

## A NOTE ON COLOR-BIAS PERFECT MATCHINGS IN HYPERGRAPHS\*

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**Abstract.** A result of Balogh et al. yields the minimum degree threshold that ensures a 2-colored graph contains a perfect matching of significant color-bias (i.e., a perfect matching that contains significantly more than half of its edges in one color). In this note we prove an analogous result for perfect matchings in  $k$ -uniform hypergraphs. More precisely, for each  $2 \leq \ell < k$  and  $r \geq 2$  we determine the minimum  $\ell$ -degree threshold for forcing a perfect matching of significant color-bias in an  $r$ -colored  $k$ -uniform hypergraph.

**Key words.** color-bias, discrepancy, perfect matchings

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**1. Introduction.** A *perfect matching* in a hypergraph  $H$  is a collection of vertex-disjoint edges of  $H$  which covers the vertex set  $V(H)$  of  $H$ . In recent decades there has been significant interest in the problem of establishing *minimum degree* conditions that force a perfect matching in a  $k$ -uniform hypergraph. More precisely, given a  $k$ -uniform hypergraph  $H$  and an  $\ell$ -element vertex set  $S \subseteq V(H)$  (where  $\ell \in [k-1]$ ), we define  $d_H(S)$  to be the number of edges containing  $S$ . The *minimum  $\ell$ -degree*  $\delta_\ell(H)$  of  $H$  is the minimum of  $d_H(S)$  over all  $\ell$ -element sets of vertices in  $H$ . We refer to  $\delta_1(H)$  as the *minimum vertex degree* of  $H$  and to  $\delta_{k-1}(H)$  as the *minimum codegree* of  $H$ .

Suppose that  $\ell, k, n \in \mathbb{N}$  such that  $\ell \leq k-1$  and  $k$  divides  $n$ . Let  $m_\ell(k, n)$  denote the smallest integer  $m$  such that every  $k$ -uniform hypergraph  $H$  on  $n$  vertices with  $\delta_\ell(H) \geq m$  contains a perfect matching.

A simple consequence of Dirac's theorem is that  $m_1(2, n) = n/2$  for all even  $n \in \mathbb{N}$ . Improving earlier asymptotically exact bounds given in [13, 19], Rödl, Ruciński, and Szemerédi [20] determined the minimum codegree threshold for perfect matchings in  $k$ -uniform hypergraphs. That is, they showed that if  $n \in \mathbb{N}$  is sufficiently large, then  $m_{k-1}(k, n) = n/2 - k + C$ , where  $C \in \{3/2, 2, 5/2, 3\}$  depends on the values of  $n$  and  $k$ .

The value of  $m_\ell(k, n)$  is known for various pairs  $(k, \ell)$  when  $n$  is sufficiently large. For example, after an earlier asymptotic result of Pikhurko [17], Treglown and Zhao [21] determined the value of  $m_\ell(k, n)$  for  $\ell \geq k/2$  and  $n$  sufficiently large. However, the minimum vertex degree case of the problem is wide open in general, and the only

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case where the asymptotic or exact value of  $m_1(k, n)$  is known is when  $k = 2, 3, 4, 5$ . See, e.g., [18, 23] for discussions on further results in the area.

Given any  $1 \leq \ell < k$ , it is known that

$$(1) \quad m_\ell(k, n) \geq \max \left\{ \frac{1}{2} - o(1), 1 - \left( \frac{k-1}{k} \right)^{k-\ell} - o(1) \right\} \binom{n}{k-\ell}.$$

See, e.g., the introduction of [22] for the two families of hypergraphs that demonstrate (1). It is widely believed that the inequality in (1) is asymptotically sharp for all choices of  $k, \ell$ ; see [12, 14]. Moreover, Treglown and Zhao [22] gave a conjecture on the exact value of  $m_\ell(k, n)$  for sufficiently large  $n \in k\mathbb{N}$ .

The aim of this paper is to study the *color-bias* version of this problem. The topic of color-bias structures in graphs was first raised by Erdős in the 1960s (see [5, 6]). Sparked by work of Balogh et al. [1], there has been renewed interest in the topic, particularly in establishing minimum degree conditions that force a color-bias copy of a graph  $F$ . More precisely, if a graph  $G$  contains a copy of  $F$ , then, however, the edges of  $G$  are 2-colored, one can clearly ensure that  $G$  contains a copy of  $F$  with at least  $e(F)/2$  edges of the same color. The question then is how large the minimum degree  $\delta(G)$  of  $G$  needs to be to guarantee that  $G$  contains a copy of  $F$  with significantly more than  $e(F)/2$  edges of the same color, no matter how one 2-colors the edges of  $G$ . The following result resolves this problem in the case when  $F$  is a Hamilton cycle.

