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The number of odd spanning trees in the complete graphs ☆

Yong-De Feng *, Yawen Chen, Baoyindureng Wu

College of Mathematics and Systems Science, Xinjiang University, Urumqi, Xinjiang 830017, China

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ABSTRACT

An odd graph is a graph G for which every vertex $v \in V(G)$ satisfies $d_G(v) \equiv 1 \pmod 2$. An odd spanning tree T of G is a spanning tree such that $d_T(v) \equiv 1 \pmod 2$ for all $v \in V(T)$. It is known that for any complete graph K_n of even order has an odd spanning tree. In this paper, we establish the exact number of labeled odd spanning trees in K_n. By employing the classical Prüfer sequence and constructing the corresponding generating function, we prove that the number of labeled odd spanning trees in K_n is given by $\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(2k-n)^{n-2}$ (where n is even).

1. Introduction

In this paper, all graphs under consideration are assumed to be simple. The degree of a vertex v in a graph G, denoted by $d_G(v)$, is the number of edges of G incident with v. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices of G. For any graph G, we denote the numbers of vertices and edges in G by $v(G)$ and $e(G)$; these two basic parameters are called the order and size of G, respectively. Let K_n denote the complete graph on n vertices. For terminology and notations not explicitly defined here, readers are referred to the standard literature [2,3,9].

A graph is connected if, for every partition of its vertex set into two nonempty sets X and Y, there is an edge with one end in X and one end in Y. A spanning subgraph of a graph G is a subgraph obtained by edge deletions only. A connected acyclic graph is called a tree. Moreover, a subtree of a graph is a subgraph which is a tree. Finally, if this tree is a spanning subgraph, it is called a spanning tree of the graph. Building upon foundational concepts, Albertson, Berman, Hutchinson, and Thomassen [1] (1990) studied homeomorphically irreducible spanning tree (HIST) — spanning tree T of a graph G containing no degree-two vertices (i.e., $d_T(v) \neq 2$ for all $v \in V(T)$) [1,4,7]. Subsequently, Wu et al. [12] introduced the notation of odd spanning tree which is a spanning tree T of G that every vertex $v \in V(G)$ satisfies $d_T(v) \equiv 1 \pmod 2$ in 2025. Trivially, an odd spanning tree of G must be a HIST. For even order graphs, Wu et al. established the following fundamental result.

Theorem 1.1 (*[12]*)**.** *Let n be a positive even number. If G is a connected graph of order n with $\delta(G) \geq \frac{n}{2}+1$, then G has an odd spanning tree.*

Every complete graph of even order K_n has at least one odd spanning tree by Theorem 1.1. There are many studies on the number of spanning trees [5,6,8,10]. A natural question is to investigate the exact number of labeled odd spanning trees in K_n. We derive an explicit enumeration formula for odd spanning trees in K_n. In order to show this result, we first present the following fundamental concepts and formula.

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* Corresponding author.
E-mail address: ydfengxj@163.com (Y.-D. Feng).

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Theorem 1.2 ([3] Cayley's Formula)**.** *The number of labeled trees on n vertices is* n^{n-2}.

There is a bijection between labeled trees on n vertices and labeled spanning trees of the complete graph K_n. This formula can be established through several proof methods, among which the bijective argument using Prüfer sequences provides a particularly elegant combinatorial demonstration.

Prüfer sequences, is the sequences studied by Heinz Prüfer (1918), furnish a bijection that compute Cayley's Formula for labeled trees on n vertices. Let $[n] := \{1, 2, 3, \dots, n\}$.

Definition 1.3 ([11])**.** Let T be a tree on $[n]$, with $n \geq 3$. Cut off the leaf of T that has the smallest label, and write down its single neighbor. Then continue this same procedure on the remaining tree until there are only two vertices (and one edge) left. This procedure results in a sequence of elements of $[n]$ that has length $n-2$, called the Prüfer sequence, or Prüfer code of T.

