

Algorithmic game theory – Tutorial 5*

1 The Lemke–Howson algorithm

The *best response polyhedron* for player 1 in G is a polyhedron $\bar{P}$ is defined as

$$\bar{P} = \{(x, v) \in \mathbb{R}^m \times \mathbb{R} : x \geq \mathbf{0}, \mathbf{1}^\top x = 1, N^\top x \leq \mathbf{1}v\}.$$

Similarly, we define the best response polyhedron for player 2 in G as

$$\bar{Q} = \{(y, u) \in \mathbb{R}^n \times \mathbb{R} : y \geq \mathbf{0}, \mathbf{1}^\top y = 1, My \leq \mathbf{1}u\}.$$

A point (x, v) of $\bar{P}$ has *label* $i \in A_1 \cup A_2$ if either $i \in A_1$ and $x_i = 0$ or if $i \in A_2$ and $(N^\top)_i x = v$, where $(N^\top)_i$ is the i th row of $N^\top$. A point (y, u) has a label $i \in A_1 \cup A_2$ if either $i \in A_1$ and $(M)_i y = u$, where $(M)_i$ is the i th row of M , or if $i \in A_2$ and $y_i = 0$.

Assume that the matrices M and $N^\top$ are non-negative and have no zero column. The *best response polytope* for player 1 in G is a polytope P defined as

$$P = \{x \in \mathbb{R}^m : x \geq \mathbf{0}, N^\top x \leq \mathbf{1}\}.$$

Similarly, the best response polytope for player 2 in G is a polytope Q defined as

$$Q = \{y \in \mathbb{R}^n : y \geq \mathbf{0}, My \leq \mathbf{1}\}.$$

The labels in P and Q are defined analogously as in $\bar{P}$ and $\bar{Q}$.

Nash equilibria in a non-degenerate game correspond to completely labeled pairs of vertices from $P \times Q \setminus \{(\mathbf{0}, \mathbf{0})\}$.

Exercise 1. Draw the best response polyhedron and normalized best response polytope for the Battle for Gotham's soul. Then, find all completely labeled pairs of vertices that correspond to Nash equilibria.

	Cooperate (3)	Detonate (4)
Cooperate (1)	(1, 1)	(1, 2)
Detonate (2)	(2, 1)	(1, 1)

Table 1: Battle for Gotham's soul.

Exercise 2. Use the Lemke–Howson algorithm and compute a Nash equilibrium of the following bimatrix game

$$M = \begin{pmatrix} 0 & 6 \\ 2 & 5 \\ 3 & 3 \end{pmatrix} \quad a \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 3 \end{pmatrix}.$$

Start the algorithm by choosing the label 2.

A *configuration graph* with the vertex set formed by those pairs (x, y) of vertices from $P \times Q$ that are *k-almost completely labeled*, meaning that every label from $A_1 \cup A_2 \setminus \{k\}$ appears as a label of x or of y . Two vertices (x, y) and (x', y') of the configuration graph are connected by an edge if either $x = x'$ and yy' is an edge of Q or if xx' is an edge of P and $y = y'$.

Exercise 3. Show that the Lemke–Howson algorithm does not terminate in a vertex of the form $(x, \mathbf{0})$ or $(\mathbf{0}, y)$ in the configuration graph.

*Information about the course can be found at <http://kam.mff.cuni.cz/~sychrovsky/>