**THEOREM 1.1** (Balogh et al. [1]). *Let  $0 < c < 1/4$  and  $n \in \mathbb{N}$  be sufficiently large. If  $G$  is an  $n$ -vertex graph with*

$$\delta(G) \geq (3/4 + c)n,$$

*then given any 2-coloring of  $E(G)$ , there is a Hamilton cycle in  $G$  with at least  $n/2 + cn/32$  edges of the same color. Moreover, if  $n \in 4\mathbb{N}$ , there is an  $n$ -vertex graph  $G'$  with  $\delta(G') = 3n/4$  and a 2-coloring of  $E(G')$  for which every Hamilton cycle in  $G'$  has precisely  $n/2$  edges in each color.*

Note that Theorem 1.1 shows that the minimum degree threshold for forcing a *color-bias* Hamilton cycle in a graph is significantly higher than the threshold for forcing only a Hamilton cycle. Indeed, Dirac's theorem tells us that any  $n$ -vertex graph  $G$  with  $\delta(G) \geq n/2$  contains a Hamilton cycle.

Since a Hamilton cycle on an even number of vertices is the union of two perfect matchings, Theorem 1.1 implies the following result.

**THEOREM 1.2** (Balogh et al. [1]). *Let  $0 < c < 1/4$  and  $n \in 2\mathbb{N}$  be sufficiently large. If  $G$  is an  $n$ -vertex graph with*

$$\delta(G) \geq (3/4 + c)n,$$

*then given any 2-coloring of  $E(G)$ , there is a perfect matching in  $G$  with at least  $n/4 + cn/64$  edges of the same color. Moreover, if  $n \in 4\mathbb{N}$ , there is an  $n$ -vertex graph  $G'$  with  $\delta(G') = 3n/4$  and a 2-coloring of  $E(G')$  for which every perfect matching in  $G'$  has precisely  $n/4$  edges in each color.*

Let  $n \in 4\mathbb{N}$ . We define the graph  $G'$  in Theorem 1.2 as follows:  $V(G')$  consists of the disjoint union of two vertex classes  $A$  and  $B$  of sizes  $n/4$  and  $3n/4$ , respectively;  $E(G')$  contains all possible red edges whose endpoints are both in  $B$  and all possible

blue edges with one endpoint in  $A$  and one endpoint in  $B$ . Thus,  $\delta(G') = 3n/4$ , and every perfect matching in  $G'$  has precisely  $n/4$  edges in each color.

Since [1] appeared, a number of analogues of Theorem 1.1 have been established for other types of spanning structures. Given graphs  $G$  and  $F$ , an  $F$ -factor in  $G$  is a collection of vertex-disjoint copies of  $F$  in  $G$  that together cover  $V(G)$ . In [2], the minimum degree threshold for forcing a color-bias  $K_r$ -factor was determined.<sup>1</sup> More recently, this result was extended to  $F$ -factors for every fixed graph  $F$ ; see [4]. For  $k \geq 2$ , the minimum degree threshold for forcing a color-bias  $k$ th power of a Hamilton cycle in a graph was established in [3].

Other variants of the problem have also been studied. In [7, 10] an  $r$ -color version of Theorem 1.1 was proven: in this setting, now one  $r$ -colors  $E(G)$  and seeks a Hamilton cycle with significantly more than  $n/r$  edges of the same color. Color-bias problems have also been considered for random graphs [9]. Recently, Mansilla Brito [16] gave a minimum codegree result for forcing a color-bias copy of a tight Hamilton cycle in a 3-uniform hypergraph. We remark that all of these color-bias results can be phrased in the equivalent language of *discrepancy*; see, e.g., [1, 2, 3, 4, 10].

