A PROOF OF THE 3/4-CONJECTURE FOR THE TOTAL DOMINATION GAME*

JULIEN PORTIER† AND LEO V. VERSTEEGEN†

Abstract. The total domination game is a game played on a graph G by 2 players, Dominator and Staller, who alternate in selecting vertices such that each newly selected vertex increases the number of vertices that are totally dominated by the set of selected vertices. The game stops when the set of selected vertices is a totally dominating set of G. Dominator's aim is to arrive at this state in as few moves as possible, while Staller wants to achieve the opposite. In this paper we prove the 3/4-conjecture by Henning, Klavžar, and Rall, stating that Dominator has a strategy for finishing the total domination game on G within $3n/4$ moves, where n is the number of vertices of G. We also prove that if the minimum degree of G is at least 2, then Dominator can finish the game within $5(n+1)/7$ moves.

1. Introduction. Let G be a graph without isolated vertices. A vertex is *totally dominated* by a set $A \subset V(G)$ if it has a neighbor in A. A totally dominating set of G is a set that totally dominates every vertex of G. The total domination game was introduced as a natural variant of the domination game by Henning, Klavžar, and Rall in [20]. The two players, Dominator (D) and Staller (S), alternately select vertices of a graph G without isolated vertices. Each selected vertex must strictly increase the number of vertices that are totally dominated by the set of selected vertices. The game stops when the set of selected vertices is a totally dominating set of G. Dominator's aim is to minimize the number of vertices in this set at the end of the game, while Staller's aim is to maximize it. The *game total domination number* $\gamma_{tg}(G)$ of G is defined as the number of vertices in the resulting set when Dominator starts the game and both players play optimally. In [21], Henning, Klavžar, and Rall proved that $\gamma_{tg}(G) \leq \frac{4}{5}n$ for every graph G without isolated vertices or edges, and they conjectured that the following holds.

CONJECTURE 1.1. *For every graph G without isolated vertices or edges, $\gamma_{tg}(G) \leq \frac{3}{4}n$.*

Conjecture 1.1 is plainly called the 3/4-*conjecture,* and it has been verified for graphs of minimum degree at least 2 in [10] and for graphs in which $\deg(u) + \deg(v) \geq 4$ for every edge $uv \in E(G)$ and in which no two vertices of degree 1 are at distance 4 in [22]. The best bound for arbitrary graphs without isolated vertices or edges so far was due to Bujtás [7], who proved for every such graph the bound $\gamma_{tg}(G) \leq \frac{11}{14}n$. In this paper we prove Conjecture 1.1.

†Department of Pure Mathematics and Mathematical Statistics (DPMMS), University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom (jp899@cam.ac.uk, lversteegen.math@gmail.com).

THEOREM 1.2. *For every graph G without isolated vertices or edges, $\gamma_{tg}(G) \leq \frac{3}{4}n$.*

Note that Theorem 1.2 is best possible, as the example of the union of disjoint copies of P_4 shows. Also note that the admission of isolated edges adds nothing of interest since both vertices of such edges will ultimately have to be played.

As mentioned above, in [10] Bujtás, Henning, and Tuza verified Conjecture 1.1 for graphs of minimum degree at least 2 and, in fact, proved that for every such graph G, $\gamma_{tg}(G) < \frac{8}{11}n$. In this paper, we improve on their result as follows.

THEOREM 1.3. *For every graph G with minimum degree at least 2, $\gamma_{tg}(G) \leq \frac{5}{7}n + \frac{5}{7}$.*

2. Preliminaries. Before we go into the details of the proofs of Theorem 1.3 in section 3 and of Theorem 1.2 in section 4, we want to convey their central idea. For a graph G on n vertices, we can make the following two observations:

1. At any point in the game, the number of totally dominated vertices is at most n.
2. At any point in the game, the sum of the number of played vertices and the number of vertices that have not been played is n.

Moreover, if at some point in the game a vertex v has not been played and the last neighbor of v becomes totally dominated, then v is *unplayable* from there on out. To exploit the two observations above, we would like Dominator to play in a way such that the sum of the numbers of totally dominated and unplayable vertices grows as quickly as possible. Whether it is easier to totally dominate many new vertices or to make many of them unplayable varies over the course of the game. For this reason, the strategies of D consist of a number of different *phases* that follow one another linearly and come with separate sets of instructions for D. In each phase, we will claim and prove bounds on the number of new totally dominated and unplayable vertices, depending on the number of moves that were played during the phase that D can achieve by playing according to our instructions. The transition is controlled by an end condition for each phase. Once this condition is met, the current phase ends, and we progress into the next one. At the end of the proofs of Theorem 1.2 and Theorem 1.3, we will then combine these bounds to show that $\gamma_{tg}(G) \leq 3n/4$ in the general case and $\gamma_{tg}(G) \leq 5(n+1)/7$ in the minimum-degree-at-least-2 case, respectively.

2.1. Describing the state of the game. Intuitively, S should always wish that as few vertices as possible get totally dominated, but suboptimal moves played by S or unexpectedly good opportunities for D can often lead to more complicated case distinctions later on. For this reason, we will sometimes want to pretend that certain vertices are not yet dominated when they in fact are. To leave no doubt about the validity of such arguments, we substitute some natural notions of the game, such as a vertex being totally dominated by artificial counterparts that track how often we explicitly registered these things during the game. These artificial concepts are not part of the game but rather auxiliary pieces of data kept by us as observers of the game.

Specifically, we will think of all vertices as *colored* in one of the colors *white* or *black,* and in addition we may mark them as *depleted* or *dependent.* We impose on ourselves the following restrictions with respect to how vertices may be colored or marked:

- If a vertex is not totally dominated, it has to be white.
- Once colored black, a vertex is not allowed to be recolored white.

