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Note

Kempe equivalence of 4-colorings of graphs on non-orientable surfaces

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ABSTRACT

Two vertex colorings of a graph are Kempe equivalent if they can be transformed into each other by a sequence of Kempe changes which interchange the colors used on a component of the subgraph induced by two color classes. Fisk showed that every two vertex 4-colorings of a 3-colorable triangulation on the sphere are Kempe equivalent, and then Mohar extended this result to any 3-colorable planar graph. Fisk also verified that there are 4-chromatic triangulations on the sphere and 3-colorable triangulations on the torus such that some two 4-colorings of them are not Kempe equivalent. In this paper, we show that every two vertex 4-colorings of a 3-colorable triangulation on the projective plane or the Klein bottle are Kempe equivalent. Our result is best possible in terms of all conditions; 3-colorability, the genus of a non-orientable surface, a triangulation (i.e., it cannot be replaced with a graph).

1. Introduction

All graphs considered in this paper are finite, undirected and simple unless we particularly mention it. A *Kempe change* was essentially introduced by Alfred Kempe to try proving the four color theorem. Each Kempe change interchanges the colors used on a component of the subgraph induced by two color classes. Even today, this remains one of the fundamental and most powerful tools in graph coloring theory. Two colorings are *Kempe equivalent* if they are transformed into each other by a sequence of Kempe changes. Kempe equivalence of vertex colorings of graphs is deeply and widely studied, since it is not only an important research subject in combinatorial reconfiguration but also is related to many other subjects in graph theory; for example, random coloring [9], perfectly contractile graph [1], a recoloring version of Hadwiger's conjecture [3,6] and so on. It is easy to see that every two k-colorings of any bipartite graph with $k \geq 2$ are Kempe equivalent (cf. [7]). On the other hand, it is PSPACE-complete to determine whether two given k-colorings of a graph G are Kempe equivalent for any fixed $k \geq 3$ [2], where a *k-coloring* is a proper vertex coloring with k colors.

In this paper, we consider Kempe equivalence of 4-colorings of graphs on surfaces. Fisk [5] showed that every two 4-colorings of a 3-colorable triangulation on the sphere are Kempe equivalent. Mohar [7] extended this result to any 3-colorable planar graph. Moreover, Fisk observed that:

(a) There exists a 4-chromatic triangulation G on the sphere with two 4-colorings which are not Kempe equivalent.

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(b) There exists a 3-colorable triangulation T_0 on the torus (shown in Fig. 1) with two 4-colorings which are not Kempe equivalent.

By the above results, Kempe equivalence of 4-colorings of graphs on *orientable* surfaces is solved with respect to its genus. To deal with Kempe equivalence of 4-colorings of graphs on orientable surfaces, Fisk [5] used a "degree" of 4-colorings, which is defined by regarding a 4-coloring of a graph G on a surface with a simplicial map from G to the surface of a tetrahedron (which can also be considered as a triangulation on the sphere). Mohar and Salas [8] further investigated the relation between "degree" and Kempe equivalence of 4-colorings of 3-colorable triangulations on orientable surfaces. However, since the "degree" needs the orientability of a surface, it cannot be applied to Kempe equivalence of 4-colorings of graphs on *non-orientable* surfaces. Hence we consider the similar problem for non-orientable surfaces, and then we have the following. The projective planar case is already proved in [5] by using some topological fact, but Fisk did not precisely mention it.

Theorem 1.1. *Let G be a 3-colorable triangulation on the projective plane or the Klein bottle. Then every two 4-colorings of G are Kempe equivalent.*

Theorem 1.1 is best possible in terms of each condition:

(c) A "triangulation" cannot be replaced with a "graph",
(d) the non-orientable genus cannot be extended to 3 or more, and
(e) the 3-colorability is necessary.

The claim (e) immediately follows from (a). The others (c) and (d) follow from (i) and (ii) of the following theorem, respectively.

Theorem 1.2.

(i) *There exists a 3-colorable graph G on the projective plane with two 4-colorings which are not Kempe equivalent.*
(ii) *There exists a 3-colorable triangulation G on the non-orientable surface with genus 3 with two 4-colorings which are not Kempe equivalent.*

By our results together with known results for the orientable case, Kempe equivalence of 4-colorings of graphs on closed surfaces is solved with respect to its genus.

