A Simple Analysis of Ranking in General Graphs

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We provide a simple combinatorial analysis of the Ranking algorithm, originally introduced in the seminal work by Karp, Vazirani, and Vazirani [KVV90], demonstrating that it achieves a $(1/2 + c)$-approximate matching for general graphs for $c \geq 0.005$.

1 Introduction

In this work, we study a randomized greedy matching algorithm called Ranking for general graphs. This algorithm was first introduced in the seminal work of Karp, Vazirani, and Vazirani [17] in 1990 for online bipartite matching with one-sided vertex arrivals and was subsequently extended to general graphs [12] and bipartite graphs [19] with random vertex arrivals. The algorithm belongs to a class of matching algorithms called vertex-iterative (VI) randomized greedy matching algorithms. These algorithms first draw a permutation π over the vertices uniformly at random. They then iterate over the vertices in the order of π, matching each vertex to one of its available neighbors (if any) according to a specific neighbor selection policy. In Ranking for general graphs, this policy selects the first unmatched neighbor in the order given by the same permutation π used for the initial iteration.

Besides their simplicity and applicability to settings such as online matching, a significance of VI randomized greedy matching algorithms is that they outperform deterministic greedy algorithms and the randomized edge-iterative greedy algorithm[1] [8], achieving an approximation ratio larger than 0.5. The approximation ratio of vertex-iterative randomized greedy matching algorithms is well-understood for bipartite graphs, with the best approximation ratio being between 0.696 [19] and 0.75 [12]. However, despite a long line of work [1, 21, 3, 22, 12, 6], our understanding of their approximation ratio for general graphs remains limited, with the best-known lower and upper bounds being 0.5469 [6] and 0.75 [12]. Aronson, Dyer, Frieze, and Suen [1] were the first to show that VI randomized greedy algorithms surpass 0.5 for general graphs by designing a VI algorithm called the modified randomized greedy (MRG)[2] and proving a lower bound of $0.5 + 1/400,000$ for its approximation ratio. This approximation ratio was later improved to $0.5 + 1/256$ [21]. Both of these papers involve quite complicated combinatorial analyses. Later, in [6] and [22], authors proved 0.5469 and 0.531 approximation guarantees for Ranking and MRG, respectively, using factor-revealing linear programs.

[1] The algorithm that draws a permutation π over the edges uniformly at random and greedily includes the edges according to π.

[2] In this algorithm, the neighbor selection policy selects one of the unmatched neighbors uniformly at random.

In this work, we take a fresh look at the Ranking algorithm for general graphs and provide a significantly simpler combinatorial analysis demonstrating that it achieves an approximation ratio of at least 0.505. Our approach is not only more straightforward and intuitive compared to all previous work, but our approximation guarantee also surpasses existing guarantees obtained without solving factor-revealing linear programs.

Our Techniques. Our proof idea starts with a few simple, well-known observations. Let OPT be the maximum matching of the input graph, which we can assume is a perfect matching w.l.o.g. [21], and let R be the output of Ranking. If R has an approximation ratio smaller than $1/2 + c$ for a constant $c > 0$, then $R \oplus \mathrm{OPT}$ contains at least $(1/2 - 3c)n/2$ augmenting paths of length three. Furthermore, since R is a maximal matching, the endpoints of these augmenting paths are unmatched in R, and they form an independent set. Following these observations, we define a graph structure called a *k-wasteful independent set* (k-WIS) in Definition 2.1. Given a subset of $2k$ vertices, they form a k-WIS if they are the $2k$ endpoints of k augmenting paths of length three in $R \oplus \mathrm{OPT}$. Our analysis then involves a double counting of these k-WIS for $k = (1/2 - 3c)n/2$. We provide combinatorial lower and upper bounds for the expected number of k-WIS in the output of Ranking (respectively in Lemma 2.2 and Lemma 2.5). We then show that for $c \leq 0.005$, our bounds reach a contradiction, hence proving that Ranking in expectation has an approximation ratio of at least 0.505.

Further Related Work. After its introduction in the seminal work of Karp, Vazirani, and Vazirani [17], Ranking and its extensions have been studied extensively in various settings such as online matching with vertex arrivals [9, 11, 2, 7, 13, 14, 16, 6], oblivious matching [19, 22, 3], and stochastic matching with query commits [10, 5]. See the excellent survey on online matching [15] for a more detailed discussion of related work.

1.1 Preliminaries

Notation and Definitions. Throughout the paper, we denote the input graph as $G = (V, E)$, where $|V| = n$. A maximal matching in G is a matching M such that there is no edge $e \in E \setminus M$ for which $M \cup e$ is also a matching. A maximum matching in G is a matching with the maximum number of edges, denoted by $\mu(G)$. Let OPT denote a fixed maximum matching of G. As discussed in [12, 21] (see Corollary 2 of [21]), one can assume that G contains a perfect matching when analyzing the ratio of the Ranking algorithm.