Example 1.4. T_1 and T_2 (see Fig. 1) are the odd spanning trees of K_6, their Prüfer sequences are

$$P(T_1) = (3, 3, 3, 3), P(T_2) = (3, 4, 3, 4).$$

The Prüfer sequence possesses the following property.

Property 1.5 ([11])**.**
1. Bijection: There exists a one-to-one correspondence between the set of labeled trees with n vertices and the set of Prüfer sequences of length $n-2$ *over* $[n]$.
2. Vertex Degree: The number of times a vertex $i \in [n]$ *appears in the Prüfer sequence equals* $d_T(i) - 1$.

By exploiting the bijective correspondence between Prüfer sequences and labeled spanning trees, we obtain the following Theorem 1.6.

Theorem 1.6 ([11] Cayley's Formula via Prüfer Sequences)**.** *The Prüfer sequence establishes an explicit bijection between the set of labeled trees on n vertices and the set of all sequences of length* $n-2$ *with entries from* $[n]$*, thereby proving that the number of such trees is* n^{n-2}.

Inspired by the proof technique of Cayley's Formula, we address the problem of enumerating odd spanning trees in complete graphs, as introduced earlier in this paper. Our approach leverages the classical Prüfer sequence method and constructs an associated generating function. For an arbitrary graph G, let $\tau_o(G)$ denote the number of odd spanning trees of G. Our main result is stated as follows.

Theorem 1.7. *Let n be a positive number. Then*

$$\tau_o(K_n) = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (2k-n)^{n-2} & \text{if } n \text{ is even.} \end{cases}$$

In this section, we introduce some fundamental concepts. We prove our main result (Theorem 1.7) in next section. Finally in the last section, we provide a related question.

2. Proof of Theorem 1.7

By the Handshaking Lemma, the number of odd degree vertices for any connected graph is even. Therefore, every connected graph with odd order has no odd spanning subgraphs, we obtain Remark 2.1.

Remark 2.1. If G is a connected graph with odd order, then $\tau_o(G) = 0$.

In what follows, we only consider graphs of even order. We shall introduce several essential concepts and auxiliary results that form the foundation of our argument before we give the proof of Theorem 1.7.

Definition 2.2 ([9])**.** Given a sequence a_n $(n \geq 0)$, its generating function $G(x)$ is defined as

$$G(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where x is a formal variable (usually its convergence is not of concern and is only discussed within the context of formal power series), and the coefficient a_n can be an integer, a real number, a complex number, or even a more general mathematical object.

Definition 2.3 ([9]). Let x be an abstract symbol, and $a_n(n = 0, 1, 2, \dots)$ be any real number sequence. The formal power series

$$g(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots + a_n \frac{x^n}{n!} + \cdots$$

is called the exponential generating function of the sequence $a_n(n = 0, 1, 2, \dots)$.

Next, we introduce the notion of multisets.

Theorem 2.4 ([9]). *Let the multiset $S = \{M_1a_1, M_2a_2, \dots, M_ka_k\}$, where the multiplicities $M_1, M_2, \dots, M_k$ are all positive integers or ∞. Let h_r denote the number of r-permutations of S. Then, the exponential generating function for h_r $(r = 0, 1, 2, \dots)$, with $h_0 = 1$, is given by:*

$$g(x) = f_{M_1}(x) f_{M_2}(x) \cdots f_{M_k}(x),$$

where for each $i = 1, 2, \dots, k$,

$$f_{M_i}(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{M_i}}{M_i!}.$$

Theorem 2.5 ([9]). *Suppose r distinct balls are to be distributed into k distinct boxes $a_1, a_2, \dots, a_k$, where the capacity of each box a_i is constrained by a set N_i $(1 \le i \le k)$. Then, the exponential generating function for the number of possible distribution arrangements is given by:*

$$g(x) = \prod_{i=1}^{k} \left(\sum_{m \in N_i} \frac{x^m}{m!} \right).$$

By Property 1.5, we obtain the following lemma.