2. Definitions

For fundamental terminology and notation undefined in this paper, we refer the reader to [4]. In this section, we introduce some terminology and notation that are frequently used in this paper but are not entirely standard.

For a graph G and a subset $S \subseteq V(G)$, $G[S]$ denotes the subgraph induced by S. For a vertex colored graph G, $G(i, j)$ denotes the subgraph induced by all vertices colored with i or j. Every connected component D of $G(i, j)$ is called a *Kempe component* (or K-*component*), and such a component D is also called an (i, j)-*component*. An edge $e \in E(G(i, j))$ is called an (i, j)-*edge*. When we simultaneously consider 3- and 4-colorings of G, an (i, j)-edge of G is defined with respect to the 4-coloring. By switching the colors i and j on D, a new k-coloring can be obtained. This operation is called a *Kempe change* (or K-*change*). In particular, we call a K-change on a component of $G(i, j)$ an (i, j)-*change*. If a K-change is applied on a component containing a vertex v, then such a K-change is also called a K-change *concerning* v. Two k-colorings c_1 and c_2 of a graph G are *Kempe equivalent* (or K-*equivalent*) without a $(k+1)$-th extra color, if c_1 can be obtained from c_2 by a sequence of Kempe changes, possibly involving more than one pair of colors in successive Kempe changes.

A *surface* is a two-dimensional closed manifold. An *embedding* of a graph G on a surface $\mathbb{F}$ is a drawing of G on $\mathbb{F}$ with no pair of crossing edges. In this paper, we simply say an embedding of G on $\mathbb{F}$ as G *on* $\mathbb{F}$. A cycle C of a graph (or a closed curve) on $\mathbb{F}$ is 1-*sided* (resp. 2-*sided*) if the regular neighborhood of C is homeomorphic to a Möbius band (resp. an annulus). In addition, C is *separating* (resp. *non-separating*) if C separates (resp. does not separate) $\mathbb{F}$.

A *face* f of G on a surface $\mathbb{F}$ is a connected component of $\mathbb{F} - G$ and a *region* of G is a union of several faces. For a vertex v of G on $\mathbb{F}$, the *link* of v is the boundary cycle (or walk) of the region formed by all faces incident to v. A *facial path* $F_0F_1 \dots F_k$ of a triangulation G on a surface is a sequence of faces F_i of G such that for any $i \in \{0, 1, \dots, k-1\}$, F_i and F_{i+1} share exactly one edge if they are distinct (note that a facial path may traverse the same face more than once). We say that the facial path *passes* an edge shared by F_i and F_{i+1} if F_i and F_{i+1} are distinct. Similarly, a facial path $\mathcal{P}$ passes a path or a cycle of G if $\mathcal{P}$ passes an edge of it.

3. Proof of Theorem 1.1

Prior to the proof of the theorem, we present several facts and lemmas. Let T_0 be the 12-vertex 6-regular triangulation on the torus shown in Fig. 1. In the figure, each label ij ($i \in \{1,2,3,4\}$ and $j \in \{a,b,c\}$) denotes a combination of two colors defined by a 4-coloring $f : V(T_0) \to \{1,2,3,4\}$ and a (unique) 3-coloring $g : V(T_0) \to \{a,b,c\}$.

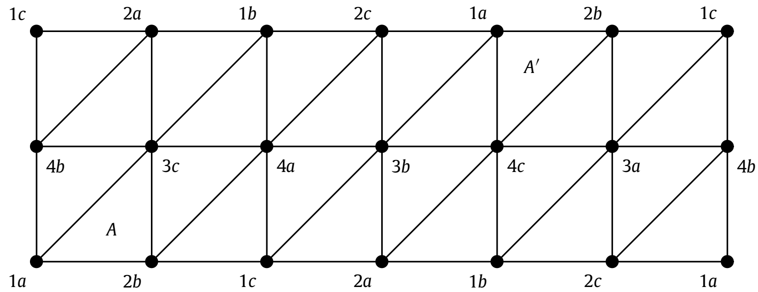

Fig. 1. The 6-regular triangulation T_0 and two faces A and A'.

