Solutions to Tutorial 4

1 Differential equations

(a) Consider an infectious disease that cannot be cured and that currently affects N people. The source of new infections is people who are already infected. Thus, the number of new infections during a given time interval is proportional to the number of currently infected people. Determine the number of infected people as a function of time, $N(t)$.

Let $\alpha > 0$ be the proportionality constant. Then

$$\frac{dN}{dt} = \alpha N.$$

Separating variables and integrating gives

$$\frac{dN}{N} = \alpha\, dt, \qquad \ln N = \alpha t + C.$$

With $N(0) = N_0$, the solution is

$$\boxed{N(t) = N_0 e^{\alpha t}.}$$

Thus, this model predicts exponential growth.

(b) Modify the previous model by assuming that our population has a finite size N_{tot}. Define the fraction of infected people by

$$n = \frac{N}{N_{\text{tot}}}.$$

The number of new infections decreases when almost everyone has already contracted the disease; it is therefore proportional to $n(1-n)$. Determine the fraction of infected people as a function of time, $n(t)$.

Let $\beta > 0$ denote the growth-rate constant. We obtain the logistic differential equation

$$\frac{dn}{dt} = \beta n(1-n).$$

For $0 < n < 1$, separation of variables yields

$$\int \frac{dn}{n(1-n)} = \int \beta\, dt, \qquad \ln \frac{n}{1-n} = \beta t + C.$$

If $n(0) = n_0$ with $0 < n_0 < 1$, then

$$\boxed{n(t) = \frac{1}{1 + \frac{1-n_0}{n_0} e^{-\beta t}}.}$$

This approaches 1 as $t \to \infty$. The constant solutions $n(t) \equiv 0$ and $n(t) \equiv 1$ cover the boundary initial values.

(c) Divide the population into three categories:

- S: susceptible, i.e., people who have not yet had the disease;
- I: infected, i.e., people who currently have the disease;
- R: removed, i.e., people who have recovered and are immune.

We can again define population fractions, for example $s = S/N_{\text{tot}}$. Formulate a system of differential equations for these variables, assuming that:

- s decreases at a rate proportional to si, as in the previous exercise;
- r increases at a rate proportional to the number of infected people, i.e., a certain fraction of infected people recovers each day;

- i increases at a rate proportional to si and decreases at a rate proportional to the number of infected people.

Let $\beta > 0$ be the infection-rate parameter and $\gamma > 0$ the recovery-rate parameter. The standard SIR system for the population fractions is

$$\boxed{\begin{aligned} \frac{ds}{dt} &= -\beta si, \\ \frac{di}{dt} &= \beta si - \gamma i, \\ \frac{dr}{dt} &= \gamma i. \end{aligned}}$$

Here $s + i + r = 1$ if the population is closed and all individuals belong to exactly one category. Indeed,

$$\frac{d}{dt}(s + i + r) = 0,$$

so the total population fraction is conserved. The number of infected people initially grows when $\beta s > \gamma$ and decreases when $\beta s < \gamma$.

2 Black–Scholes: the strike-payment term

(a) Under the risk-neutral measure Q, suppose that the stock price at time T is given by

$$S_T = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) T + \sigma\sqrt{T}\, Z\right), \qquad Z \sim N(0,1),$$

where $S_0 > 0$, $K > 0$, $T > 0$, and $\sigma > 0$. The price of a European call contains the strike-payment term $Ke^{-rT}Q(S_T > K)$, where

$$Q(S_T > K) = \mathbb{E}^Q\left[\mathbf{1}_{\{S_T > K\}}\right].$$

Derive an explicit formula for this term. Recall that if $Y \sim N(m, \tau^2)$, its probability density is

$$f_Y(y) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{(y-m)^2}{2\tau^2}\right), \qquad y \in \mathbb{R}.$$

In particular:

1. Find the distribution of $X = \ln S_T$ and write the probability $Q(S_T > K)$ as an integral of the lognormal density of S_T from K to ∞.
2. Substitute $x = \ln s$ in this integral, carefully transforming the differential and integration bounds.
3. Standardize using $z = (x - \mu)/v$, where μ and v are the mean and standard deviation of X. Express the answer using the standard normal CDF Φ.

