Solutions to Tutorial 5

1 Newton's method by hand

Starting from $x_0 = 1$, use Newton's method to find the positive solution of

$$\cos x = x^3$$

by computing x_1, x_2, and x_3. Use the table at the end of the sheet for the required numerical values.

Put

$$f(x) = \cos x - x^3, \qquad f'(x) = -\sin x - 3x^2.$$

The key idea is geometric. At the current guess x_n, replace the curve by its tangent line. The point where this line crosses the horizontal axis becomes the next guess. Thus

$$0 \approx f(x_n) + f'(x_n)(x_{n+1} - x_n),$$

which yields Newton's formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

For the present equation this becomes

$$\boxed{x_{n+1} = x_n + \frac{\cos x_n - x_n^3}{\sin x_n + 3x_n^2}.}$$

Using the reference values at the end of the sheet,

$$\begin{aligned} x_1 &= 1 + \frac{0.540302 - 1}{0.841471 + 3} \approx 0.880333, \\ x_2 &= 0.880333 + \frac{0.636895 - 0.682246}{0.770951 + 3(0.774986)} \approx 0.865684, \\ x_3 &= 0.865684 + \frac{0.648119 - 0.648752}{0.761539 + 3(0.749409)} \approx 0.865474. \end{aligned}$$

Near the root, the curve is well approximated by its tangent, so the tangent's intercept rapidly approaches the true root.

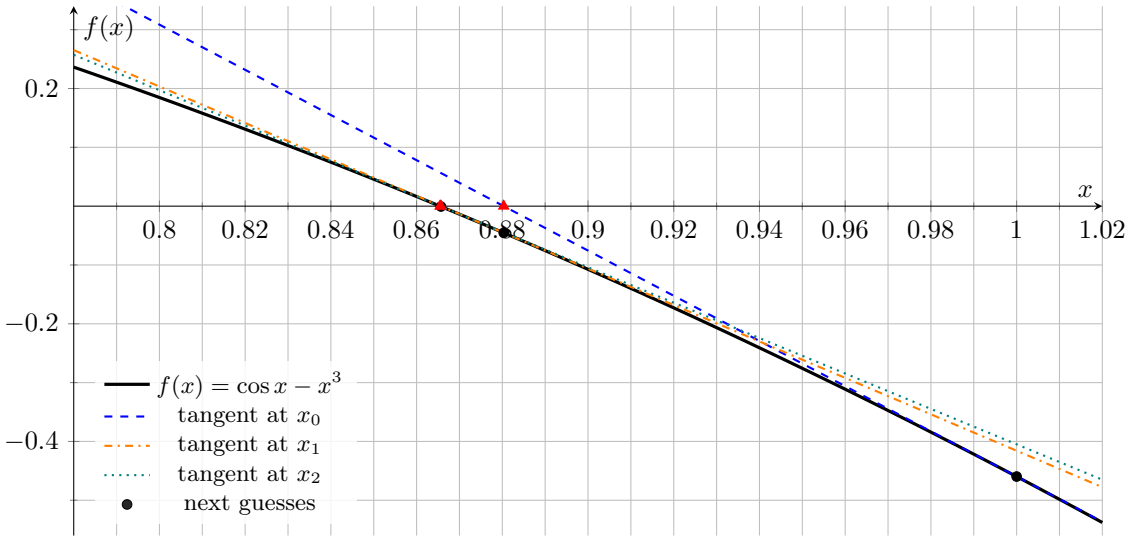


Figure 1: Each Newton step replaces the curve by a tangent. The tangent at x_n crosses the horizontal axis at the next guess x_{n+1}.

2 Newton's method can cycle

Apply Newton's method to

$$f(x) = x^3 - 2x + 2$$

with $x_0 = 0$. Compute x_1, x_2, x_3, and x_4.

We have

$$f'(x) = 3x^2 - 2, \qquad x_{n+1} = x_n - \frac{x_n^3 - 2x_n + 2}{3x_n^2 - 2}.$$

Therefore

$$x_1 = 1, \qquad x_2 = 0, \qquad x_3 = 1, \qquad x_4 = 0.$$

The iterates remain in the two-cycle

$$0 \longmapsto 1 \longmapsto 0 \longmapsto 1 \longmapsto \cdots$$

and never approach the real root $x^* \approx -1.769292$.

Nothing is wrong with the local convergence theorem. The function is smooth and the root is simple, since $f'(x^*) \approx 7.391 \neq 0$. The missing hypothesis is that the initial guess must be sufficiently close to the root; $x_0 = 0$ is outside the neighborhood for which convergence is guaranteed.

A common interval version of the convergence proof also cannot be applied. Between x^* and x_0, the derivative

$$f'(x) = 3x^2 - 2$$

vanishes at $x = -\sqrt{2/3}$. Thus the assumption that f' stays nonzero, and hence that the Newton step remains well controlled on the interval, is not satisfied.