Observation 3.1. The vertex set of T_0 is the product of the set of four colors $\{1,2,3,4\}$ and that of three colors $\{a,b,c\}$. The face set of T_0 consists of all 24 tuples (i_1j_1, i_2j_2, i_3j_3) of three elements in $\{ij : i \in \{1,2,3,4\}, j \in \{a,b,c\}\}$ such that $i_s \neq i_t$ and $j_s \neq j_t$ for any $s,t \in \{1,2,3\}$ with $s \neq t$.

Lemma 3.2 *([5,8]). If a triangulation T on a surface has a 4-coloring f and a 3-coloring g, then there is a simplicial map $g \times f : T \to T_0$ such that $(g \times f)(x) = f(x)g(x) \in V(T_0)$ for each vertex $x \in V(T)$.*

Lemma 3.3. *Let A be an annulus and let x and y be two points of A. Let J and J' be two lines connecting distinct boundaries of A and suppose that they separate x and y (that is, any simple curve from x to y crosses J or J'). Then, for any simple curve γ between x and y, either*

- *γ crosses J an even number of times and J' an odd number of times, respectively, or*
- *γ crosses J an odd number of times and J' an even number of times, respectively.*

Proof. Since x and y respectively belong to distinct two regions R and R' bounded by J, J' and boundaries of A, γ has to pass J and J' alternately starting from one of x and y. Thus, the total number of crossings of γ and (J, J') is odd, which implies the lemma. □

The following is a key lemma for proving our main theorem.

Lemma 3.4. *Let $A = (1a, 2b, 3c)$ and $A' = (1a, 2b, 4c)$ be two triangular faces of T_0 (see Fig. 1). Let $\mathcal{P} = F_0F_1 \ldots F_k$ where $F_0 = A$ and $F_k = A'$ be a facial path of T_0 which passes no $(1,2)$-edge. Then there are two pairs (r,s) and (r',s') such that $\mathcal{P}$ passes both (r,s)-edges and (r',s')-edges an even number of times.*

Proof. Since $\mathcal{P}$ passes no $(1,2)$-edge, we can regard $\mathcal{P}$ as a facial path of a triangulation on the annulus. The faces A and A' are separated by sets of (i,j)-paths corresponding to J and J' in Lemma 3.3 in the following two kinds:

(1) J_1: two $(2,3)$-paths $2a3c2b, 2c3b2a$ and two $(1,4)$-paths $1b4a1c, 1a4c1b$, J_1': a $(2,3)$-path $2b3a2c$ and a $(1,4)$-path $1c4b1a$.
(2) J_2: a $(2,4)$-path $2c4a2b$ and a $(1,3)$-path $1a3b1c$, J_2': two $(2,4)$-paths $2b4c2a, 2a4b2c$ and two $(1,3)$-paths $1c3a1b, 1b3c1a$.

Note that γ passing J or J' in Lemma 3.3 an odd (resp. even) number of times corresponds to $\mathcal{P}$ passing any path of J_i or J_i' odd (resp. even) times for each $i \in \{1,2\}$. By Lemma 3.3, for each $i \in \{1,2\}$, if $\mathcal{P}$ crosses J_i is an even (resp. odd) number of times, then $\mathcal{P}$ crosses J_i' is an odd (resp. even) number of times. Moreover, the parity of the number of intersections between $\mathcal{P}$ and J_i is the same as one between $\mathcal{P}$ and J_j where $\{i,j\} = \{1,2\}$. Therefore, one of $((2,3),(1,4))$ and $((2,4),(1,3))$ is a set of desired color pairs. □

We have prepared to prove Theorem 1.1.

Proof of Theorem 1.1. Let G be a 3-colorable triangulation on the projective plane or the Klein bottle. It suffices to show that any 4-coloring of G can be transformed into a 3-coloring, since G has a unique 3-coloring up to permutation of the colors (that is, all 3-colorings of G are Kempe equivalent).