Our main result determines the minimum  $\ell$ -degree threshold for forcing a color-bias perfect matching in a  $k$ -uniform hypergraph for all  $\ell \geq 2$  and  $k \geq 3$ . To state our result, we need the following definitions: Given integers  $1 \leq \ell < k$ , let  $\mathcal{C}_{k,\ell}$  be the set of all  $c > 0$  such that  $m_\ell(k, n) \leq c \binom{n}{k-\ell}$  for all sufficiently large  $n \in k\mathbb{N}$ . Set  $c_{k,\ell}$  to be the infimum of  $\mathcal{C}_{k,\ell}$ . In particular, note that the general conjecture on the asymptotic value of  $m_\ell(k, n)$  equivalently states that

$$c_{k,\ell} = \max \left\{ \frac{1}{2}, 1 - \left( \frac{k-1}{k} \right)^{k-\ell} \right\}.$$

**THEOREM 1.3.** *Let  $k, \ell, r \in \mathbb{N}$  where  $2 \leq \ell < k$  and  $r \geq 2$ . Given any  $\eta > 0$  where  $c_{k,\ell} + \eta < 1$ , there exists an  $n_0 \in \mathbb{N}$  such that the following holds: Let  $H$  be a  $k$ -uniform hypergraph on  $n \geq n_0$  vertices, where  $n \in k\mathbb{N}$ . If*

$$\delta_\ell(H) \geq (c_{k,\ell} + \eta) \binom{n}{k-\ell},$$

*then given any  $r$ -coloring of  $E(H)$ , there is a perfect matching in  $H$  with at least  $\frac{n}{rk} + \frac{\eta n}{8r(r-1)k^k(k^2+k)}$  edges of the same color.*

We remark that Theorem 1.3 holds even in the cases in which we do not know the value of  $c_{k,\ell}$ . By definition of  $c_{k,\ell}$ , the minimum  $\ell$ -degree condition in Theorem 1.3 is essentially best possible. Indeed, for  $c < c_{k,\ell}$ , a minimum  $\ell$ -degree condition of  $\delta_\ell(H) \geq c \binom{n}{k-\ell}$  does not even guarantee a perfect matching, let alone one of significant color-bias. So in this sense the color-bias and “standard” versions of the problem are aligned when  $\ell \geq 2$ .

In contrast, the same phenomenon does not occur for the minimum vertex degree version of the problem. Indeed, Theorem 1.2 tells us that the minimum degree threshold for a color-bias perfect matching in a *graph* is different from the minimum degree threshold for a perfect matching in a graph. Furthermore, in section 4 we describe a similar phenomenon in the 3-uniform hypergraph setting.

*Remark.* While finalizing a manuscript that gave the proof of Theorem 1.3 in the case when  $\ell = k - 1$  and  $r = 2$ , we learnt of simultaneous and independent work

<sup>1</sup>Recall  $K_r$  denotes the complete graph on  $r$  vertices.

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of Gishboliner, Glock, and Sgueglia [8]. They determined the *minimum codegree threshold* for forcing a tight Hamilton cycle of significant color-bias in an  $r$ -colored  $k$ -uniform hypergraph (where  $r \geq 2$  and  $k \geq 3$ ). As an immediate consequence of their result, they also established the corresponding *minimum codegree threshold* for perfect matchings.  $\square$

We therefore decided to seek a generalization of our minimum codegree result to other degree conditions, i.e., Theorem 1.3. In doing so, we found an argument much cleaner than our original approach.

**Notation.** Let  $H$  be a hypergraph. The *neighborhood*  $N_H(X)$  of a set  $X \subseteq V(H)$  is the family of sets  $S \subseteq V(H) \setminus X$  such that  $S \cup X \in E(H)$ . If  $X = \{x\}$ , we define  $N_H(x) := N_H(X)$ . Given a vertex  $x \in V(H)$  and set  $Y \subseteq V(H)$ , we sometimes write  $xY$  or  $Yx$  to denote  $\{x\} \cup Y$ . Given a coloring  $c$  of  $E(H)$ , we call an edge  $e \in E(H)$  a  $C$ -edge if  $e$  is colored  $C$  in  $c$ . Given a set  $X \subseteq V(H)$ , we write  $H[X]$  for the *induced subhypergraph of  $H$  with vertex set  $X$* . We define  $H \setminus X := H[V(H) \setminus X]$ .

Given a hypergraph  $F$  with an  $r$ -coloring  $c: E(F) \rightarrow \{C_1, \dots, C_r\}$ , its *color profile* is  $(x_1, \dots, x_r)$ , where  $x_i$  is the number of  $C_i$ -edges in  $F$  for each  $i \in [r]$ . Two color profiles  $(x_1, \dots, x_r)$ ,  $(y_1, \dots, y_r)$  are said to be *different with respect to the color  $C_i$*  if  $x_i \neq y_i$ .

## 2. Preliminaries and useful results.

**2.1. Proof overview and key definitions.** Throughout this section, we will suppose that  $H$  is a  $k$ -uniform hypergraph on  $n$  vertices with an  $r$ -coloring  $c: E(H) \rightarrow \{C_1, \dots, C_r\}$ .