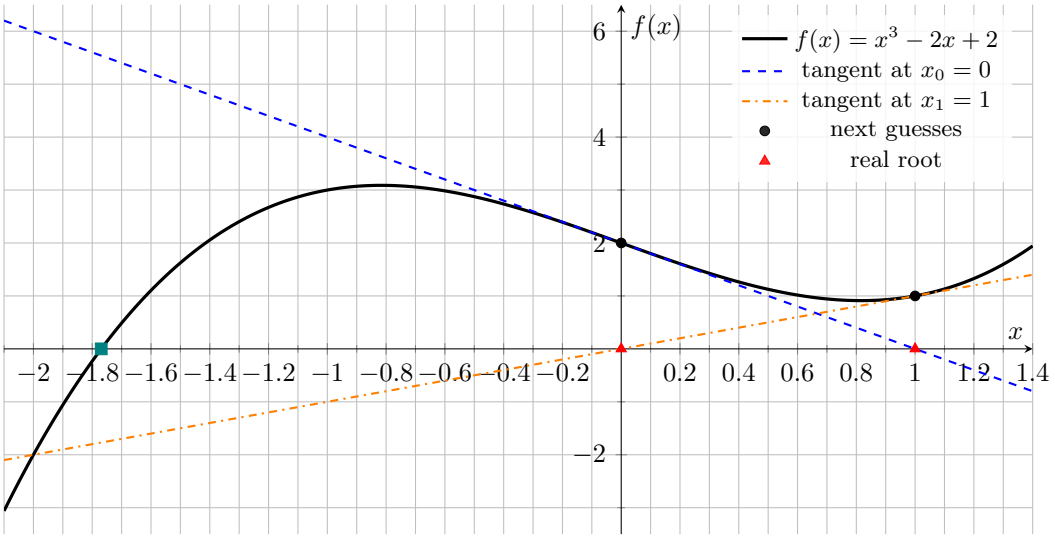


Figure 2: The tangent at $x_0 = 0$ crosses the horizontal axis at $x_1 = 1$, while the tangent at $x_1 = 1$ crosses it at $x_2 = 0$. Newton's method therefore repeats the same two steps and never reaches the real root.

3 Computing implied volatility

A non-dividend-paying stock has current price $S_0 = 100$. Consider a European call with strike $K = 100$, one year to expiry, and a continuously compounded risk-free rate $r = 5\%$. Its observed market price is $C_{\text{mkt}} = 10.4506$.

For volatility $\sigma > 0$, the Black–Scholes call price is

$$C_{\text{BS}}(\sigma) = S_0\Phi(d_1(\sigma)) - Ke^{-rT}\Phi(d_2(\sigma)),$$

where

$$d_1(\sigma) = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \qquad d_2(\sigma) = d_1(\sigma) - \sigma\sqrt{T},$$

and Φ is the standard normal distribution function.

Formulate the implied-volatility calculation as a root-finding problem for Newton's method. Specify the function, its derivative, and the iteration that we could give to a numerical solver. Use the initial guess $\sigma_0 = 0.30$ and report the resulting implied volatility.

Define the pricing residual

$$g(\sigma) = C_{\text{BS}}(\sigma) - C_{\text{mkt}}.$$

The implied volatility is the positive solution of $g(\sigma) = 0$. The market price C_{mkt} is constant with respect to σ, so

$$g'(\sigma) = \frac{\partial C_{\text{BS}}(\sigma)}{\partial \sigma}.$$

To compute this derivative, first rewrite

$$A = \ln(S_0/K) + rT, \qquad d_1(\sigma) = \frac{A}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}, \qquad d_2(\sigma) = \frac{A}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}.$$

Therefore

$$d_1'(\sigma) = -\frac{A}{\sigma^2\sqrt{T}} + \frac{\sqrt{T}}{2}, \qquad d_2'(\sigma) = -\frac{A}{\sigma^2\sqrt{T}} - \frac{\sqrt{T}}{2} = d_1'(\sigma) - \sqrt{T}.$$

The derivative of the standard normal distribution function is its density,

$$\Phi'(z) = \phi(z), \qquad \phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}.$$

Applying the chain rule to both terms in the Black–Scholes price gives

$$g'(\sigma) = S_0\phi(d_1(\sigma))d_1'(\sigma) - Ke^{-rT}\phi(d_2(\sigma))d_2'(\sigma).$$

This expression simplifies because the two normal densities satisfy

$$\begin{aligned}\frac{\phi(d_2(\sigma))}{\phi(d_1(\sigma))} &= \exp\left(\frac{d_1(\sigma)^2 - d_2(\sigma)^2}{2}\right) \\ &= \exp\left(d_1(\sigma)\sigma\sqrt{T} - \frac{\sigma^2 T}{2}\right) \\ &= \exp(\ln(S_0/K) + rT) = \frac{S_0 e^{rT}}{K}.\end{aligned}$$