Suppose G has a 4-coloring $f : V(G) \to \{1, 2, 3, 4\}$. Let e be an edge of G and let F_e and F'_e be two triangular faces sharing e. For a face p of G, we denote by $C(p)$ the set of colors used on the vertices of p. An edge e of G is *singular* (resp. *non-singular*) if $C(F_e) = C(F'_e)$ (resp. $C(F_e) \neq C(F'_e)$). For a coloring g of G, let $NS(g)$ be the set of non-singular edges of G by the coloring g. Note that if g is a 3-coloring, $NS(g) = \emptyset$, and hence we will make $NS(f) = \emptyset$ by a sequence of K-changes.

We suppose that K-changes are already applied to f to reduce $|NS(f)|$ as small as possible. We suppose to the contrary that $NS(f) \neq \emptyset$, and without loss of generality, we may assume that $NS(f)$ contains at least one $(1, 2)$-edge. Let H be the subgraph of $G[NS(f)]$ consisting of all $(1, 2)$-edges.

Claim 3.5. *The degree of any vertex in H is even.*

Proof. For any vertex v of G, it is known in [5, Lemma 2] that the length of the link of v in G is the same as the number of non-singular vertices on the link modulo 2, where a vertex of a 3-colored cycle is *non-singular* if the two neighbors of it on the cycle have distinct colors. Note that the length of the link of any vertex is even since G is Eulerian by the 3-colorability of G. Moreover, the parities of the number of non-singular vertices of each color are all the same [5, Lemma 2]. Therefore, since the number of non-singular $(1, 2)$-edges incident to v is even, the claim holds. □

Claim 3.6. *Every cycle in H is non-separating.*

Proof. If a cycle in H is separating, then by applying K-changes to all $(3, 4)$-components in one region bounded by the cycle, we can make all edges of the cycle be singular. This contradicts that $|NS(f)|$ is minimum. □

Claim 3.7. *Let C be a 1-sided cycle in H. At least one edge e in $E(H) \setminus E(C)$ is incident to C, that is, one end of e is in $V(C)$.*

Proof. Suppose to the contrary that no edge in $E(H) \setminus E(C)$ is incident to C. Let $C = x_0x_1 \dots x_{k-1}$. Let $u_{k-1}x_{k-1}x_0$, $u_kx_0x_1$ and $u_ix_ix_{i+1}$ be faces of G for each $i \in \{0, 1, 2, \dots, k-2\}$ such that for each $i \in \{1, 2, \dots, k\}$, there is a facial path $P_i = r^i_0 r^i_1 \dots r^i_{\ell_i}$ from $r^i_0 = u_ix_{i-1}x_i$ to $r^i_{\ell_i} = u_ix_ix_{i+1}$ (subscripts of x_i are modulo k). Note that the boundary of r^i_j contains x_i for each $j \in \{0, 1, \dots, \ell_i\}$ but P_i does not pass any edge of C (see Fig. 2). Let L_i be the path in the link of x_i between u_{i-1} and u_i such that all vertices of L_i are contained in boundary cycles of faces of P_i as shown in Fig. 2.

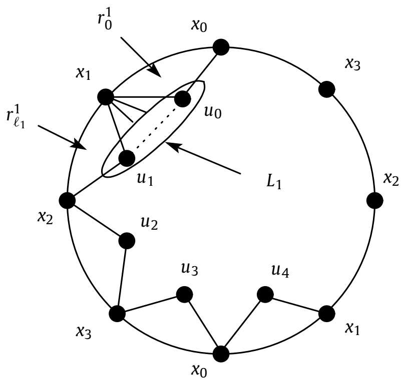

Fig. 2. A 1-sided cycle C with $k = 4$.

Observe that $f(u_i) \in \{3, 4\}$ for each $i \in \{0, 1, 2, \dots, k\}$. By the assumption that no edge in $E(H) \setminus E(C)$ is incident to C, the following two properties hold:

(i) If $f(u_{i-1}) = f(u_i)$, then $|V(L_i)|$ is odd.
(ii) If $f(u_{i-1}) \neq f(u_i)$, then $|V(L_i)|$ is even.

