Tutorial 4

1 Differential equations

(a) Consider an infectious disease that cannot be cured and that currently affects N people. The source of new infections is people who are already infected. Thus, the number of new infections during a given time interval is proportional to the number of currently infected people. Determine the number of infected people as a function of time, $N(t)$.

(b) Modify the previous model by assuming that our population has a finite size N_{tot}. Define the fraction of infected people by

$$n = \frac{N}{N_{\text{tot}}}.$$

The number of new infections decreases when almost everyone has already contracted the disease; it is therefore proportional to $n(1-n)$. Determine the fraction of infected people as a function of time, $n(t)$.

(c) Divide the population into three categories:

- S: susceptible, i.e., people who have not yet had the disease;
- I: infected, i.e., people who currently have the disease;
- R: removed, i.e., people who have recovered and are immune.

We can again define population fractions, for example $s = S/N_{\text{tot}}$. Formulate a system of differential equations for these variables, assuming that:

- s decreases at a rate proportional to si, as in the previous exercise;
- r increases at a rate proportional to the number of infected people, i.e., a certain fraction of infected people recovers each day;
- i increases at a rate proportional to si and decreases at a rate proportional to the number of infected people.

2 Black–Scholes: the strike-payment term

(a) Under the risk-neutral measure Q, suppose that the stock price at time T is given by

$$S_T = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\,Z\right), \qquad Z \sim N(0,1),$$

where $S_0 > 0$, $K > 0$, $T > 0$, and $\sigma > 0$. The price of a European call contains the strike-payment term $Ke^{-rT}Q(S_T > K)$, where

$$Q(S_T > K) = \mathbb{E}^Q\left[\mathbf{1}_{\{S_T > K\}}\right].$$

Derive an explicit formula for this term. Recall that if $Y \sim N(m, \tau^2)$, its probability density is

$$f_Y(y) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{(y-m)^2}{2\tau^2}\right), \qquad y \in \mathbb{R}.$$

In particular:

1. Find the distribution of $X = \ln S_T$ and write the probability $Q(S_T > K)$ as an integral of the lognormal density of S_T from K to ∞.
2. Substitute $x = \ln s$ in this integral, carefully transforming the differential and integration bounds.
3. Standardize using $z = (x - \mu)/v$, where μ and v are the mean and standard deviation of X. Express the answer using the standard normal CDF Φ.

3 Black–Scholes: the stock-payment term

(a) Under the same risk-neutral Black–Scholes model as in the previous exercise, derive an explicit formula for the stock-payment term

$$e^{-rT}\mathbb{E}^{Q}\left[S_T \mathbf{1}_{\{S_T>K\}}\right].$$

You may use the notation

$$\mu = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \qquad v = \sigma\sqrt{T},$$

and the lognormal density derived above. In particular:

1. Express the expectation as an integral over $s > K$.
2. Make the substitution $x = \ln s$, including the change of the differential and integration bounds. Explain why a factor e^x remains in the integrand.
3. Complete the square in the exponent and evaluate the resulting normal tail integral. Express the result in terms of the standard normal CDF Φ.