The Ranking Algorithm. We first draw a permutation π over the vertices uniformly at random. Then, we iterate over the vertices in order of π, and match each vertex to its first unmatched neighbor, again in the order of π, if any. We use $\pi(v)$ to denote the position of vertex v in the permutation selected by Ranking.

Augmenting Paths. An augmenting path P for a matching M in a graph G is a path that alternates between edges of OPT and M such that the path starts and ends with vertices that are unmatched in M. The length of the augmenting path is the number of edges it contains. The presence of an augmenting path indicates that the matching M is not maximum. Furthermore, when matching M is far from being maximum, say a 1/2-approximation, there exist many short augmenting paths.

Proposition 1.1 ([18, Lemma 1]). *Let $\alpha \geq 0$ and M be a maximal matching of G such that $|M| \leq (1/2+\alpha)\cdot \mu(G)$. Then, at least $(1/2-3\alpha)\cdot\mu(G)$ edges of M are in disjoint, length-three augmenting paths.*

Combinatorial Tools. We use the following two well-known bounds on the binomial coefficient.

Proposition 1.2 ([20, Chapter 3]). *Let $a_1, a_2, \ldots, a_n$ be n positive integers such that $\sum_{i=1}^{n} a_i = m$. Then, for a positive integer x, $\sum_{i=1}^{n}\binom{a_i}{x}$ is minimized when $\{a_1,\ldots,a_n\} = \{\lfloor m/n\rfloor, \lceil m/n\rceil\}$.*

Proposition 1.3 ([4, Chapter 11]). *Let n and αn be two positive integers and $1/n \leq \alpha \leq 1/2$. Then, we have $\binom{n}{\alpha n} \leq 2^{nH(\alpha)}$ where $H(\alpha) = -\alpha\cdot\log_2(\alpha) - (1-\alpha)\cdot\log_2(1-\alpha)$.*

2 The Analysis

Throughout the analysis, we assume that the expected approximation ratio of Ranking is at most $1/2+c$ for some rational $c \geq 0$. Then, we prove that if c is smaller than some constant, it leads to a contradiction. This implies a lower bound on the expected approximation ratio of Ranking. Moreover, we assume that, without loss of generality, $cn/2$ is an integer. To understand this, consider a graph where the approximation ratio of Ranking is $1/2+c$ and $cn/2$ is not an integer. For $c = a/b$ with integers a, b, by copying the same graph $2b$ times, the approximation guarantee of Ranking remains unchanged, but now $cn'/2$ is an integer, where $n' = 2bn$ is the number of vertices in the new graph. Our proof relies on counting a structure in the graph, referred to as a k-wasteful independent set (k-WIS), which we define formally in Definition 2.1. Next, in Lemma 2.2, we demonstrate a lower bound on the expected number of k-WIS if the approximation ratio of Ranking is at most $1/2+c$. Further, in Lemma 2.5, we prove an upper bound on the number of k-WIS. Finally, in Theorem 2.6, we combine these bounds to show a lower bound for the constant c.

Definition 2.1 (k-wasteful independent set (k-WIS)). *Given matching R the output of* Ranking*, a subset of $2k$ vertices I is a k-wasteful independent set, iff vertices in I are end-points of k length-three augmenting paths in $R \oplus OPT$. This also implies that I is an independent set of G since* Ranking *outputs a maximal matching and vertices in I are left unmatched in R. Figure 1 shows an example of a 5-*WIS*.*

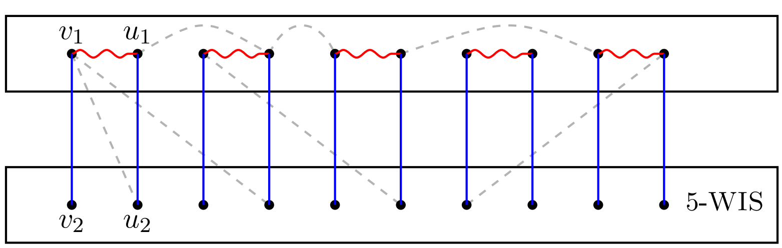

Figure 1: The red edges are from matching R outputted by Ranking and the blue ones are from OPT. The lower vertices form a 5-WIS as they are the endpoints of five length-three augmenting.