Here we give a proof to the above statement (i): Suppose $f(x_i) = 2$, $f(u_{i-1}) = f(u_i) = 4$. If L_i has no vertex with color 1, then only two colors 4 and 3 appear alternately on L_i, that is, $|V(L_i)|$ is odd since $f(u_{i-1}) = f(u_i)$. Thus we suppose L_i has a vertex with color 1. By the assumption of the claim, the neighbors of x_i with color 1 in L_i lie on the between two vertices in L_i with the same color. In this case, by regarding the color of all vertices with color 1 in L_i as color 3 or 4 suitably, only two colors 4 and 3 appear alternately on L_i. Therefore, $|V(L_i)|$ is odd in each case. (Since the proof of (ii) is similar to that of (i), we omit it.)

Moreover, since $f(u_0) \neq f(u_k)$ and the length of C is even, we have

(iii) The number of indices $i \in \{0, 1, \ldots, k-1\}$ satisfying (i) is odd.

Let $g : V(G) \to \{a, b, c\}$ be a 3-coloring of G such that $g(x_0) = a$, $g(x_1) = b$ and $g(u_0) = g(u_k) = c$. We consider the permutation of the three colors a, b, c of triangular faces $x_i x_{i+1} u_i$ in the following order

$$x_0 x_1 u_0 \to x_1 x_2 u_1 \to \cdots \to x_0 x_1 u_k.$$

For each $i \in \{0, 1, \ldots, k-1\}$, we let

$$\pi_i : (g(x_i), g(x_{i+1}), g(u_i)) \to (g(x_{i+1}), g(x_{i+2}), g(u_{i+1}))$$

be an element of the symmetric group S_3; note that $\pi_{k-2} : (g(x_{k-2}), g(x_{k-1}), g(u_{k-2})) \to (g(x_{k-1}), g(x_0), g(u_{k-1}))$ and $\pi_{k-1} : (g(x_{k-1}), g(x_0), g(u_{k-1})) \to (g(x_0), g(x_1), g(u_k))$. Since $(g(x_0), g(x_1), g(u_0)) = (g(x_0), g(x_1), g(u_k)) = (a, b, c)$, the composition of permutations

$$\Pi = \pi_{k-1} \circ \pi_{k-2} \circ \cdots \circ \pi_0$$

must be the identity map. By (i) and (ii), π_i is order 2 (resp. 3) if $|V(L_i)|$ is odd (resp. even) for $i \in \{0, 1, \ldots, k-1\}$. However, since the number of π_i's with order 2 is odd by (iii), Π is never the identity map, a contradiction. □

Remark 3.8. As a corollary of Claims 3.5–3.7, we are done for the projective plane: By Claim 3.6, every cycle in H is 1-sided when G is embedded on the projective plane. By Claims 3.5 and 3.7, there are two 1-sided cycles C_1 and C_2 in H which must cross, since every non-separating cycle on the projective plane is 1-sided. Therefore, we can find a separating cycle in H consisting of several edges of C_1 and C_2, contrary to Claim 3.6.

By Claims 3.6, 3.7 and Remark 3.8, we may assume that

- G is a 3-colorable triangulation on the Klein bottle, and
- there exists a non-separating 2-sided cycle C in H and there may exist exactly one 1-sided cycle C' in H which crosses C exactly once.

Note that as every non-separating 2-sided cycle on the Klein bottle passes two cross caps, the second assumption follows.

Let $F = xyz$ and $F' = xyw$ be two faces sharing an edge $xy \in E(C)$ where $f(x) = 1$, $f(y) = 2$, $f(z) = 3$ and $f(w) = 4$. Let $g : V(G) \to \{a, b, c\}$ be a 3-coloring of G such that $g(x) = a$, $g(y) = b$ and $g(z) = g(w) = c$. Let $\mathcal{P} = F_0 F_1 \ldots F_k$ with $F_0 = F$ and $F_k = F'$ be a facial path in G, which passes no $(1, 2)$-edge.