- A vertex is only allowed to be marked as depleted if it has not been played and it has no white neighbors.
- If a vertex has been marked as depleted or dependent, we are not allowed to remove this mark.
- A vertex is only allowed to be dependent if it has at most one white neighbor.
- A vertex is not allowed to be marked as both depleted and dependent.

In particular, every vertex is colored white at the start of the game, and at any stage of the game, the number of black vertices is always bounded from above by the number of totally dominated vertices. Note further that once a vertex has been marked as depleted, it cannot be played after that.

As usual, a *leaf* is a vertex of degree exactly 1. The *parent* of a leaf is its unique neighbor. A point t of the game refers to a situation in our analysis after t moves have been played, always assuming that D has played according to our instructions thus far. In the proofs of both Theorem 1.2 and Theorem 1.3, we wish to keep track of the number of black vertices $\beta(t)$ and the number of depleted vertices $\delta(t)$ at any given point in time t. For the general case of graphs with leaves, we will also keep track of the number of leaves that have been played or marked as depleted, which we denote by $\lambda(t)$. By the rules we have set about black, depleted, and dependent vertices, it is easy to see that β, λ, and δ are increasing in t.

Finally, we denote the number of *unplayed* dependent vertices by σ and the number of undominated neighbors of dependent vertices by ν. For the proof of Theorem 1.2, we are only interested in their difference $\chi = \sigma - \nu$. To prove the stronger bound in Theorem 1.3, however, we will also involve the quantity ν directly. We point out that χ is also monotonous.

The end conditions of most phases of the game include a conjunction of subconditions that capture the progression of the game in terms of the absence of certain configurations of white, black, and unplayed vertices. We remark here that we will not explicitly account for the possibility that the game ends before the end condition of a certain phase is met. For example, this might occur when vertices that are dominated have not been colored white or when the end condition of a phase includes the fact that it is Dominator's turn, but the last move of the game is made by D. We omit the discussion of such cases and leave it to the reader to verify that after coloring all remaining vertices white and marking all unplayed and unmarked vertices as depleted, all postulated bounds on tracked quantities for the given phase hold. We will also pretend that if a vertex v has white neighbours but no undominated neighbors, D is allowed to play v. Indeed, we may simply imagine that D has played v when they played an arbitrary legal move instead as this can never be to Dominator's disadvantage.

3. Proof of Theorem 1.3. There are going to be four phases, and for each $i \in [4]$, t_i is the last move before the end of phase i. Additionally, we let $t_0 = 0$. It can happen that phases are skipped because their end conditions are met from the start of the phase, and in this case we have $t_i = t_{i-1}$. For each $i \in [4]$, let $T_i = t_i - t_{i-1}$.

Phase 1. This phase begins at the start of the game and ends once it is Dominator's turn and no unplayed vertex has at least three white neighbors. For a set $A \subset V$, a *triple separation* is a family $(B_v)_{v \in A}$ such that for all $v \in A$, $B_v \in N(v)^{(3)}$ and for all distinct $v, w \in A$, $B_v \cap B_w = \emptyset$. The triple separation is *white* if for all $v \in A$, all elements of B_v are white, and a set A is called *white-separable* if no vertex in A has been played and A has a white triple separation. Let M denote the size of the largest set A for which there exists a triple separation.

CLAIM 3.1. *D can play in a way such that the following hold:*
- *No vertex is marked as dependent.*
- $\beta(t_1) \geq 2T_1$.
- $\beta(t_1) \geq 2M$.

Proof. Let M_t be the size of the largest white-separable set after move t. By definition of M and due to the fact that all vertices are white at the start of the game, $M_0 = M$. The phase is over exactly when $M_t = 0$, and we will ensure that $M_t \geq M - t$ after every move. Whenever it is Dominator's turn, D plays a vertex v from a largest white-separable set A, and fixing some white triple separation $(B_v)_{v \in A}$, we color the vertices in B_v black. Whenever it is Staller's turn and S plays a vertex v, we color one arbitrarily chosen neighbor of v black. In both cases, we have $M_t \geq M_{t-1} - 1$.

Note that after every two moves, four new vertices are colored black. Therefore, we have $\beta(t_1) \geq 2t_1 = 2T_1$. Since $M_{t_1} = 0$ and $M_{t_1} \geq M - t_1$, it also follows that $\beta(t_1) \geq 2M$. □

Phase 2. For $k \in \mathbb{N}$, a *k-walk* is a sequence of vertices $w_1 v_1 \ldots w_k v_k$ such that for all $1 \leq i < j \leq k$, $v_i \neq v_j$, $w_i \neq w_j$, w_i is white, v_i is adjacent to w_{i+1}, and w_j is adjacent to v_j. A k-walk is a *k-circuit* if v_k is adjacent to w_1, and a walk is *terminal* if it is a 2-walk and v_2 has no other white neighbor than w_2.

The current phase ends once it is Dominator's turn, and no 6-walk, terminal walk, or 2-circuit exists. Note that after this phase, no 6-walk or 2-circuit can be reintroduced.

CLAIM 3.2. *D can play in a way such that the following hold:*
- $\beta(t_2) - \beta(t_1) \geq 3T_2/2$.
- $4\delta(t_2) \geq T_2$.
- *No vertex is marked as dependent.*

Proof. We show that if at some point t it is Dominator's turn and the phase is not over, at least one of the following holds:
- D can play the next move so that

$$\beta(t+2) - \beta(t) \geq 3 \qquad \text{and} \qquad \delta(t+2) - \delta(t) \geq 1.$$

- D can play the next two moves so that

$$\beta(t+4) - \beta(t) \geq 6 \qquad \text{and} \qquad \delta(t+4) - \delta(t) \geq 1.$$

Once this dichotomy is proved, a simple inductive argument yields the claim. Before the first move of its phase, and after each of these sequences, we color all vertices that are white and totally dominated black.