Our general strategy for the proof of Theorem 1.3 is as follows. Our aim is to find certain *gadgets* inside of  $H$ . A gadget is just a subhypergraph of  $H$  with some given structure. A gadget  $G$  is *good* if  $G$  contains two perfect matchings that have different color profiles with respect to the  $r$ -coloring  $c$ .

For a certain well-chosen  $t \in \mathbb{N}$ , we will prove that there are  $t$  vertex-disjoint good gadgets  $G_1, \dots, G_t$  in  $H$  and a  $j \in [r]$  so that, for each good gadget  $G_i$ , the two perfect matchings  $M_i$  and  $M'_i$  in  $G_i$  have color profiles that are different with respect to the color  $C_j$ .

We will then be able to easily find a perfect matching in  $H$  of significant color-bias. Indeed, removing the vertices of  $G_1, \dots, G_t$  from  $H$  will result in a  $k$ -uniform hypergraph  $H'$  that contains a perfect matching  $M$ . The flexibility of the good gadgets then allows us to extend  $M$  into a perfect matching in  $H$  with significant color-bias, whatever the colour profile of  $M$  may be.

We next state the definitions required to formally introduce the notion of a good gadget.

**DEFINITION 2.1.** Let  $u, v \in V(H)$  be distinct and  $T \in N_H(u) \cap N_H(v)$ . We say

- $uTv$  is **S** if  $c(T \cup \{u\}) = c(T \cup \{v\})$ ,
- $uTv$  is  **$C_i C_j$**  if  $c(T \cup \{u\}) = C_i$  and  $c(T \cup \{v\}) = C_j$ .

Let  $C_i C_j(uv)$  denote the collection of sets  $T \in N_H(u) \cap N_H(v)$  for which  $uTv$  is  $C_i C_j$ . Define  $S(uv)$  analogously.

Note that  $C_i C_j(uv) = C_j C_i(vu)$  for all distinct  $u, v \in V(H)$ .

**DEFINITION 2.2.** Let  $D > 0$ , and let  $u, v \in V(H)$  be distinct. We say that  $N_H(u) \cap N_H(v)$  is

- **type S(D)** if  $|S(uv)| \geq Dn^{k-2}$ ,
- **type  $C_i C_j(D)$**  if  $i \neq j$  and  $|C_i C_j(uv)| \geq Dn^{k-2}$ .

We remark that it may be the case that  $N_H(u) \cap N_H(v)$  has more than one type.

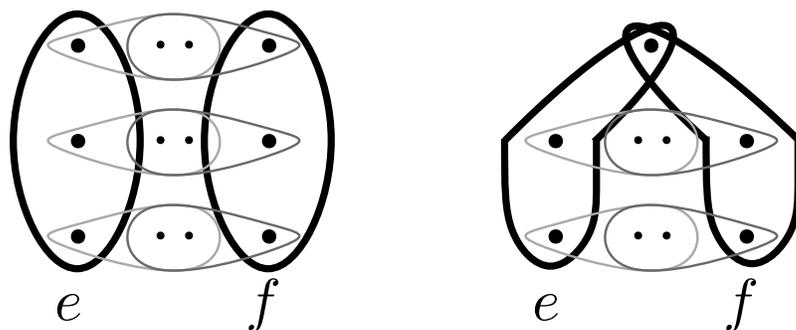


FIG. 1. On the left, a  $(12, e, f)$ -gadget. On the right, a  $(9, e, f)$ -gadget.

DEFINITION 2.3. Let  $e = \{e_1, \dots, e_k\}$  and  $f = \{f_1, \dots, f_k\}$  be two edges in  $H$ . A  $(k^2 + k, e, f)$ -gadget  $G$  is a subhypergraph of  $H$  on  $k^2 + k$  vertices so that

- $V(G)$  is the disjoint union of  $e, f$ , and  $T_1, \dots, T_k$ , where  $T_i \in N_H(e_i) \cap N_H(f_i)$  for each  $i \in [k]$ ;
- $e, f \in E(G)$ ;
- $e_i T_i, f_i T_i \in E(G)$  for all  $i \in [k]$ .

A  $(k^2 + k, e, f)$ -gadget in which every  $e_i T_i f_i$  is  $S$  will be called an  $S$ - $(k^2 + k, e, f)$ -gadget.

A  $(3k, e, f)$ -gadget  $G$  is a subhypergraph of  $H$  on  $3k$  vertices so that

- $e_i = f_i$ , for all  $i \in \{3, \dots, k\}$ ;
- $V(G)$  is the disjoint union of  $e, f_1, f_2, T_1$ , and  $T_2$ , where  $T_i \in N_H(e_i) \cap N_H(f_i)$  for each  $i \in [2]$ ;
- $e, f \in E(G)$ ;
- $e_1 T_1, f_1 T_1, e_2 T_2, f_2 T_2 \in E(G)$ .