Equivalently,

$$Ke^{-rT}\phi(d_2(\sigma)) = S_0\phi(d_1(\sigma)).$$

Using this identity together with $d_2'(\sigma) = d_1'(\sigma) - \sqrt{T}$, the terms containing $d_1'(\sigma)$ cancel:

$$\begin{aligned}g'(\sigma) &= S_0\phi(d_1(\sigma))d_1'(\sigma) - S_0\phi(d_1(\sigma))\big(d_1'(\sigma) - \sqrt{T}\big) \\ &= \boxed{S_0\phi(d_1(\sigma))\sqrt{T}}.\end{aligned}$$

Newton's iteration is therefore

$$\boxed{\sigma_{n+1} = \sigma_n - \frac{C_{\mathrm{BS}}(\sigma_n) - C_{\mathrm{mkt}}}{S_0\phi(d_1(\sigma_n))\sqrt{T}}.}$$

For the numerical inputs in the question, this becomes

$$d_1(\sigma) = \frac{0.05}{\sigma} + \frac{\sigma}{2}, \qquad d_2(\sigma) = \frac{0.05}{\sigma} - \frac{\sigma}{2},$$

and a solver starting from $\sigma_0 = 0.30$ produces

n	σ_n	$C_{\mathrm{BS}}(\sigma_n)$	$g(\sigma_n)$
0	0.300000000	14.231254786	3.780654786
1	0.200360394	10.464107657	0.013507657
2	0.200000455	10.450600635	0.000000635
3	0.200000438	10.450600000	0.000000000

Thus the implied volatility is

$$\boxed{\sigma_{\mathrm{impl}} \approx 0.20 = 20\%.}$$

The denominator in the Newton step is precisely the residual derivative,

$$g'(\sigma) = S_0\sqrt{T}\,\phi(d_1(\sigma)), \qquad \phi(d_1(\sigma)) = \frac{1}{\sqrt{2\pi}}e^{-d_1(\sigma)^2/2}.$$

The normal density becomes very small when $|d_1(\sigma)|$ is large. To interpret this in terms of the contract parameters, introduce the forward price $F_0 = S_0e^{rT}$. Then

$$d_1(\sigma) = \frac{\ln(F_0/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

Thus $|d_1(\sigma)|$ is typically large when the strike is far from the forward price relative to the total volatility $\sigma\sqrt{T}$. In other words, the option is deep in or out of the money. A short maturity or low volatility makes a given moneyness gap more extreme. At the other extreme, a very large $\sigma\sqrt{T}$ also sends $d_1(\sigma)$ far into the positive tail.

There is one additional source of a small residual derivative: the explicit factor $\sqrt{T}$. Even for an approximately at-the-money-forward option, for which $d_1(\sigma)$ is near zero and $\phi(d_1(\sigma))$ is large, $g'(\sigma)$ tends to zero as expiry approaches. In all these cases the price curve is flat in σ, so a modest pricing error can produce an unreasonably large Newton step. A practical solver should keep $\sigma > 0$ and use a bracketed fallback such as bisection when $|g'(\sigma)|$ is too small.

4 Using second derivatives

Let $f \in C^2$. Derive Halley's method for finding a root of f:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}.$$

Write

$$f_n = f(x_n), \qquad f'_n = f'(x_n), \qquad f''_n = f''(x_n).$$

If $x_{n+1} = x_n + h_n$ is to approximate a root, the second-order Taylor expansion gives

$$0 = f(x_n + h_n) = f_n + f'_n h_n + \frac{1}{2} f''_n h_n^2 + O(h_n^3).$$

The first-order approximation gives the Newton correction

$$h_n^{\mathrm{N}} = -\frac{f_n}{f'_n}.$$

Near a simple root, $h_n^{\mathrm{N}} = h_n + O(h_n^2)$. Consequently,

$$h_n^2 = h_n h_n^{\mathrm{N}} + O(h_n^3),$$

so replacing one factor of h_n by the Newton correction changes only the third-order remainder. The Taylor equation therefore becomes

$$0 \approx f_n + f'_n h_n + \frac{1}{2} f''_n h_n \left(-\frac{f_n}{f'_n}\right).$$

Collecting the terms containing h_n gives

$$f_n + h_n \left(f'_n - \frac{f_n f''_n}{2f'_n}\right) \approx 0.$$

Hence

$$h_n = -\frac{f_n}{f'_n - f_n f''_n/(2f'_n)} = -\frac{2f_n f'_n}{2(f'_n)^2 - f_n f''_n}.$$

Since $x_{n+1} = x_n + h_n$, we obtain Halley's update

$$\boxed{x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2(f'(x_n))^2 - f(x_n)f''(x_n)}.}$$

The formula is defined provided its denominator is nonzero. If the second-derivative term is omitted, it reduces to Newton's method.

Reference values for the hand calculation

Values are rounded to six decimal places.

n	x_n	$\cos x_n$	$\sin x_n$	x_n^2	x_n^3
0	1.000000	0.540302	0.841471	1.000000	1.000000
1	0.880333	0.636895	0.770951	0.774986	0.682246
2	0.865684	0.648119	0.761539	0.749409	0.648752
3	0.865474	0.648279	0.761403	0.749045	0.648279