Suppose first there exists a 6-walk $w_1 \ldots v_6$. Then D plays v_1. If in the subsequent move S plays some vertex x that is not adjacent to w_3 or w_4, D responds by playing v_3, after which S plays some second move y. We then color w_1, w_2, w_3, w_4, and all white neighbors of x and y black and mark v_2 as depleted. Similarly, if S responds by playing a vertex that is adjacent to w_3, D plays v_4 next and we can mark v_3 as depleted. Finally, if S plays a vertex adjacent to w_4, D can play v_5 next.

Suppose next there exists a 2-circuit $w_1 v_1 w_2 v_2 w_1$. D simply plays v_1, we color w_1 and w_2 black, and we mark v_2 as depleted. Whichever vertex Staller plays, it is now sufficient to color all its neighbors black to arrive at the first outcome.

Suppose finally that there exists a terminal walk $w_1 v_1 w_2 v_2$. Here, D plays v_1, we color w_1 and w_2 black, and we mark v_2 as depleted. Once again, no matter which vertex Staller plays, we arrive at the first outcome. □

Phase 3. This phase ends once it is Dominator's turn, no terminal walk exists, no white vertex with degree larger than 2 and part of a 2-walk exists, and no 5-walk exists. Recall that M is the largest set with a triple separation.

CLAIM 3.3. *D can play in a way such that the following holds:*
- $\beta(t_3) - \beta(t_2) \geq 3T_3/2$.
- $4(\delta(t_3) - \delta(t_2)) + \chi(t_3) \geq T_3$.
- $\nu(t_3) \leq (\chi(t_3) + M)/2$.
- *All dominated vertices are black.*

Proof. Throughout this phase, we will construct an increasing sequence of sets $\emptyset = A_{t_2} \subset \cdots \subset A_{t_3}$ with a triple separation. As during the second phase, we will show that if at some point t it is Dominator's turn and the phase is not over, at least one of the following holds:
- D can play the next move so that $\beta(t+2) \geq \beta(t) + 3$, $\delta(t+2) \geq \delta(t) + 1$ and $\nu(t+2) \leq \nu(t)$.
- D can play the next two moves so that $\beta(t+4) - \beta(t) \geq 6$, $\delta(t+4) \geq \delta(t) + 1$, and $\nu(t+4) \leq \nu(t)$.
- D can play the next move so that $\beta(t+2) \geq \beta(t) + 3$, $\chi(t+2) \geq \chi(t) + 2$, and $\nu(t+2) \leq \nu(t) + 1$.
- D can play the next move so that $\beta(t+2) \geq \beta(t) + 3$, $\chi(t+2) \geq \chi(t) + 2$, $\nu(t+2) \leq \nu(t) + 2$, and $|A_{t+2}| \geq |A_t| + 2$.

Assuming that this is true, it is easy to deduce the claim inductively. In the following, all dominated white vertices are to be colored black immediately without explicit mention.

We now show that Dominator can always play as propositioned above. If there is a terminal walk, we instruct D to play as in the previous phase, leading to the first outcome.

Suppose next there was a white vertex u with three neighbors v_1, v_2, and v_3 and that v_1 has another white neighbor, which we call w_1. Because no terminal walk exists, v_2 and v_3 must both also have a second white neighbor, and we denote them by w_2 and w_3, respectively. Since no 2-circuit exists, w_1, w_2, and w_3 are distinct. As G has minimum degree at least 2, w_1, w_2 and w_3 must all have a second neighbor, and because no vertex has at least three white neighbors, at least two of them must be distinct. Without loss of generality, we can therefore assume that w_1 and w_2 have distinct neighbors x_1 and x_2, respectively. Note that x_1 and x_2 cannot be adjacent to u. In this setup from this setup we have to distinguish between a few different cases. For each case, we will only state the moves that D will play, and we leave it up to the reader to verify that we arrive at one of the four outcomes listed above.

Suppose first that x_1 and x_2 are both adjacent to w_3. In this scenario, D can play v_1. If S replies by playing a vertex that is not adjacent to w_2 or w_3, D can play x_2 in the next move, and we will be able to mark both v_3 and v_2 as depleted. Similarly, if S plays a vertex that is not v_2 but is adjacent to w_2 or w_3, we will be able to mark v_2 or v_3 as depleted immediately. This leaves open the possibility that S will reply by playing v_2. In this case, we mark x_1, x_2, and v_3 as dependent, noting that this increases σ by three and ν only by one.

Next, we assume that x_1 is adjacent to w_3 but x_2 is not. Since no terminal walk exists, x_2 must have a different white neighbor y. If $y = w_1$, the situation is isomorphic to that above as can be seen by relabeling w_1 as w_3 and vice versa. Assume therefore that $y \neq w_1$. Because y is not a leaf and no vertex has three white neighbors, it has a second neighbor z distinct from v_1, v_2, v_3, x_1, and x_2. Again, z must have a second
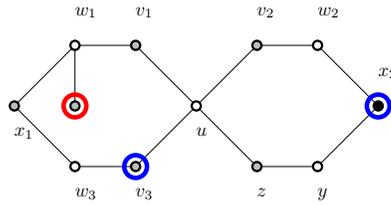
Fig. 1. *The case where $a = u$.*
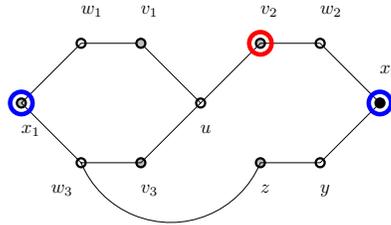
Fig. 2. *The case where $a = w_3$.*

white neighbor a different from y, and a must be one of w_1, w_3, or u. Indeed, if a were not, $azyx_2w_2v_2uv_3w_3x_1w_1v_1$ would be a 6-walk.