Given  $t \in \{3k, k^2 + k\}$ , we say that a  $(t, e, f)$ -gadget  $G$  is good if it contains two perfect matchings with different color profiles (with respect to the  $r$ -coloring of  $G$  induced by the  $r$ -coloring  $c$  of  $H$ ).

Note that  $e$  and  $f$  are vertex-disjoint in a  $(k^2 + k, e, f)$ -gadget but intersect in  $k - 2$  vertices in a  $(3k, e, f)$ -gadget; see Figure 1.

2.2. Tools for the proof of Theorem 1.3. The following well-known result allows one to deduce a lower bound on  $\delta_\ell(H)$  given a lower bound on  $\delta_{\ell'}(H)$  for any  $\ell \leq \ell'$ .

PROPOSITION 2.4. Let  $1 \leq \ell \leq \ell' < k$ , and let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices. If  $\delta_{\ell'}(H) \geq x \binom{n-\ell'}{k-\ell'}$  for some  $0 \leq x \leq 1$ , then  $\delta_\ell(H) \geq x \binom{n-\ell}{k-\ell}$ .

The next result gives a sufficient condition for finding a good  $(3k, e, f)$ -gadget in a  $k$ -uniform hypergraph of large minimum 2-degree.

LEMMA 2.5. Let  $k \geq 3$  and  $D := 3k$ . Let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices with an  $r$ -coloring  $c : E(H) \rightarrow \{C_1, \dots, C_r\}$ . Suppose there exists  $i \neq j \in [r]$  and distinct  $v_1, v_2, v_3, v_4 \in V(H)$  such that  $N_H(v_1) \cap N_H(v_2)$  and  $N_H(v_3) \cap N_H(v_4)$  are both type  $C_i C_j(D)$ . If

$$\delta_2(H) > \frac{1}{2} \binom{n}{k-2},$$

then there exists a good  $(3k, e, f)$ -gadget in  $H$  for some  $e, f \in E(H)$ .

*Proof.* By the minimum 2-degree condition, there exists a set  $X \subseteq V(H)$  of size  $k - 2$  such that  $A = X \cup \{v_1, v_3\}$  and  $B = X \cup \{v_2, v_4\}$  are both in  $E(H)$ . We show that we can construct a  $(3k, A, B)$ -gadget, and then we prove that it is good.

Given that  $N_H(v_1) \cap N_H(v_2)$  is type  $C_i C_j(D)$ , there are at least  $3kn^{k-2}$  sets  $T_{1,2} \in N_H(v_1) \cap N_H(v_2)$  such that  $c(v_1 T_{1,2}) = C_i$  and  $c(v_2 T_{1,2}) = C_j$ . As  $|A \cup B| = k + 2 < 3k$ , we may choose such a set  $T_{1,2}$  so that it is also vertex-disjoint from  $A \cup B$ . Similarly, there is a set  $T_{3,4} \in N_H(v_3) \cap N_H(v_4)$  such that  $c(v_3 T_{3,4}) = C_i$ ,  $c(v_4 T_{3,4}) = C_j$  and  $T_{3,4}$  is vertex-disjoint from  $A, B$ , and  $T_{1,2}$ .

Then, define a gadget  $G$  as follows:

- $V(G)$  is the union of  $A, B, T_{1,2}$ , and  $T_{3,4}$ ;
- $A, B, v_1 T_{1,2}, v_2 T_{1,2}, v_3 T_{3,4}$ , and  $v_4 T_{3,4}$  are in  $E(G)$ .

By definition,  $G$  is a  $(3k, A, B)$ -gadget.

To prove that  $G$  is good, we need to find two perfect matchings in  $G$  with different color profiles. Define  $M_A := \{A, v_2 T_{1,2}, v_4 T_{3,4}\}$  and  $M_B := \{B, v_1 T_{1,2}, v_3 T_{3,4}\}$ . Both  $M_A$  and  $M_B$  are perfect matchings in  $G$ . While  $M_A$  has at least two  $C_j$ -edges ( $v_2 T_{1,2}$  and  $v_4 T_{3,4}$ ),  $M_B$  has at least two  $C_i$ -edges ( $v_1 T_{1,2}$  and  $v_3 T_{3,4}$ ). Thus,  $M_A$  and  $M_B$  have different color profiles, as desired.  $\square$

The next lemma ensures that a hypergraph  $H$  as in Theorem 1.3 contains a good gadget or a perfect matching of huge color-bias.