Lemma 2.2 (Lower Bound on the Expected Number of k-WIS). *If the approximation ratio of* Ranking *is at most $1/2+c$ for $0 \leq c \leq 1/6$, then the expected number of k-WIS in the output of* Ranking *is at least one, when $k = (1/2-3c)\cdot n/2$.*

Proof. First, if the size of the matching produced by Ranking for some permutation is $(1/2+c)\cdot n/2$, then there exist at least $(1/2-3c)\cdot n/2$ disjoint length-three augmenting paths by Proposition 1.1. Let $\mu_1, \ldots, \mu_{n!}$ be the sizes of the matchings produced by Ranking for all different permutations such that $\mu_i = (1/2+c_i)\cdot n/2$ for some $c_i \geq 0$ (such c_i always exists since the output of Ranking is a maximal matching). For the ith permutation, there exist at least $(1/2-3c_i)\cdot n/2$ length-three augmenting paths. Therefore, the total number of length-three augmenting paths in all permutations, denoted by s, is at least

$$s = \sum_{i=1}^{n!} (\frac{1}{2} - 3c_i) \cdot \frac{n}{2} = \frac{n \cdot n!}{4} - \frac{3n}{2} \sum_{i=1}^{n!} c_i \geq (n!) \cdot \left(\frac{n}{4} - \frac{3nc}{2} \right),$$

where the last inequality is followed by the fact that the approximation ratio of the algorithm is at least $1/2+c$. Let a_i be the number of length-three augmenting paths in the ith permutation. There exist at least $\binom{a_i}{k}$ number of k-WIS in the ith permutation. It follows that the total number of k-WIS is $\sum_{i=1}^{n!} \binom{a_i}{k}$. Moreover, $\sum_{i=1}^{n!} \binom{a_i}{k}$ is minimized when all a_i's are equal due to Proposition 1.2. Therefore, the total number of k-WIS is at least

$$\sum_{i=1}^{n!} \binom{a_i}{k} \geq n! \cdot \binom{s/n!}{k} = n! \cdot \binom{n/4 - 3nc/2}{n/4 - 3nc/2} = n!,$$

which implies that the expected number of k-WIS is at least one. □

Claim 2.3. *There exist at most $\binom{n/2}{2k} \cdot 3^k$ different k-WIS for all possible outputs of* Ranking *on input graph G.*

Proof. First, we select $2k$ edges from OPT that will create k length-three augmenting paths in Definition 2.1. This selection has $\binom{n/2}{2k}$ possible combinations. To determine which endpoints of these edges will form I (the $k-$WIS) in Definition 2.1, there are 2^{2k} possible choices. We further show that this is at most 3^k. Consider the collection of k length-three alternating paths denoted by L. Suppose there is an augmenting path (v_2, v_1, u_1, u_2) in L (for instance, see Figure 1). In this case, we can rule out the possibility of both v_1 and u_1 being present in I simultaneously, since there is an edge between v_1 and u_1, and I must be an independent set. Now apply this process to all k augmenting paths in L. For each pair of edges in OPT that appear in the same length-three augmenting path in L, there are 3 possible choices for which endpoint can be included in I, since the endpoints of the middle edge in the augmenting path cannot both be in I. Consequently, the total number of different possible k-WIS is at most $\binom{n/2}{2k} \cdot 3^k$. □

Claim 2.4. *Let $I = \{v_1, v_2, \ldots, v_{2k}\}$ be a set of independent vertices in G. Then, the vertices in I form a k-WIS with probability at most $1/2^{2k}$ in the output of* Ranking.

Proof. Let E_I denote the set of edges in OPT that have at least one endpoint in I. Since OPT is a perfect matching and I is an independent set, we have $|E_I| = 2k$, and E_I covers all vertices in I. Let $V(E_I)$ be the set of endpoints of E_I and $\bar{I} = V(E_I) \setminus I$. For the edges in E_I to form length-three augmenting paths, they must be partitioned into pairs, with each pair creating one length-three augmenting path.

Let (v_1, v_2) and (u_1, u_2) be such a pair with v_2 and u_2 in I (for instance, see Figure 1). Then there must exist an edge between v_1 and u_1 to form the augmenting path. We claim that $\pi(v_1) < \pi(u_2)$ if π is the permutation chosen by Ranking. To prove this by contradiction, assume that $\pi(v_1) > \pi(u_2)$.

When u_2 is processed by Ranking, since it is wasted (unmatched in R), it must be available and have no free neighbor. In particular, this means u_1 is matched and has a match with rank less than $\pi(u_2)$. However, this implies that $\pi(\text{match of } u_1) < \pi(u_2) < \pi(v_1)$, contradicting the fact that u_1 is matched with v_1 in R. Thus, we have $\pi(v_1) < \pi(u_2)$. We call u_2 in this case the *counterpart* of v_1 and let $C(v_1)$ denote the counterpart of v_1. For each vertex in $\bar{I}$, its counterpart is a unique vertex in I. Hence, the set of all counterparts of vertices in $\bar{I}$ is I. Using the same argument, we can show that for all vertices $v \in \bar{I}, \pi(v) < \pi(C(v))$.