By Lemma 3.2, the map $(g \times f)$ (specifically) gives a facial map from $\mathcal{P}$ to a facial path $\mathcal{P}_{T_0}$ in T_0 from A to A' shown in Fig. 1. Note that since all edges of T_0 are non-singular, if two consecutive faces in $\mathcal{P}$ share a singular (resp. non-singular) edge, then they are mapped to the same face (resp. two adjacent faces) of T_0. By Lemma 3.4, there are two pairs (r, s) and (r', s') such that $\mathcal{P}_{T_0}$ passes both (r, s)- and (r', s')-edges of T_0 an even number of times. This implies that

(*) $\mathcal{P}$ passes both (r, s)- and (r', s')-cycles of $G[NS(f)]$ an even number of times.

We see that $\mathcal{P}$ must pass a $(3, 4)$-edge of $G[NS(f)]$ at least once since $\mathcal{P}_{T_0}$ passes a $(3, 4)$-edge of T_0 at least once. Note that a simple closed curve γ passing only xy and edges passed by $\mathcal{P}$ is 1-sided since $\mathcal{P}$ passes no $(1, 2)$-edge. Therefore, if there is a non-separating 2-sided (i, j)-cycle in $G[NS(f)]$ for $(i, j) \neq (1, 2)$, then

(**) $\mathcal{P}$ passes each non-separating 2-sided (i, j)-cycle of $G[NS(f)]$ an odd number of times

since a non-separating 2-sided curve intersects a 1-sided curve an odd number of times on the Klein bottle. Moreover, there is neither 1-sided $(1, 2)$- nor $(3, 4)$-cycle in $G[NS(f)]$ since there are non-separating 2-sided $(1, 2)$- and $(3, 4)$-cycles in $G[NS(f)]$ (the latter follows from that $\mathcal{P}$ must pass a $(3, 4)$-edge of $G[NS(f)]$ at least once).

By (*) and (**), there are at least two pairs (i, j)'s such that there is a 1-sided (i, j)-cycle in $G[NS(f)]$. By Claim 3.7, if there is a 1-sided (i, j)-cycle in $G[NS(f)]$, then there is a non-separating 2-sided (i, j)-cycle in $G[NS(f)]$ (since two crossing 1-sided (i, j)-cycles produce a separating (i, j)-cycle formed by several edges of those 1-sided cycles). Therefore, there are exactly two such pairs, and hence, without loss of generality, we may assume that $(1, 3)$ and $(2, 3)$ are such pairs.

Remember that a simple closed curve γ passing only xy and edges passed by $\mathcal{P}$ is 1-sided. On the Klein bottle, we can take two non-homotopic 1-sided simple closed curves. Thus, we can take $\mathcal{P}$ so that at least one of 1-sided $(1, 3)$- and $(2, 3)$-cycles of $G[NS(f)]$ is non-homotopic to γ, that is, $\mathcal{P}$ passes those non-homotopic cycles an even number of times. Therefore, there exists at most one pair (i, j) such that $\mathcal{P}$ passes (i, j)-edges of $G[NS(f)]$ an even number of times, which contradicts (*). This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

In this section, we construct graphs for statements (i) and (ii) in Theorem 1.2. As in Fig. 1, for a graph G in each figure, each label ij ($i \in \{1,2,3,4\}$ and $j \in \{a,b,c\}$) denotes a combination of two colors defined by a 4-coloring $f : V(G) \to \{1,2,3,4\}$ and a 3-coloring $g : V(G) \to \{a,b,c\}$.

See the graph G_1 shown in Fig. 3. As $G_1(i,j)$ is connected for any pair (i,j) ($i,j \in \{1,2,3,4\}$), one must change the color of all i-colored vertices if one changes the color of any of them. So the only K-changes permute colors. Thus no 4-coloring can be changed to a 3-coloring by K-changes. This implies (i) in Theorem 1.2 (we can obtain infinitely many such graphs from G_1 by a repeated application of even subdivision, which is defined later).

