Tutorial 3

1 One-step replication

A share trades at $S_0 = 100$. At expiry it will be worth either $S_u = 120$ or $S_d = 80$. Interest is zero. Consider a European call with strike $K = 100$.

(a) Compute the call payoff in each state.
(b) Find Δ shares and B dollars of cash whose terminal value $\Delta S_T + B$ matches the call in both states.
(c) Use the cost of the replicating portfolio to find C_0.
(d) Find the risk-neutral probability q from the stock price and verify the same call price by risk-neutral valuation.

2 A two-step tree

Now allow two moves. At each step the share is multiplied by 1.2 or 0.8; the middle paths recombine:

time	0	1	2
upper node	100	120	144
middle node			96
lower node		80	64

Interest is zero and $q = 1/2$ at every branch. Price a European call with $K = 100$.

(a) Write the three terminal call payoffs.
(b) Work backward to find the call values at the 120 and 80 nodes, and then find C_0.
(c) Find (Δ, B) at each pre-expiry node. Is it optimal to rebalance the hedge at each node? Why?

3 Non-zero interest rates

Use the same two-step stock tree and call, but let one dollar in the cash account grow to $R = 1.05$ dollars during each step. The stock pays no dividends.

(a) At a node with current stock price S and next prices S_u, S_d, show that

$$S = \frac{qS_u + (1-q)S_d}{R}, \qquad q = \frac{RS - S_d}{S_u - S_d}.$$

(b) If $q = 1/2$ were retained at the root, what value of S_0 would the next prices 120 and 80 imply? Then use the observed $S_0 = 100$ to compute q at all three pre-expiry nodes and check that the model is free of arbitrage.
(c) Propagate the call values backward from the leaves.

4 A cash dividend and an American call

Return to zero interest and $q = 1/2$ at every branch. At time 1, immediately after the holder's exercise decision, the share pays a cash dividend $D = 10$. The dividend creates one jump from the cum-dividend price S_{1^-} to the ex-dividend price $S_{1^+} = S_{1^-} - D$; it is not subtracted again elsewhere:

S_{1^-}	S_{1^+}	possible prices at expiry
120	110	132, 88
80	70	84, 56

The share starts at $S_0 = 100$. Consider an American call with $K = 100$.

(a) Compute the four terminal call payoffs and the continuation value at each $t = 1^+$ node.

(b) Just before the dividend, compare immediate exercise with continuation at both $t = 1^-$ nodes. Where is early exercise optimal?

(c) What would the otherwise identical European call be worth? Explain the difference.

5 The corresponding put

Use the same dividend tree and strike, but replace the call by an American put.

(a) Starting from the terminal put payoffs, work backward at a high level and identify whether early exercise is optimal at any node.

(b) Find P_0.

(c) Are there any parts of the pricing mechanism we considered for a call which are not relevant for the put? If yes, what should we consider for a put that is not relevant for the call?