Suppose that $a = u$ (see Figure 1). In this case, D can play x_2. If S replies by playing a vertex that is adjacent to u, we can mark either v_2 or z as depleted. If S plays x_1, we can mark v_1, v_2, v_3, and z as dependent, thus increasing χ by at least three while increasing ν by at most one. If S does not play x_1 or a vertex that is adjacent to u, D can next play one of v_1 and v_3, and we mark both v_2 and z as depleted. The cases $a = w_1$ and $a = w_3$ are symmetric, so we will only consider $a = w_3$ (see Figure 2). D can still play x_2, and if S does not reply by playing a vertex that is adjacent to u or w_3, D will play v_3 next, and we can mark both v_2 and z as depleted. The cases of S playing a vertex adjacent to u or w_3 are again symmetric, so we may assume that S plays a vertex adjacent to u. If the vertex that S has played is not v_2, then we can mark v_2 as depleted, and if it is, D can play x_1 next, upon which we mark all of v_1, v_2, and z as depleted.

We are now left with the possibility that neither x_1 nor x_2 will be adjacent to w_3 and have other neighbors y_1 and y_2, respectively. We will assume first that y_1 and y_2 are distinct. In this case, they must have neighbors $z_1, z_2 \notin \{v_1, v_2, v_3, x_1, x_2\}$, respectively. Note that $z_1 \neq z_2$, as otherwise $w_3v_3uv_1w_1x_1y_1zy_2x_2w_2v_2$ would be a 6-walk. In this situation, D plays v_3 and, in the next move, depending on Staller's reply, plays one of x_1 and x_2 to mark v_1 or v_2 as depleted.

Assume that $y := y_1 = y_2$. Let $x_3 \notin \{x_1, x_2\}$ be the second neighbor of w_3. Once more, x_3 must have a second white neighbor a, which must be different from u. On the other hand, we must have $a \in \{w_1, w_2, y\}$, as otherwise there would be a 6-walk. If $a = w_1$, we have a situation that is isomorphic to an earlier one as can be seen by relabeling x_1 as z and x_3 as x_1 as well as swapping the labels of v_1 and w_1 with those of v_3 and w_3, respectively (see Figure 2). An analogous isomorphism can be found if $a = w_2$.

Finally, we have to consider the case where $a = y$ (see Figure 3). Here, D can play v_1. If S replies by playing a vertex that is not adjacent to y, D can play one of x_2 or x_3 next, marking x_1 as depleted in either case. If S replies by playing a
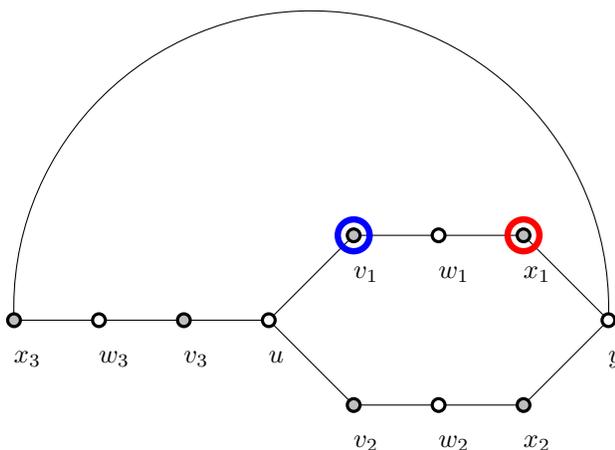
FIG. 3. *The case where $y = a$.*

vertex that is adjacent to y but is not x_1, we can mark x_1 as depleted immediately. Lastly, if S replies by playing x_1, we can mark v_2, v_3, x_2, and x_3 as dependent, set $B_u = \{v_1, v_2, v_3\}$, $B_y = \{x_1, x_2, x_3\}$, and let $A_{t+2} = A_t \cup \{u, y\}$.

If neither a terminal walk nor a vertex like u exists, but the phase is not over, there must be a 5-walk $w_1 \dots v_5$. As $w_4 v_4 w_5 v_5$ is not a terminal walk, v_5 must have a second white neighbor w_6, and in fact, we must have $w_6 = w_1$. Indeed, because w_2, w_3, and w_4 have only two neighbors, w_6 must be distinct from them. If we also had $w_6 \neq w_1$, w_6 would have another neighbor $v_6 \notin \{v_1, \dots v_5\}$, and we would have found a 6-walk, which cannot be.

Hence, $w_1 \dots v_5$ is a 5-circuit and D can play v_1. If S replies by playing v_2, D plays v_4 next, and we mark v_3 and v_5 as depleted. If S replies by playing v_3 or v_4, we can mark v_2 or v_5, respectively, as depleted immediately. Lastly, if S plays neither of these vertices, w_3 and w_4 will still be white after Staller's move and D can play v_3, after which we mark v_2 as depleted.   □

At the end of Phase 3, no terminal walk, no 5-walk, and no white vertex with degree larger than 2 and part of a 2-walk exist, and thus the remaining white vertices must lie in very specific configurations, as the following lemma shows.