LEMMA 2.6. *Let  $2 \leq \ell < k$  and  $\eta > 0$ . There exists an  $n_0 \in \mathbb{N}$  such that the following holds for all  $n \geq n_0$  with  $n \in k\mathbb{N}$ : Let  $H$  be a  $k$ -uniform hypergraph on  $n$  vertices with an  $r$ -coloring  $c: E(H) \rightarrow \{C_1, \dots, C_r\}$  and*

$$\delta_\ell(H) \geq (c_{k,\ell} + \eta) \binom{n}{k - \ell}.$$

*Suppose that  $H$  does not have a perfect matching containing at least  $n/k - \binom{r}{2}$  edges of the same color. Then*

- *there exists a good  $(3k, e, f)$ -gadget in  $H$  for some  $e, f \in E(H)$ ; or*
- *there exists a good  $(k^2 + k, e, f)$ -gadget in  $H$  for some  $e, f \in E(H)$ .*

*Proof.* Let  $H$  and  $c$  be as in the lemma, and suppose  $n$  is sufficiently large. Let  $D := k^2 + k \geq 3k$ . Note that, given our minimum  $\ell$ -degree condition, Proposition 2.4 implies that

$$(2) \quad \begin{aligned} \delta_1(H) &\geq (c_{k,\ell} + \eta) \binom{n-1}{k-1} > \left(\frac{1}{2} + \frac{\eta}{2}\right) \binom{n}{k-1} \quad \text{and} \\ \delta_2(H) &\geq (c_{k,\ell} + \eta) \binom{n-2}{k-2} > \frac{1}{2} \binom{n}{k-2}. \end{aligned}$$

Here the inequalities follow as  $c_{k,\ell} \geq 1/2$  by (1).

As  $n$  is sufficiently large, and by definition of  $c_{k,\ell}$ , the minimum  $\ell$ -degree condition ensures a perfect matching  $M$  in  $H$ .

Let  $L := \binom{r}{2} + 1$ . By the hypothesis of the lemma,  $M$  does not contain  $n/k - \binom{r}{2}$  edges of the same color; so, there exist distinct edges  $e_1, \dots, e_L, f_1, \dots, f_L \in M$  such that  $c(e_i) \neq c(f_i)$  for each  $i \in [L]$ .

Given any distinct  $x, y \in V(H)$ , (2) implies that  $|N_H(x) \cap N_H(y)| \geq \eta \binom{n}{k-1}$ . In particular, this means that  $N_H(x) \cap N_H(y)$  is of type  $S(D)$  or of type  $C_i C_j(D)$  for some distinct  $i, j \in [r]$ .

Suppose there exists  $i \neq j \in [r]$  and distinct  $x, y, z, w \in V(H)$  such that  $N_H(x) \cap N_H(y)$  and  $N_H(z) \cap N_H(w)$  are both type  $C_i C_j(D)$ . Then by Lemma 2.5, there exists a good  $(3k, e, f)$ -gadget in  $H$  for some  $e, f \in E(H)$ .

So, we may assume no such  $i \neq j \in [r]$  and  $x, y, z, w \in V(H)$  exist. In particular, for each of the  $\binom{r}{2} = L - 1$  choices for  $i \neq j \in [r]$ , there is at most one pair  $(e_s, f_s)$  such that there exist  $u \in e_s$  and  $v \in f_s$  so that either  $N_H(u) \cap N_H(v)$  or  $N_H(v) \cap N_H(u)$  is type  $C_i C_j(D)$ . Thus, the following claim holds.

CLAIM 2.7. *There is a pair  $(e_s, f_s)$  such that for each  $u \in e_s$  and  $v \in f_s$ , we have that  $N_H(u) \cap N_H(v)$  is type  $S(D)$ .*

Let  $e_s = \{u_1, \dots, u_k\}$  and  $f_s = \{v_1, \dots, v_k\}$ . For each  $i \in [k]$ , we choose a set  $T_i$  so that

- (i)  $T_i \in S(u_i v_i)$ ;
- (ii)  $T_1, \dots, T_k, e_s, f_s$  are all vertex-disjoint.

Note that we can guarantee (ii) since  $|S(u_i v_i)| \geq Dn^{k-2} = (k^2 + k)n^{k-2}$  for each  $i \in [k]$ .

