^2+1}\ \mathrm{d}z=2\left[z-\arctan(z)\right]_0^1=2(1-\arctan(1)).$$

(b)

$$\int_0^1 x\ln(x)\ \mathrm{d}x\overset{pp.}{=}\left[\frac{x^2}{2}\ln(x)\right]_0^1-\int_0^1\frac{x^2}{2}\frac{1}{x}\ \mathrm{d}x=\left[\frac{x^2}{2}\ln(x)\right]_0^1-\int_0^1\frac{x}{2}\ \mathrm{d}x=\left[\frac{x^2}{2}\ln(x)-\frac{x^2}{4}\right]_0^1=-\frac{1}{4},$$

kde jsme použili

$$\lim_{x\to 0^+}\frac{x^2}{2}\ln(x)=\frac{1}{2}\lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x^2}}\overset{l'H}{=}\frac{1}{2}\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{2}{x^3}}=-\frac{1}{2}\lim_{x\to 0^+}\frac{x^2}{2}=0.$$

(c)

$$I=\int_0^\infty e^{-ax}\cos(bx)\,\mathrm{d}x \overset{pp.}{=} -\left[\frac{e^{-ax}}{a}\cos(bx)\right]_0^\infty+\int_0^\infty \frac{e^{-ax}}{a}b\sin(bx)\,\mathrm{d}x=\frac{1}{a}+\frac{b}{a}\int_0^\infty e^{-ax}\sin(bx)\,\mathrm{d}x \overset{pp.}{=}$$
$$\frac{1}{a}-\frac{b}{a}\left[\frac{e^{-ax}}{a}\sin(bx)\right]_0^\infty-\frac{b^2}{a^2}\int_0^\infty \frac{e^{-ax}}{a}\cos(bx)\,\mathrm{d}x=\frac{1}{a}-\frac{b}{a^2}-\frac{b^2}{a^2}I,$$

tedy

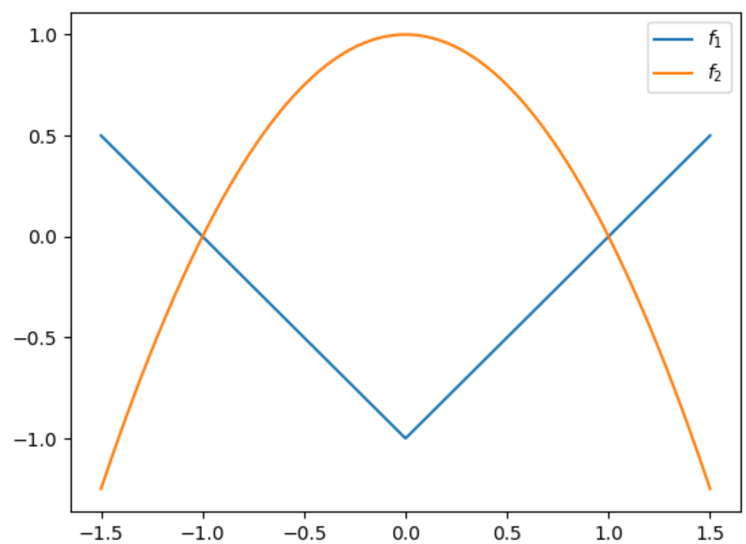
$$I=\frac{\frac{1}{a}-\frac{b}{a^2}}{1+\frac{b^2}{a^2}}=\frac{\frac{a-b}{a^2}}{\frac{a^2-b^2}{a^2}}=\frac{1}{a+b}.$$

(d)

$$\int_0^\infty x^3e^{-\frac{x^2}{2}}\,\mathrm{d}x=\left\{\begin{array}{c}\frac{x^2}{2}=y\\ x\mathrm{d}x=\mathrm{d}y\end{array}\right\}=2\int_0^\infty ye^{-y}\,\mathrm{d}y \overset{pp.}{=} 2\left[-ye^{-y}\right]_0^\infty+2\int_0^\infty e^{-y}\,\mathrm{d}y=2\left[-ye^{-y}-e^{-y}\right]_0^\infty=2.$$