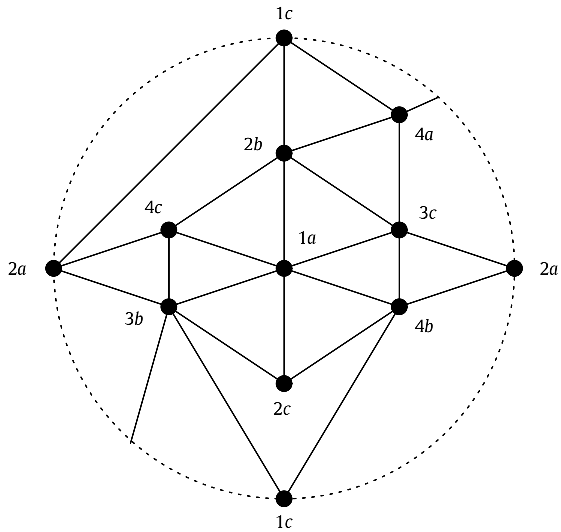

Fig. 3. A 3-colorable graph G_1 on the projective plane which has two distinct non-Kempe-equivalent 4-colorings.

Secondly, we construct a desired 3-colorable triangulation on the non-orientable surface with genus 3. Let T_0^* denote the graph on the torus obtained from T_0 shown in Fig. 1 by deleting the edge $1b3c$. After that, we add one cross cap into the 4-region bounded by $2a1b4a3c$, that is, we remove a disc (from the interior of the 4-region) and identify opposite points on the boundary. It is well known that the torus with one cross cap is homeomorphic to the non-orientable surface with genus 3. Thus, it suffices to make a desired 3-colorable triangulation on this surface.

We can obtain a 3-colorable triangulation H with multiple edges from T_0^* by replacing the 4-region $2a1b4a3c$ of T_0^* with one shown in Fig. 4. The graph H has three pairs of multiple edges for $1b4a$, $1b3c$ and $2a1b$. Note that $H(i,j)$ is connected for any pair (i,j) ($i,j \in \{1,2,3,4\}$ with $i \neq j$) in H.

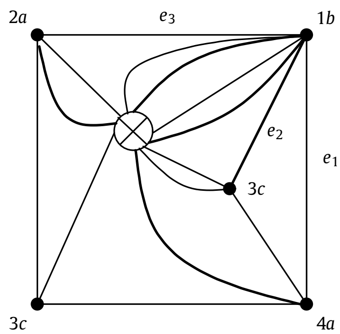

Fig. 4. A graph in the 4-region $2a1b4a3c$.

Here we introduce a local transformation (originally introduced by Fisk [5]), called an *even subdivision*, to obtain a desired 3-colorable triangulation G_2 on the non-orientable surface with genus 3 from the above triangulation H with multiple edges.

Let uvw and uvw' be two triangular faces. An *even subdivision* of an edge uv is to place uv with a path $uabv$ and to add edges aw, bw, aw', bw'. In this operation, if uv is a non-singular edge in H, then $H'(i,j)$ is connected for any pair (i,j) ($i,j \in \{1,2,3,4\}$) in the resulting triangulation H' (which may have multiple edges); for example, as shown in Fig. 5, we can easily see that the connectivity of $G(i,j)$ is preserved. (Note that 3- and 4-colorings of H' are uniquely determined by

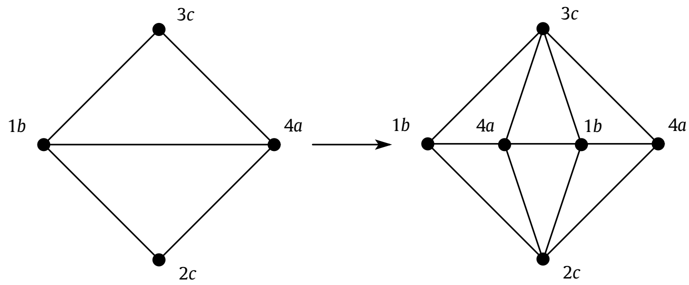

Fig. 5. An even subdivision of a non-singular edge.

g and f of H.) Therefore, by applying three even subdivisions to three edges e_1, e_2 and e_3 each of which is one of multiple edges, we have a desired 3-colorable triangulation G_2 on the non-orientable surface with genus 3. Moreover, we can obtain infinitely many such graphs from G_2 by a repeated application of even subdivision. This implies (ii) in Theorem 1.2. $\square$

Declaration of competing interest

To the best of our knowledge, the named authors have no conflict of interest, financial or otherwise.

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Data availability

No data was used for the research described in the article.

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