We construct a  $(k^2 + k, e_s, f_s)$ -gadget  $G$  as follows:

- $V(G)$  is the union of  $e_s, f_s, T_1, \dots, T_k$ ;
- $e_s$  and  $f_s$  are edges in  $G$ ;
- $u_i T_i, v_i T_i$  are edges in  $G$  for all  $i \in [k]$ .

By definition,  $G$  is an  $S-(k^2 + k, e_s, f_s)$ -gadget with  $c(e_s) \neq c(f_s)$ . This implies that  $G$  is a good  $(k^2 + k, e_s, f_s)$ -gadget. Indeed,  $M_e := \{e_s, v_1 T_1, \dots, v_k T_k\}$  and  $M_f := \{f_s, u_1 T_1, \dots, u_k T_k\}$  are perfect matchings in  $G$  with different color profiles.

**3. Proof of Theorem 1.3.** Let  $H$  be a sufficiently large  $n$ -vertex  $k$ -uniform hypergraph as in the statement of the theorem. Let  $c : E(H) \rightarrow \{C_1, \dots, C_r\}$  be an  $r$ -coloring of  $E(H)$ . If  $H$  contains a perfect matching with at least  $n/k - \binom{r}{2}$  edges of the same color, then we are done.

So, suppose no perfect matching in  $H$  contains at least  $n/k - \binom{r}{2}$  edges of the same color. By Lemma 2.6, we can find either a good  $(3k, e, f)$ -gadget or a good  $(k^2 + k, e, f)$ -gadget in  $H$ . Call this gadget  $G_1$ .

Next, consider  $H_1 := H \setminus V(G_1)$ . Clearly  $\delta_\ell(H_1) \geq (c_{k,\ell} + \eta/2) \binom{n}{k-\ell}$ . Suppose  $H_1$  contains a perfect matching  $M_1$  with at least  $|H_1|/k - \binom{r}{2}$  edges of the same color. Thus, by taking any perfect matching in  $G_1$  and adding it to  $M_1$ , we obtain a perfect matching in  $H$  containing at least  $|H_1|/k - \binom{r}{2} \geq n/k - |G_1|/k - \binom{r}{2} \geq n/k - k - 1 - \binom{r}{2}$  edges of the same color, as desired.

Hence, we may assume  $H_1$  does not contain such a perfect matching  $M_1$ . By Lemma 2.6, we can find either a good  $(3k, e, f)$ -gadget or a good  $(k^2 + k, e, f)$ -gadget in  $H_1$ . Call this gadget  $G_2$ , and set  $H_2 := H_1 \setminus V(G_2)$ .

Repeating this argument, we obtain either a perfect matching in  $H$  of significant color-bias, or a collection of  $t := \frac{\eta n}{4k^k(k^2+k)}$  vertex-disjoint gadgets  $G_1, \dots, G_t$  where, given any  $i \in [t]$ ,  $G_i$  is either a good  $(3k, e, f)$ -gadget or a good  $(k^2 + k, e, f)$ -gadget in  $H$ . In particular, note that each gadget we select has size at most  $k^2 + k$ , and if we remove  $t(k^2 + k)$  vertices from  $H$ , we still have that  $\delta_\ell(H) \geq (1/2 + \eta) \binom{n}{k-\ell} - t(k^2 + k)n^{k-\ell-1} \geq (1/2 + \eta/2) \binom{n}{k-\ell}$ . Thus, we can indeed repeatedly apply Lemma 2.6 to obtain these gadgets  $G_1, \dots, G_t$ .

Set  $\mathcal{G} := \{G_1, \dots, G_t\}$ . For each color  $C_i$ , consider the set  $\mathcal{G}_i$  of all the gadgets in  $\mathcal{G}$  that contain two perfect matchings with different color profiles with respect to the color  $C_i$ . Clearly, there exists some  $j \in [r]$  such that  $\mathcal{G}_j$  contains at least  $t/r$  gadgets.

For each gadget  $G_i$  in  $\mathcal{G}_j$ , consider the perfect matching  $M_i$  in  $G_i$  with the largest possible number of edges colored  $C_j$ ; let  $M'_i$  be the perfect matching in  $G_i$  with the fewest possible edges colored  $C_j$ . So,  $M_i$  has at least one more  $C_j$ -edge than  $M'_i$ .

Let  $M^+$  denote the union of all these  $M_i$ , and let  $M^-$  denote the union of all these  $M'_i$ . So,  $M^+$  contains at least  $t/r = \frac{\eta n}{4rk^k(k^2+k)}$  more  $C_j$ -edges than  $M^-$ .