Lemma 2.6 (*Properties of Prüfer sequences for odd spanning Trees*). *Let T be an odd spanning tree of the complete graph K_n (where n is even and any vertex $i \in [n]$). Let $P(T) = (p_1, p_2, \dots, p_{n-2})$ be the Prüfer sequence of T. Then the following properties hold:*

1. Even Times: *For any vertex $i \in [n]$, its multiplicity in $P(T)$ equals $d_i - 1$, where d_i is the degree of i in T. Since d_i is odd, $d_i - 1$ is even. Thus, every labeled vertex in $P(T)$ appears an even number of times.*

2. Sequence Length: *The length of $P(T)$ is $n - 2$.*

Within the framework of the aforementioned definitions and supported by Lemma 2.6, we now give the proof of our main result.

Theorem 1.7. *Let n be a positive number. Then*

$$\tau_o(K_n) = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (2k - n)^{n-2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. By Lemma 2.6 (the bijection between Prüfer sequences and odd spanning trees), the problem reduces to counting the number of Prüfer sequences of length $n - 2$ in which each vertex i appears an even number of times (some vertices may occur zero times). To facilitate this enumeration, we construct the corresponding generating function. Let c_i denote the number of occurrences of vertex i in the Prüfer sequence. Then c_i satisfies

$$\sum_{i=1}^{n} c_i = n - 2.$$

We have $c_i = d_i - 1$ by Lemma 2.6. Since d_i is odd in an odd spanning tree, c_i is even. To properly handle the enumeration problem, we employ exponential generating functions. Moreover, since each count c_i must be even, we specifically utilize hyperbolic cosine functions (a special class of exponential generating functions) for enumeration

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots, \tag{1}$$

then from Theorem 2.5, the generating function for n vertices is given by

$$G(x_1, \dots, x_n) = \prod_{i=1}^{n} \cosh(x_i) = \prod_{i=1}^{n} \left(\sum_{k=0}^{\infty} \frac{x_i^{2k}}{(2k)!} \right). \tag{2}$$

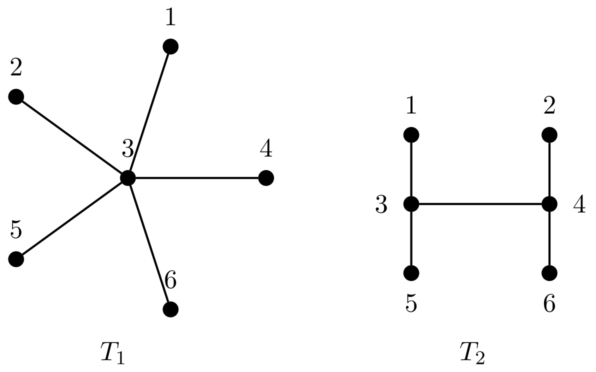

Fig. 1. The odd spanning trees of K_6.

Our goal is to calculate the number of odd spanning trees. Because all the labeled vertex in the complete graph are symmetrical, let $x_i = x$. So we first need to extract the coefficient of x^{n-2} from Eq. (2). By combining this with the properties of exponential generating functions, we obtain

$$\tau_o(K_n) = (n-2)! \left[\cosh^n(x)\right]_{\sum_{i=1}^{n} c_i = n-2}. \tag{3}$$

Moreover

$$G(x) = G(x_1, \ldots, x_n) = \cosh^n(x) = \left(\frac{e^x + e^{-x}}{2}\right)^n, \tag{4}$$

expand $(e^x + e^{-x})^n$ through binomial:

$$(e^x + e^{-x})^n = \sum_{k=0}^{n} \binom{n}{k} e^{kx} \cdot e^{-(n-k)x} = \sum_{k=0}^{n} \binom{n}{k} e^{(2k-n)x}. \tag{5}$$