LEMMA 3.4. *Once Phase 3 ends, every vertex with two white neighbors lies inside a 3- or 4-circuit.*

Proof. Suppose a vertex v_1 has two white neighbors w_1 and w_2. Because w_2 is not a leaf, w_2 has an additional neighbor v_2, and since no terminal walk or 2-circuit exists, v_2 has another white neighbor w_3 which must in turn have a second neighbor $v_3 \neq v_1$. Repeating this argument, we know that v_3 has a second white neighbor w_4 that has itself a second neighbor $v_4 \neq v_2$. If $w_4 = w_1$ (and consequently $v_4 = v_1$), v_1 is in in a 3-circuit, so we will assume that $w_4 \neq w_1$. Repeating the argument again, v_4 has a second white neighbor w_5 with a second neighbor v_5. We know that $w_5 \neq w_2$ since w_2 can only have two neighbors, and therefore we must have $w_5 = w_1$, as otherwise we would have found a 5-walk. Thus, v_1 is in a 4-circuit.   □

At this point, it is possible that there are vertices v, w, x such that w is undominated and adjacent to both v and x, and v is marked as dependent but x is not. In these situations, x cannot have a second white neighbor, as that would yield a terminal walk, and x cannot be marked as depleted by definition. Furthermore, by

marking all such x's that are neighbors of white neighbors w of dependent vertices v as dependent, we do not increase ν. After doing so, we can make the following observation.

LEMMA 3.5. *We have* $\nu(t_3) \leq \chi(t_3)$.

Phase 4. Phase 4 ends once the game is over.

CLAIM 3.6. *Let $C(t)$ be the set of vertices that have a single white neighbor after move t. Dominator can play in a way so that there exists nonnegative integers a and b such that the following hold:*

- *$T_4 \leq 2a + b + 1$.*
- *$\beta(t_4) - \beta(t_3) \geq 3a + b$.*
- *$\delta(t_4) - \delta(t_3) \geq a + b - \nu(t_3)$.*
- *For all $t \geq t_3$, $C(t) \subset C(t_3)$*
- *No new vertices are marked as dependent.*
- *Every vertex with two white neighbors lies inside a 3- or 4-circuit.*

Note that the last point holds at the start of the phase by Lemma 3.4.

Proof. We denote by b the number of moves in which a vertex in $C(t_3)$ is played. Whenever either player does so, we color its unique white neighbor w black. Because w is not a leaf, it has a second neighbor x. If x is marked as dependent, ν has certainly decreased by one. If x is not marked as dependent, we can mark it as depleted. Overall, the number of depleted vertices introduced from such moves must be at least $b - \nu(t_3)$.

We can now give Dominator's strategy for this phase. If $C(t_3) \neq \emptyset$, then on the first move there has to exist a vertex v with a single white neighbor w. After D plays v, we can color w black. Note also that $C(t_3 + 1) \subset C(t_3)$. If $C(t_3) = \emptyset$ but the game is still ongoing, D plays an arbitrary vertex with two white neighbors, and we do not color or mark any vertices.

From here on out Dominator plays reactively depending on Staller's previous move. We distinguish the following cases:

- S plays a vertex v_i inside a 4-circuit $w_1 \ldots v_4$, and without loss of generality, $i = 1$. After D responds by playing v_3, we color all of $w_1, \ldots, w_4$ black and mark v_2 and v_4 as depleted. Because all of $w_1, \ldots, w_4$ have degree 2, $C(t + 2) = C(t)$. We have $\beta(t + 2) = \beta(t) + 4$ and $\delta(t + 2) = \delta(t) + 2$.
- S plays a vertex v_i inside a 3-circuit $w_1 \ldots v_3$, and without loss of generality, $i = 1$. After D responds by playing v_2, we color w_1, w_2, and w_3 black and mark v_2 and v_4 as depleted. Again, $C(t + 3) = C(t)$. We have $\beta(t + 2) = \beta(t) + 3$ and $\delta(t + 2) = \delta(t) + 1$.
- S plays a vertex in $v \in C(t)$. If $C(t + 1)$ contains another vertex u, D plays it and we color its white neighbor black as well. If $C(t + 1)$ is empty but the game is not over, D plays an arbitrary vertex with two white neighbors, and we do not color or mark any vertex. Note that this situation can occur at most once and only if $C(t_3) \neq \emptyset$.

Denoting the combined number of moves played by Staller of the first and second types by a, we obtain the claim. □

Bounding the number of moves. Let $T = t_4 = T_1 + T_2 + T_3 + T_4$. To prove Theorem 1.3, it is enough to show that $T \leq 5(n + 1)/7$. We will do so by combining the bounds on β, δ, and χ that we have established during the previous sections. As mentioned before, we have two principal angles of attack. First, we know that the

number $\beta(t_4)$ of black vertices at the end of the game is at most n. Summing the lower bounds on the number of vertices that have been colored black during each of the phases, we get that

$$(1) \qquad \frac{\beta(t_1)}{6} + \frac{5T_1}{3} + \frac{3T_2}{2} + \frac{3T_3}{2} + 3a + b \leq \beta(t_4) \leq n.$$

Second, we can divide the vertices up into those that have been played and those that have not been played. Of the former kind there are exactly T many. The set of not-played vertices will include all $\delta(t_4)$ depleted vertices and at least $\chi(t_4)$ dependent vertices. Therefore, we have

$$(2) \qquad T + \delta(t_4) + \chi(t_4) \leq n.$$

By the bounds on δ and χ we observed in Claims 3.2, 3.3, and 3.6 and the monotonicity of χ, we have

$$(3) \qquad \delta(t_4) + \chi(t_4) \geq a + b - \nu(t_3) + \frac{T_3}{4} + \frac{3\chi(t_3)}{4} + \frac{T_2}{4}.$$

Adding the bounds $\nu(t_3) \leq \chi(t_3)$ from Lemma 3.5 and $\nu(t_3) \leq (\chi(t_3) + M)/2$ from Claim 3.3, we get $\nu(t_3) \leq 3\chi(t_3)/4 + M/4$. Inserting this bound and $\beta(t_1) \geq 2M$ from Claim 3.1 into (3), we get

$$(4) \qquad \delta(t_4) + \chi(t_4) \geq a + b + \frac{T_2 + T_3}{4} - \frac{\beta(t_1)}{8}.$$