If we consider a permutation of the vertices in $I \cup \bar{I}$, it must satisfy the property described in the above argument for I to be a k-WIS. In the remainder of the proof, we will focus on calculating the probability of this property being satisfied. Let σ be a permutation over vertices in $\bar{I}$. In each iteration, we add a new vertex from I to the permutation such that it does not violate the property. Let u be the last vertex in σ. In the permutation of vertices in I and $\bar{I}$, $C(u)$ can appear only after u, and since u is the last element in $\bar{I}$, the relative place of $C(u)$ with respect to σ is only after its last element. For $C(u)$ to be after all elements in σ, which has length $2k$, it has a probability of $1/(2k+1)$ since the permutation that Ranking chooses is uniformly at random.

Let u_i be the ith vertex from the end in σ. When we add the counterpart of u_i, there are $2i-1$ positions (from the end of the permutation to the one right after u_i[3]) among the available $(2k+i)$ positions where we can place it. Therefore,

$$\begin{aligned}\Pr[\ I \text{ is a } k\text{-WIS in output of Ranking }] \leq& \prod_{i=1}^{2k} \frac{2i-1}{2k+i} \\ =& \frac{(4k)!/(\prod_{i=1}^{2k} 2i)}{\prod_{i=1}^{2k}(2k+i)} \\ =& \frac{(4k)!/((2k)! \cdot 2^{2k})}{(4k)!/(2k)!} = \frac{1}{2^{2k}}.\end{aligned}$$

□

Lemma 2.5 (Upper Bound on the Expected Number of k-WIS)**.** *The expected number of* k-WIS *in the output of* Ranking *is at most* $\binom{n/2}{2k} \cdot (3/4)^k$.

Proof. By Claim 2.3, there exist at most $\binom{n/2}{2k} \cdot 3^k$ different k-WIS. Also, each of them has a probability $1/(2^{2k})$ to be in the output of Ranking by Claim 2.4. Thus, by the linearity of the expectation, the expected number of k-WIS in the output of Ranking is at most $\binom{n/2}{2k} \cdot (3/4)^k$. □

Theorem 2.6. *The expected approximation ratio of* Ranking *is at least* 0.505 *in general graphs.*

Proof. Suppose that the approximation ratio of Ranking is at most $1/2 + c$ for $c \leq 0.005$. Let $k = (1/2 - 3c) \cdot n/2$. By Lemma 2.2, the expected number of k-WIS is at least one. On the other hand, by Lemma 2.5, the expected number of k-WIS is at most $\binom{n/2}{2k} \cdot (3/4)^k$. Hence, it must hold that $\binom{n/2}{2k} \cdot (3/4)^k \geq 1$. Now, we show that this cannot happen for $c \leq 0.005$. In particular, we have

$$\binom{n/2}{2k} \cdot \left(\frac{3}{4}\right)^k$$

[3] As the counterparts of the last $i-1$ vertices are all placed after u_i, there are, in total, $2(i-1)+1 = 2i-1$ slots where we can put $C(u_i)$ to make $\pi(u_i) < \pi(C(u_i))$ hold.

$$
\begin{aligned}
&= \binom{n/2}{(1-6c)\cdot n/2} \cdot \left(\frac{3}{4}\right)^{(1/2-3c)\cdot n/2} && \text{(Since } k = (1/2-3c)\cdot n/2\text{)} \\
&= \binom{n/2}{6c \cdot n/2} \cdot \left(\frac{3}{4}\right)^{(1/2-3c)\cdot n/2} && \text{(By the identity of binomial coefficients)} \\
&\le 2^{(n/2)\cdot H(6c)} \cdot \left(\frac{3}{4}\right)^{(1/2-3c)\cdot n/2} && \text{(By Proposition 1.3)} \\
&= 2^{(n/2)\cdot H(6c)} \cdot 2^{(n/2)\cdot \log_2(\frac{3}{4})\cdot(1/2-3c)} \\
&= 2^{(n/2)\cdot [H(6c)+\log_2(\frac{3}{4})\cdot(1/2-3c)]} \\
&< 1 && \text{(When } c \le 0.005,\ H(6c) + \log_2(\frac{3}{4})\cdot(1/2-3c) < 0\text{)},
\end{aligned}
$$

which is a contradiction. Therefore, c cannot be smaller than 0.005 as otherwise, we obtain $\binom{n/2}{2k} \cdot (3/4)^k < 1$ which completes the proof. □

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