Write

$$\mu = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T, \qquad v = \sigma\sqrt{T}.$$

Then $X = \ln S_T = \mu + vZ \sim N(\mu, v^2)$. The probability of exercise is

$$Q(S_T > K) = \int_K^\infty f_{S_T}(s)\, ds.$$

Since S_T is lognormal, its density is

$$f_{S_T}(s) = \frac{1}{sv\sqrt{2\pi}} \exp\left(-\frac{(\ln s - \mu)^2}{2v^2}\right), \qquad s > 0.$$

Consequently,

$$Q(S_T > K) = \int_K^\infty \frac{1}{sv\sqrt{2\pi}} \exp\left(-\frac{(\ln s - \mu)^2}{2v^2}\right) ds.$$

For $x = \ln s$ we have $s = e^x$ and $ds = s\,dx$; the lower limit becomes $\ln K$. The factor s from the Jacobian cancels the $1/s$ in the density:

$$Q(S_T > K) = \int_{\ln K}^{\infty} \frac{1}{v\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2v^2}\right) dx.$$

Next set $z = (x-\mu)/v$, so $dx = v\,dz$. We obtain

$$Q(S_T > K) = \int_{(\ln K-\mu)/v}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz = 1 - \Phi\left(\frac{\ln K - \mu}{v}\right).$$

By symmetry of the standard normal distribution, $1 - \Phi(a) = \Phi(-a)$. Define

$$d_2 = \frac{\mu - \ln K}{v} = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}.$$

Therefore the strike-payment term is

$$\boxed{Ke^{-rT}Q(S_T > K) = Ke^{-rT}\Phi(d_2).}$$

3 Black–Scholes: the stock-payment term

(a) Under the same risk-neutral Black–Scholes model as in the previous exercise, derive an explicit formula for the stock-payment term

$$e^{-rT}\mathbb{E}^Q\left[S_T \mathbf{1}_{\{S_T>K\}}\right].$$

You may use the notation

$$\mu = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \qquad v = \sigma\sqrt{T},$$

and the lognormal density derived above. In particular:

1. Express the expectation as an integral over $s > K$.
2. Make the substitution $x = \ln s$, including the change of the differential and integration bounds. Explain why a factor e^x remains in the integrand.
3. Complete the square in the exponent and evaluate the resulting normal tail integral. Express the result in terms of the standard normal CDF Φ.

For a nonnegative stock price, the expectation can be written as

$$\mathbb{E}^Q\left[S_T \mathbf{1}_{\{S_T>K\}}\right] = \int_K^{\infty} s f_{S_T}(s)\,ds = \int_K^{\infty} \frac{1}{v\sqrt{2\pi}} \exp\left(-\frac{(\ln s - \mu)^2}{2v^2}\right) ds.$$

Now let $x = \ln s$, so that $s = e^x$, $ds = e^x\,dx$, and $s > K$ becomes $x > \ln K$. Thus

$$\mathbb{E}^Q\left[S_T \mathbf{1}_{\{S_T>K\}}\right] = \int_{\ln K}^{\infty} \frac{1}{v\sqrt{2\pi}} \exp\left(x - \frac{(x-\mu)^2}{2v^2}\right) dx.$$

The factor e^x comes from the Jacobian $ds = e^x\,dx$; unlike in the strike-payment integral, it does not cancel, because the extra factor s in the expectation has already cancelled the $1/s$ of the lognormal density.

Complete the square:

$$x - \frac{(x-\mu)^2}{2v^2} = -\frac{(x-\mu-v^2)^2}{2v^2} + \mu + \frac{v^2}{2}.$$

Hence

$$\mathbb{E}^Q\left[S_T \mathbf{1}_{\{S_T>K\}}\right] = e^{\mu+v^2/2} \int_{\ln K}^{\infty} \frac{1}{v\sqrt{2\pi}} \exp\left(-\frac{(x-\mu-v^2)^2}{2v^2}\right) dx.$$

With $z = (x-\mu-v^2)/v$ and $dx = v\,dz$, this becomes

$$e^{\mu+v^2/2} \int_{(\ln K-\mu-v^2)/v}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz = e^{\mu+v^2/2}\Phi\left(\frac{\mu + v^2 - \ln K}{v}\right).$$

Finally, the definitions of μ and v imply

$$e^{\mu+v^2/2} = S_0 e^{rT}, \qquad d_1 = \frac{\mu + v^2 - \ln K}{v} = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = d_2 + \sigma\sqrt{T}.$$

Discounting gives the stock-payment term

$$\boxed{e^{-rT}\mathbb{E}^Q\left[S_T \mathbf{1}_{\{S_T > K\}}\right] = S_0 \Phi(d_1).}$$