Adding (1) to 2/3 times (2), we obtain

$$\frac{\beta(t_1)}{6} + \frac{7T_1}{3} + \frac{13(T_2 + T_3)}{6} + \frac{2(T_4 + \delta(t_4) + \chi(t_4))}{3} + 3a + b \leq \frac{5n}{3}.$$

Combining this with (4), we get

$$\frac{\beta(t_1)}{24} + \frac{7T_1}{3} + \frac{7(T_2 + T_3)}{3} + \frac{2T_4 + 11a + 5b}{3} \leq \frac{5n}{3}.$$

However, $\frac{\beta(t_1)}{24}$ is nonnegative, and by Claim 3.6 we have $T_4 \leq 2a + b + 1$, which gives $7T/3 - 5/3 \leq 5n/3$. It follows that $T \leq 5(n+1)/7$, as desired.

4. Proof of Theorem 1.2. Suppose towards a contradiction that Theorem 1.2 were false. For the entirety of this section, we fix a counterexample G that is minimal with respect to the number of its edges. As before, let n be the number of vertices of G.

4.1. Simplifications. Before we describe the actual strategy for Dominator, we will show that we may make a few simplifying assumptions about the structure of G.

LEMMA 4.1. *No parent in G has more than one leaf.*

Proof. Suppose G has a parent x with two leaves y and z. By minimality of n, D must have a strategy Δ for the game on $G - z$ to end in $3n/4$ moves. Since both y and z can never be played in the game on G, D can simply follow Δ for the game on G, pretending when S plays the vertex z that they played y instead. □

LEMMA 4.2. *Let u be a parent in G with a leaf w and another neighbor v. Then $G - uv$ has an isolated edge.*

Proof. We know that $G-uv$ cannot have an isolated vertex as v cannot be another leaf of u, and we assume for contradiction that $G - uv$ has no K_2 component either. By minimality of G, we know that D has a strategy Σ to play for a totally dominating set with at most $3n/4$ vertices in $G - uv$. We claim that this must be true in G as well.

Indeed, the only way D could fail to apply Σ in G is if S plays a vertex that would be forbidden in $G - uv$ because all its neighbors are already totally dominated. The only vertices that have additional neighbors in G are u and v, but u can always be played because of its leaf. If, on the other hand, S can play v in G but not in $G - uv$, we must have, first, that w has not yet been played and, second, that all neighbors of v except u have already been otherwise totally dominated. But then D may pretend that S plays w instead of v as this is still allowed in $G - uv$, and by doing so D can play according to Σ for the rest of the game.   □

In the rest of this section, we will make the following distinction between leaves. A *leaf of type A* is a leaf whose parent is not adjacent to another parent, and a *leaf of type B* is a leaf whose parent is adjacent to at least one other parent.

Let x be a leaf of type B, and suppose that the parent u of x is adjacent to $k \geq 1$ other parents $v_1, \ldots, v_k$ with leaves $y_1, \ldots, y_k$, respectively. Then we remark that by Lemma 4.2, the component of G containing u consists only of $C = G[\{u, x, v_1, y_1, \ldots, v_k, y_k\}]$.

On the other hand, if x is a leaf of type A, then the parent y of x has exactly one other neighbor z. By Lemma 4.1, z is not a leaf, and by definition of a leaf of type A, z is not a parent. Abusing language, we say that z is the *grandparent* of both x and y. Let L denote the set of leaves (of either type), let P denote the set of parents, and let F denote the set of grandparents. Furthermore, let $R = V(G) \setminus (L \cup P \cup F)$.

As in the minimum-degree-at-least-2 case, the strategy is divided into consecutive *phases,* each starting after the last one ends by meeting a predefined end condition. We track the number of moves played so far by a variable t. There are going to be six phases, and for each $i \in [6]$, t_i is the last move before the end of phase i, and we let $t_0 = 0$. It can happen that phases are skipped because their end conditions are met from the start of the phase, and in this case we have $t_i = t_{i-1}$. For each $i \in [6]$, let $T_i = t_i - t_{i-1}$.

Phase 1. The first phase begins at the start of the game and ends once it is Dominator's turn, no grandparent is adjacent to two or more white parents, and no parent is adjacent to one or more white parents.

CLAIM 4.3. *D can play in a way such that there exists some nonnegative integers r_F and r_P such that the following hold:*
  - *$\beta(t_1) \geq \frac{3}{2}T_1$.*
  - *$\delta(t_1) \geq 2r_F + r_P$.*
  - *$r_F + r_P \geq T_1/2$.*
  - *$\lambda(t_1) \leq \delta(t_1) + T_1/2$.*
  - *No vertex has been marked as dependent.*
  - *Each leaf of type B has been played or marked as depleted.*
  - *There are at most $L - r_P$ undominated leaves left.*

Proof. For as long as possible, D plays grandparents v with two white neighbors in P, and we color all white parents of v in P black and mark their leaves as depleted. If no such vertex v exists, then D plays parents w with at least one white neighbor u in P, and we color all white neighbors of w black and mark the leaf of u as depleted.

With regard to moves by Staller, whichever vertex z they play, we color all neighbors of z black. As z must have had at least one undominated neighbor before Staller's move, we color at least three vertices black every two moves, and the first part follows.

Let r_F and r_P be, respectively, the number of times that D plays a grandparent and a parent during Phase 1. The second part of the claim holds because we mark at least 2 or 1 leaves as depleted whenever D plays a grandparent or parent, respectively. As D makes at least $T_1/2$ moves, the third part follows immediately. The fourth point follows from the fact that D never plays a leaf during this phase and S plays at most $T_1/2$ moves.