Let  $V(\mathcal{G}_j)$  denote the set of vertices in  $H$  that lie in one of the gadgets in  $\mathcal{G}_j$ . Note that  $\delta_\ell(H \setminus V(\mathcal{G}_j)) \geq (c_{k,\ell} + \eta/2) \binom{n}{k-\ell}$ , so there exists a perfect matching  $M$  in  $H \setminus V(\mathcal{G}_j)$ . Thus,  $M \cup M^+$  and  $M \cup M^-$  are both perfect matchings in  $H$ .

If  $M \cup M^-$  contains at least  $\frac{n}{rk} + \frac{\eta n}{8r(r-1)k^k(k^2+k)}$  edges of the same color, then the theorem holds. Thus, we may assume this is not the case. This immediately implies the following claim.

**CLAIM 3.1.** *For every  $i \in [r]$ , the number of  $C_i$ -edges in  $M \cup M^-$  is at least  $\frac{n}{rk} - \frac{\eta n}{8rk^k(k^2+k)}$ .*

In particular,  $M \cup M^-$  contains at least  $\frac{n}{rk} - \frac{\eta n}{8rk^k(k^2+k)}$   $C_j$ -edges. Since there are at least  $\frac{\eta n}{4rk^k(k^2+k)}$  more  $C_j$ -edges in  $M^+$  than in  $M^-$ , we obtain that  $M \cup M^+$  contains at least  $\frac{n}{rk} + \frac{\eta n}{8rk^k(k^2+k)}$   $C_j$ -edges, as desired.

**4. Concluding remarks.** In this paper we have determined the minimum  $\ell$ -degree threshold for forcing a color-bias perfect matching in a  $k$ -uniform hypergraph for all  $2 \leq \ell < k$ . The only remaining open case of the problem is the minimum *vertex* degree version.

A result of Hàn, Person, and Schacht [12] yields that  $m_1(3, n) = (5/9 + o(1)) \binom{n-1}{2}$ . The following example shows that the corresponding color-bias problem has a significantly higher minimum vertex degree threshold.

*Example 4.1.* Given any  $n \in 6\mathbb{N}$ , there exists an  $n$ -vertex 3-uniform hypergraph  $H$  with

$$\delta_1(H) \geq \frac{3}{4} \binom{n-1}{2}$$

and with a 2-coloring of  $E(H)$  so that every perfect matching in  $H$  has precisely  $n/6$  edges in each color.

*Proof.* Define  $H$  so that (i)  $V(H)$  is the disjoint union of two vertex classes  $A$  and  $B$ , both of size  $n/2$ ; and (ii)  $E(H)$  consists of all those 3-uniform edges containing at least one vertex from each of  $A$  and  $B$ . Thus,

$$\delta_1(H) = \binom{n/2}{2} + \frac{n}{2} \left( \frac{n}{2} - 1 \right) \geq \frac{3}{4} \binom{n-1}{2}.$$

Color each edge containing two vertices from  $A$  red and each edge containing two vertices from  $B$  blue. It is easy to see that every perfect matching in  $H$  uses the same number of red and blue edges.  $\square$

We suspect that this example is extremal for the minimum vertex degree problem in 3-uniform hypergraphs.

**QUESTION 4.2.** *Given any  $\eta > 0$ , does there exist a  $\gamma > 0$  so that the following holds for all sufficiently large  $n \in 3\mathbb{N}$ ? Suppose that  $H$  is an  $n$ -vertex 3-uniform hypergraph with*

$$\delta_1(H) \geq \left( \frac{3}{4} + \eta \right) \binom{n-1}{2}.$$

*Then given any 2-coloring of  $E(H)$ , there is a perfect matching in  $H$  with at least  $n/6 + \gamma n$  edges of the same color.*

*Remark.* Since this paper has been accepted, Question 4.2 has been answered in the affirmative; see [11, 15]. In fact, this new work resolves the minimum vertex degree problem fully (i.e., for all choices of the uniformity  $k \geq 3$  and number of colors  $r \geq 2$ ).  $\square$

By tweaking the proof of Theorem 1.3, one can show that given any  $k \geq 3$  and  $r \geq 2$ , there is a constant  $C$  such that every sufficiently large  $r$ -colored  $n$ -vertex  $k$ -uniform hypergraph  $H$  with  $\delta_{k-1}(H) \geq n/2 + C$  contains a perfect matching with at least  $(n/rk) + 1$  edges of the same color. Moreover, the lower bound on the color-bias grows linearly as one increases the minimum codegree further. The Ph.D. thesis of the third author will contain a rigorous proof of this.

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