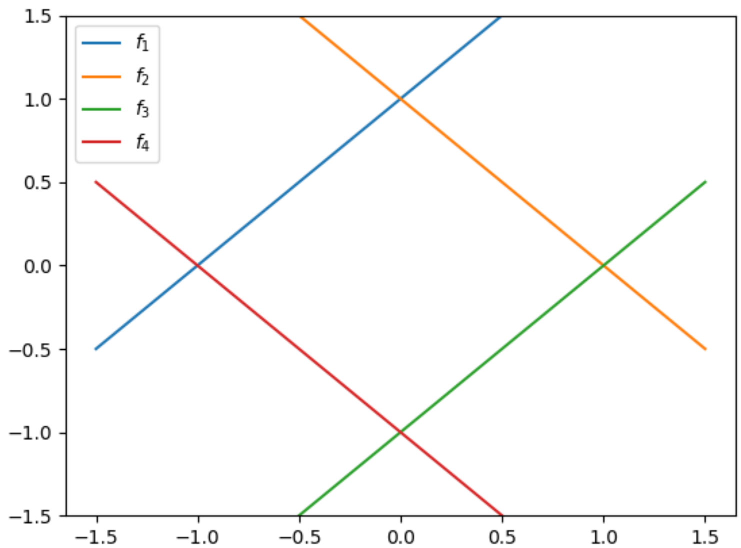
3 Oblasti mezi křivkami

(a) $f_1(x)=|x|-1,$
$f_2(x)=1-x^2$



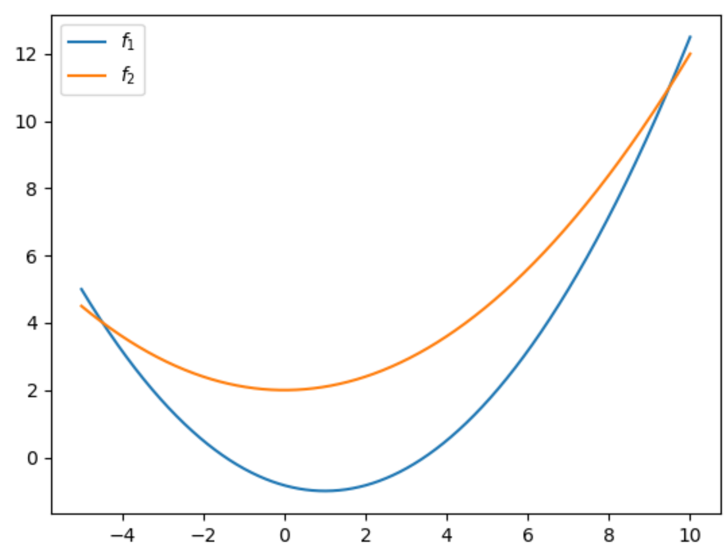
$$S=\int_{-1}^1 \left||x|-1-(1-x^2)\right|\,\mathrm{d}x=\int_{-1}^1 1-x^2-|x|+1\,\mathrm{d}x=\int_{-1}^0 2-x^2+x\,\mathrm{d}x+\int_0^1 2-x^2-x\,\mathrm{d}x=$$
$$\left[2x-\frac{x^3}{3}+\frac{x^2}{2}\right]_{-1}^0+\left[2x-\frac{x^3}{3}-\frac{x^2}{2}\right]_0^1=2-\frac{1}{3}-\frac{1}{2}+2-\frac{1}{3}-\frac{1}{2}=3-\frac{2}{3}.$$

(b) $f_1(x)=x+1,$
$f_2(x)=-x+1,$
$f_3(x)=x-1,$
$f_4(x)=-x-1,$



$$S = \int_{-1}^{0} |(x+1) - (-x-1)| \ \mathrm{d}x + \int_{0}^{1} |(-x+1) - (x-1)| \ \mathrm{d}x = \int_{-1}^{0} |2+2x| \ \mathrm{d}x + \int_{0}^{1} |2-2x| \ \mathrm{d}x = \int_{-1}^{0} 2+2x \ \mathrm{d}x + \int_{0}^{1} 2-2x \ \mathrm{d}x = [2x+x^2]_{-1}^{0} + [2x-x^2]_{0}^{1} = 2.$$