Once the phase is over, we mark all leaves whose parents are colored black as depleted, from which the last point follows. Indeed, if a leaf y of type B has neither been played nor marked as depleted, then its parent u is white. As y is a leaf of type B, u must have a neighbor in P, which we call v, and since u is white, we know that v has not been played. Hence v has a white neighbor in P, which contradicts that Phase 1 has ended. □

Phase 2. Phase 2 ends once no grandparent is adjacent to a white parent and at least three white vertices overall and it is Dominator's turn.

CLAIM 4.4. *D can play in a way such that the following hold:*
- *No vertex has been marked as dependent.*
- $\lambda(t_2) - \lambda(t_1) \leq T_2$.
- $\beta(t_2) - \beta(t_1) \geq 2T_2$.
- $\delta(t_2) - \delta(t_1) \geq T_2/2$.

Proof. If the phase is ongoing and D has to make a move, they play a grandparent u that is adjacent to a white parent and at least three white vertices overall. After they have done so, we color all white neighbors of v black. One of these neighbors is a parent, and we mark its leaf as depleted.

We handle Staller's moves as in the first phase and leave it to the reader to verify the claimed bounds. □

Phase 3. During this phase, we will decide after each move of Staller whether or not to set a *reaction flag*. The phase ends when it is Dominator's turn, the reaction flag is not set, there exists no parent so that both its leaf and grandparent are white and there exists no grandparent that is adjacent to a white parent and a second white vertex.

CLAIM 4.5. *D can play in a way such that there exist some nonnegative integers $m_1, m_2, m_3, p,$ and q such that the following hold:*
- *No leaf has been marked as dependent.*
- $\beta(t_3) - \beta(t_2) \geq \frac{3}{2}T_3$.
- $\delta(t_3) - \delta(t_2) = m_1 + m_3 + q$.
- $\chi(t_3) \geq m_2$.
- $\lambda(t_3) - \lambda(t_2) = m_1 + m_2 + m_3 + q$.
- $T_3/2 = m_3 + p + q$.
- *There are at most $L - p - r_P$ undominated leaves left.*

Proof. Throughout this phase, before each move by Staller we color all white totally dominated vertices black. In particular, since a leaf can only be totally dominated if its parent has been played, a grandparent of a black leaf must be black.

If it is Dominator's turn and the reaction flag is not set, D will play a parent with a white leaf and a white grandparent where possible, after which we color both

the leaf and the grandparent black. If no such parent exists and the phase is still ongoing, there must be a grandparent u that is adjacent to a white parent x and some other white neighbor. Then D plays u, and we color the parent as well as the other neighbor black and mark the leaf of x as depleted. For those cases where the reaction flag is set, we instruct D to play dynamically depending on Staller's previous move.

If S plays any vertex that is not a leaf, we do not set the reaction flag and simply color all its white neighbors black. Whenever S does play a leaf with a parent x and a grandparent y, we categorize this move into one of three different types as follows:

- y does not have any white neighbor other than x.
- y has one other white neighbor $z \notin P$, and z has no other neighbor having another white neighbor.
- y has one white neighbor z, and z has at least one neighbor u with another white neighbor v.

Due to the end conditions of the previous phases, no grandparent is adjacent to more than one white parent, so the above list is exhaustive. Let m_1, m_2, m_3 be the respective number of moves of the above listed types played by S during Phase 3. Furthermore, we denote by p the number of times that the reaction flag was not set and D played a parent, and by q the number of times the reaction flag was not set and D played a grandparent.

If Staller plays a leaf, we always color its parent black and proceed depending on the type $m_1, m_2,$ or m_3 of the leaf. If the leaf has type m_1, we can mark its grandparent y as depleted (note that y cannot have been played before, as that would make it illegal for S to play the leaf) and do not set the reaction flag.

Similarly, if S plays a leaf of type m_2, we consider the unique white neighbor $z \notin P \cup L$ of its grandparent y. Since z is not a leaf, it must have some other neighbor u, and by definition of type m_2, u has no other white neighbor than z. We can therefore mark y as dependent and u as well if it has not been marked as such already. In either case, we increase χ by at least 1. We do not set the reaction flag.

If S plays a leaf of type m_3, we do set the reaction flag and instruct D to play their next move as follows. For Staller's move to have type m_3, the grandparent y must have a white neighbor $z \notin P \cup L$ which in turn must have a neighbor u with another white neighbor v. Here, D plays u and we can mark z as depleted.   □

Before moving on to the next phase, we make an important observation about the leaves of the graph. Let I denote the set of leaves that have not been played or marked as depleted. As documented in Claim 4.3, all such leaves must be of type A, and by definition of $\lambda(t)$, I has at least $|L| - \lambda(T_3) \geq |L| - m_1 - m_2 - m_3 - q - T_2 - \lambda(t_1)$ elements. Now let J be the set of grandparents of leaves in I, and let $C \subset J$ be those among them that have not been played or marked as depleted or dependent. We now mark all vertices in $I \cup C$ as dependent. This increases χ by at least $|I|$. Indeed, if the grandparent of a leaf $x \in I$ is in C, they share the parent of x as their only undominated neighbor. If the grandparent of x is not in I, either the parent of x is dominated or the grandparent of x was already marked as dependent at some earlier point. Therefore, we have the following.

LEMMA 4.6. *We have* $\chi(t_3) \geq |L| - m_1 - m_3 - q - T_2 - \lambda(t_1)$.

Phase 4. Phase 4 ends once it is Dominator's turn and no vertex has three or more white neighbors.

CLAIM 4.7. *D can play in a way such that the following holds:*
- $\beta(t_4) - \beta(t_3) \geq 2T_4$.

Proof. Dominator keeps playing vertices y with at least three white neighbors, and after Staller plays a vertex z with at least one white neighbor, we color all neighbors of z and y black. The claimed bound follows.   □

Phase 5. Phase 5 ends once it is Dominator's turn and no vertex has two white neighbors.

