Solutions to Tutorial 3

1 One-step replication

A share trades at $S_0 = 100$. At expiry it will be worth either $S_u = 120$ or $S_d = 80$. Interest is zero. Consider a European call with strike $K = 100$.

(a) Compute the call payoff in each state.
(b) Find Δ shares and B dollars of cash whose terminal value $\Delta S_T + B$ matches the call in both states.
(c) Use the cost of the replicating portfolio to find C_0.
(d) Find the risk-neutral probability q from the stock price and verify the same call price by risk-neutral valuation.

(a) $(C_u, C_d) = (20, 0)$.
(b) Solve

$$120\Delta + B = 20, \qquad 80\Delta + B = 0.$$

Hence $\Delta = 1/2$ and $B = -40$.
(c) The replicating portfolio costs

$$C_0 = \Delta S_0 + B = \tfrac{1}{2}(100) - 40 = 10.$$

(d) The stock price gives

$$100 = q(120) + (1-q)80, \qquad q = \tfrac{1}{2}.$$

Therefore $C_0 = q(20) + (1-q)0 = 10$.

2 A two-step tree

Now allow two moves. At each step the share is multiplied by 1.2 or 0.8; the middle paths recombine:

time	0	1	2
upper node	100	120	144
middle node			96
lower node		80	64

Interest is zero and $q = 1/2$ at every branch. Price a European call with $K = 100$.

(a) Write the three terminal call payoffs.
(b) Work backward to find the call values at the 120 and 80 nodes, and then find C_0.
(c) Find (Δ, B) at each pre-expiry node. Is it optimal to rebalance the hedge at each node? Why?

(a) At $S_2 = 144, 96, 64$, the payoffs are $44, 0, 0$.
(b) Backward induction gives

$$C_u = \tfrac{1}{2}(44) + \tfrac{1}{2}(0) = 22, \qquad C_d = 0, \qquad C_0 = \tfrac{1}{2}(22) = 11.$$

(c) At time 0,

$$\Delta_0 = \frac{22 - 0}{120 - 80} = \frac{11}{20}, \qquad B_0 = 11 - \frac{11}{20}(100) = -44.$$

At the 120 node,

$$\Delta_u = \frac{44 - 0}{144 - 96} = \frac{11}{12}, \qquad B_u = 22 - \frac{11}{12}(120) = -88.$$

At the 80 node, $(\Delta_d, B_d) = (0, 0)$. Yes: the required delta depends on the realized node. Rebalancing to the node-specific portfolio preserves exact replication; keeping the original hedge would no longer match the call payoff in both subsequent states.

3 Non-zero interest rates

Use the same two-step stock tree and call, but let one dollar in the cash account grow to $R = 1.05$ dollars during each step. The stock pays no dividends.

(a) At a node with current stock price S and next prices S_u, S_d, show that

$$S = \frac{qS_u + (1-q)S_d}{R}, \qquad q = \frac{RS - S_d}{S_u - S_d}.$$

(b) If $q = 1/2$ were retained at the root, what value of S_0 would the next prices 120 and 80 imply? Then use the observed $S_0 = 100$ to compute q at all three pre-expiry nodes and check that the model is free of arbitrage.

(c) Propagate the call values backward from the leaves.

(a) Under the pricing probability, a traded share must earn the risk-free rate:

$$RS = qS_u + (1-q)S_d.$$

Rearranging gives the stated formulas.

(b) Keeping $q = 1/2$ would imply

$$S_0 = \frac{\frac{1}{2}(120) + \frac{1}{2}(80)}{1.05} = \frac{2000}{21} \approx 95.24,$$

rather than 100. Using the observed stock prices, at the root and the two time-1 nodes,

$$q_0 = \frac{105-80}{120-80}, \quad q_u = \frac{126-96}{144-96}, \quad q_d = \frac{84-64}{96-64};$$

all three equal 5/8. Since $0 < q < 1$ (equivalently $0.8 < 1.05 < 1.2$), there is no stock–cash arbitrage.

(c) The terminal payoffs remain $44, 0, 0$. Thus

$$C_u = \frac{(5/8)44}{1.05} = \frac{550}{21}, \qquad C_d = 0,$$

and

$$C_0 = \frac{(5/8)(550/21)}{1.05} = \frac{6875}{441} \approx 15.59.$$

This exceeds the zero-rate value 11: discounting makes the future strike payment cheaper, while the stock price implies a larger up pricing weight.

4 A cash dividend and an American call

Return to zero interest and $q = 1/2$ at every branch. At time 1, immediately after the holder's exercise decision, the share pays a cash dividend $D = 10$. The dividend creates one jump from the cum-dividend price S_{1^-} to the ex-dividend price $S_{1^+} = S_{1^-} - D$; it is not subtracted again elsewhere:

S_{1^-}	S_{1^+}	possible prices at expiry
120	110	132, 88
80	70	84, 56

The share starts at $S_0 = 100$. Consider an American call with $K = 100$.

(a) Compute the four terminal call payoffs and the continuation value at each $t = 1^+$ node.

(b) Just before the dividend, compare immediate exercise with continuation at both $t = 1^-$ nodes. Where is early exercise optimal?

(c) What would the otherwise identical European call be worth? Explain the difference.

(a) The terminal payoffs at $132, 88, 84, 56$ are $32, 0, 0, 0$. Hence the continuation values at the ex-dividend nodes 110 and 70 are 16 and 0.

(b) At every exercise date, compare the two possible actions directly:

$$C = \max\left\{S - K, \frac{qC_u + (1-q)C_d}{R}\right\}.$$

Immediately before the dividend,

S_{1^-}	exercise	continue	decision
120	20	16	exercise
80	0	0	either

Exercising at the upper node turns the option into a share in time to receive the dividend. The time-1 American values are therefore 20 and 0. At time 0, immediate exercise is worth 0, whereas continuation is

$$\tfrac{1}{2}(20) + \tfrac{1}{2}(0) = 10.$$

Thus $C_0 = 10$.

(c) A European holder cannot exercise before the dividend, so

$$C_0^{\mathrm{E}} = \tfrac{1}{2}(16) + \tfrac{1}{2}(0) = 8.$$

The American exercise right is worth \$2 more than the European one in this tree.

5 The corresponding put

Use the same dividend tree and strike, but replace the call by an American put.

(a) Starting from the terminal put payoffs, work backward at a high level and identify whether early exercise is optimal at any node.

(b) Find P_0.

(c) Are there any parts of the pricing mechanism we considered for a call which are not relevant for the put? If yes, what should we consider for a put that is not relevant for the call?

(a) The terminal put payoffs at $132, 88, 84, 56$ are $0, 12, 16, 44$. At $t = 1^+$ the continuation values are 6 at $S = 110$ and 30 at $S = 70$. Just before the dividend,

S_{1^-}	exercise	continue
120	0	6
80	20	30

so early exercise is not optimal before the dividend. At the lower $t = 1^+$ node, exercise and continuation are both worth 30; the holder is indifferent at zero rates.

(b) At the root, exercise is worth 0 and continuation is

$$P_0 = \tfrac{1}{2}(6) + \tfrac{1}{2}(30) = 18.$$

(c) The call's incentive to exercise immediately before a dividend is not relevant for a put. The dividend lowers the share price, which helps the put, so exercising beforehand would surrender that benefit. For a put we must instead consider the benefit of receiving the strike early when interest rates are positive, especially when the put is deep in the money. This motive is not relevant for a call: exercising a call requires paying the strike early and giving up interest on that cash.