(c) $f_1(x) = \frac{(x-1)^2}{6} - 1$,
$f_2(x) = \frac{x^2}{10} + 2$, Průsečíky spočteme pomocí



$$\frac{(x-1)^2}{6} - 1 = \frac{x^2}{10} + 2 \overset{\text{Wolfram}}{\Rightarrow} x = \frac{5}{2} \pm \frac{\sqrt{195}}{2}.$$

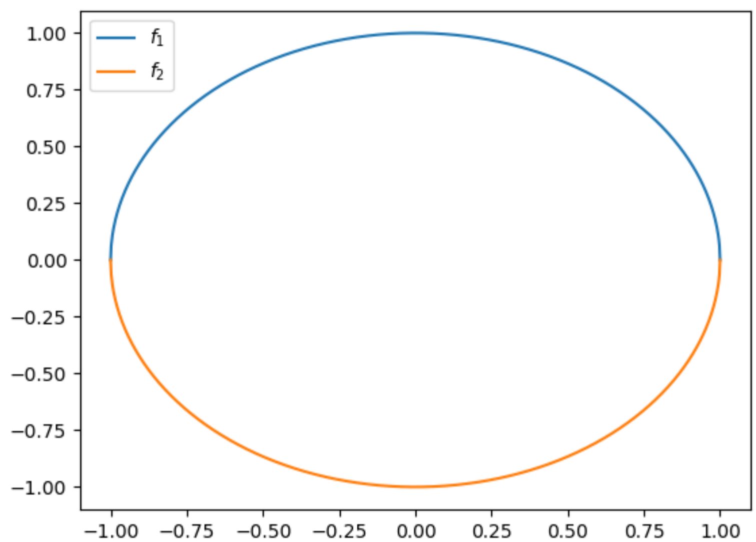
Pak (označme $x_\pm := \frac{5}{2} \pm \frac{\sqrt{195}}{2}$)

$$S = \int_{x_-}^{x_+} \left| \left(\frac{(x-1)^2}{6} - 1 \right) - \left(\frac{x^2}{10} + 2 \right) \right| \, \mathrm{d}x = \int_{x_-}^{x_+} \frac{x^2}{10} + 2 - \frac{(x-1)^2}{6} + 1 \, \mathrm{d}x =$$

$$\int_{x_-}^{x_+} \left(\frac{1}{10} - \frac{1}{6} \right) x^2 + \frac{x}{3} + 3 - \frac{1}{6} \, \mathrm{d}x = \left[\left(\frac{1}{10} - \frac{1}{6} \right) \frac{x^3}{3} + \frac{x}{6} + \left(+3 - \frac{1}{6} \right) x \right]_{x_-}^{x_+} \overset{\text{Wolfram}}{=}$$

$$\frac{13}{2} \sqrt{\frac{65}{3}} \approx 30.26.$$

(d) $f_1(x) = \sqrt{1-x^2}$,
$f_2(x) = -\sqrt{1-x^2}$,



$$S = \int_{-1}^{1} |\sqrt{1-x^2} - (-\sqrt{1-x^2})| \, \mathrm{d}x = 2 \int_{-1}^{1} \sqrt{1-x^2} \, \mathrm{d}x = \left\{ \begin{array}{c} x = \sin(y) \\ \mathrm{d}x = \cos(y)\mathrm{d}y \end{array} \right\} =$$

$$2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2(x)} \cos(x) \, \mathrm{d}x = 2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} |\cos(x)| \cos(x) \, \mathrm{d}x = -2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos^2(x) \, \mathrm{d}x,$$

kde

$$I = \int \cos^2(x) \, \mathrm{d}x \overset{pp.}{=} \cos(x)\sin(x) + \int \sin^2(x) \, \mathrm{d}x = \cos(x)\sin(x) + \int 1 - \cos^2(x) \, \mathrm{d}x = \cos(x)\sin(x) + x - I + c,$$

tedy $\int \cos^2(x) \, \mathrm{d}x = \frac{1}{2}(\cos(x)\sin(x) + x) + c$ a

$$S = -2 \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos^2(x) \, \mathrm{d}x = -2\frac{1}{2} \left[\cos(x)\sin(x) + x\right]_{\frac{3\pi}{2}}^{\frac{\pi}{2}} = -\left(0 + \frac{\pi}{2} - 0 - \frac{3\pi}{2} \right) = \pi.$$