Therefore:

$$G(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{(2k-n)x}, \tag{6}$$

expand $e^{(2k-n)x}$ into a Taylor series:

$$e^{(2k-n)x} = \sum_{m=0}^{\infty} \frac{(2k-n)^m x^m}{m!}. \tag{7}$$

From (4), (5), (6) and (7), we obtain

$$G(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \sum_{m=0}^{\infty} \frac{(2k-n)^m x^m}{m!}. \tag{8}$$

Next, we obtain the final number of labeled odd spanning trees, where $[x^{n-2}]G(x)$ denotes the coefficient of x^{n-2},

$$\tau_o(K_n) = (n-2)! \cdot [x^{n-2}]G(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (2k-n)^{n-2}. \quad \square \tag{9}$$

Next, we provide simple examples for Theorem 1.7. It is easy to verify for the cases $n = 2$ and $n = 4$. When $n = 6$, there are exactly two non-isomorphic unlabeled odd spanning trees. Specifically, for the complete graph K_6, all non-isomorphic odd spanning trees have the following structures:

1. Star $K_{1,5}$: A central vertex (degree 5) connected to 5 leaves (see T_1 in Fig. 1).
2. Double-star: Two adjacent central vertices (both of degree 3), each connected to 2 leaves (see T_2 in Fig. 1).

We can enumerate the labeled trees on 6 vertices by considering two types. There are 6 labeled odd spanning trees which are isomorphic to T_1. There are 90 labeled odd spanning trees which are isomorphic to T_2.

Moreover, applying Theorem 1.7 yields:

$n = 6$:

$$\tau_o(K_6) = \frac{1}{64} \sum_{k=0}^{6} \binom{6}{k} (2k-6)^4 = \frac{6144}{64} = 96 \quad .$$

Clearly, this agrees with the result obtained by direct computation.

3. Concluding remarks

In this paper, we investigate the enumeration and provide a rigorous proof of the counting formula for odd spanning trees in complete graphs K_n. Our approach employs the classical Prüfer sequence method to construct the corresponding generating function, leading to an explicit expression for the number of labeled odd spanning trees (Theorem 1.7).

This paper investigates the enumeration of odd spanning trees with labeled vertices. Subsequently, we will further consider the counting problem of odd spanning trees with unlabeled vertices.

Question 3.1. *Up to isomorphism, how many odd spanning trees in complete graphs K_n?*

Data availability

No data was used for the research described in the article.

References

[1] M.O. Albertson, D.M. Berman, J.P. Hutchinson, C. Thomassen, Graphs with homeomorphically irreducible spanning trees, J. Graph Theory 14 (1990) 247–258.
[2] N. Biggs, Algebraic Graph Theory, vol. 67, Cambridge University Press, 1993.
[3] J.A. Bondy, U.S.R. Murty, Graph theory, in: Graduate Texts in Math, Springer, 2008.
[4] G. Chen, S. Shan, Homeomorphically irreducible spanning trees, J. Combin. Theory Ser. B 103 (2013) 409–414.
[5] F. Dong, J. Ge, Counting spanning trees in a complete bipartite graph which contain a given spanning forest, J. Graph Theory 101 (2022) 79–94.
[6] H. Gong, Y. Gong, J. Ge, The number of spanning trees in K_n-complement of a bipartite graph, J. Algebraic Combin. 60 (2024) 541–554.
[7] F. Harary, G. Prins, The number of homeomorphically irreducible trees, and other species, Acta Mathematica 101 (1–2) (1959) 141–162.
[8] J. Li, W. Chee Shiu, A. Chang, The number of spanning trees of a graph, Appl. Math. Lett. 3 (2010) 286–290.
[9] R. Merris, Combinatorics, John Wiley & Sons, Inc, Hoboken, New Jersey, 2003.
[10] J.W. Moon, Counting Labelled Trees, Canadian Mathematical Congress, 1970.
[11] H. Prüfer, Neuer beweis eines satzes uber permutationen, Arch. Math. Phys. 8 (1918) 742–744.
[12] J. Zheng, B. Wu, Odd spanning trees of a graph, 2025, arXiv 2503.17676.