LEMMA 4.8. *D can play in a way such that the following hold:*
- *No vertex that has been marked as dependent ceases to be marked as dependent.*
- $\beta(t_5) - \beta(t_4) \geq \frac{3}{2}T_5$.
- $4(\delta(t_5) - \delta(t_4)) + 2(\chi(t_5) - \chi(t_4)) \geq T_5$.

Proof. Similarly to Phases 2 and 3 of the proof of Theorem 1.3, we will show that whenever it is Dominator's turn, at least one of the following hold.
- D can play the next move so that

$$\beta(t+2) - \beta(t) \geq 3 \qquad \text{and} \qquad \delta(t+2) - \delta(t) + \chi(t+2) - \chi(t) \geq 1.$$

- D can play the next two moves so that

$$\beta(t+4) - \beta(t) \geq 6 \qquad \text{and} \qquad \delta(t+4) - \delta(t) \geq 1.$$

Once we have established this dichotomy, the claim follows again from a simple inductive argument.

Note that while G can now have leaves, no leaf can be part of any k-walk for $k \geq 2$ at this point. Indeed, if we take any k-walk $w_1 \ldots v_k$, only w_1 and v_k can possibly be leaves. But if w_1 were a leaf, then v_1 would be a parent with a white leaf and a white neighbor, which cannot exist after the end of Phase 3. Likewise, if v_k were a leaf, v_{k-1} would be either a parent or a grandparent with a white parent and a second white neighbor, which cannot exist after the end of Phase 3 either.

For this reason, D can play first as in Phase 2 of section 3 and then as in the Phase 3 of section 3 until every vertex with two white neighbors lies in a 3- or 4-circuit (see Lemma 3.4).

Suppose now that it is Dominator's turn and there is a 3-circuit $w_1 \ldots v_3$. Here, D can play v_1, and we color w_1 and w_2 black and mark v_2 and v_3 as dependent. Should there be instead a 4-circuit $w_1 \ldots v_k$, D can still play v_1, but we need to consider two different continuations. If S plays a vertex that is adjacent to w_3 or w_4 next, without loss of generality adjacent to w_3, we color w_1, w_2, and w_3 black and mark v_3 and v_4 as dependent. If S plays a vertex that is not adjacent to either of them, D can play v_3 next, after which we color w_1, w_2, w_3, and w_4 black and mark v_2 and v_4 as depleted.   □

Phase 6. Phase 6 ends once no undominated vertex is left. From this point on, we color all dominated vertices black immediately. Also, we mark all vertices that have one white neighbor as dependent.

CLAIM 4.9. *Regardless of what moves either player makes, we have $\beta(t_6) - \beta(t_5) \geq T_6$ and $T_6 \leq \chi(t_5) + L - p - r_P$.*

Proof. Every vertex that is played during this phase either is dependent or dominates a leaf. By Claim 4.5, there are at most $L - p - r_P$ undominated leaves left at

the start of the phase, and since each neighbor of dependent parents can only be dominated once and no new dependent vertices are introduced, at most $\chi(t_5)$ dependent vertices can be played. $\square$

4.2. Bounding the number of moves. Let $T = T_1 + \cdots + T_6$. To prove Theorem 1.2, it is enough to show that $T \leq 3n/4$. We will do so by combining the bounds on β, δ, λ, and χ we have established during the previous sections.

As in section 3, we sum the lower bounds on the number of new black vertices from each of the phases to obtain

$$(5) \qquad 3(T_1 + T_3 + T_5)/2 + 2(T_2 + T_4) + T_6 \leq \beta(T) \leq n.$$

We also divide the vertices up into those that have been played and those that have not been played, which gives

$$(6) \qquad T + \delta(T) + \chi(T) \leq n.$$

Moreover, we combine our bounds for the increases of δ and χ during the different phases. For δ, we obtain

$$(7) \qquad \delta(T) \geq \lambda(t_1) - T_1/2 + T_2/2 + m_1 + m_3 + q + \delta(t_5) - \delta(t_4)$$

or, by using $m_3 + p + q = T_3/2$ from Claim 4.5 and $\delta(t_1) \geq 2r_F + r_P$ from Claim 4.3,

$$(8) \qquad \delta(T) \geq 2r_F + r_P + T_2/2 + T_3/2 - p + \delta(t_5) - \delta(t_4).$$

Using Lemma 4.6, we have

$$(9) \qquad \chi(T) \geq |L| - m_1 - m_3 - q - T_2 - \lambda(t_1) + \chi(t_5) - \chi(t_4).$$

Recall also that we have from Claim 4.9 the inequality

$$(10) \qquad \chi(T) \geq T_6 - |L| + p + r_P.$$

Adding up the inequalities (7), (8), (9), and (10) and using that $2(r_P + r_F) \geq T_1$, we obtain

$$2\delta(T) + 2\chi(T) \geq T_1/2 + T_3/2 + T_6 + 2(\delta(t_5) - \delta(t_4)) + \chi(t_5) - \chi(t_4).$$

By Lemma 4.8, $2(\delta(t_5) - \delta(t_4)) + \chi(t_5) - \chi(t_4) \geq T_5/2$, so we must have

$$(11) \qquad 2\delta(T) + 2\chi(T) \geq T_1/2 + T_3/2 + T_5/2 + T_6.$$

To get a bound on T, we linearly combine four times (6) together with two times (5) to obtain

$$7T_1 + 8T_2 + 7T_3 + 8T_4 + 7T_5 + 6T_6 + 4\delta(T) + 4\chi(T) \leq 6n,$$

and by inserting (11) into this we arrive at $8T \leq 6n$, which completes the proof.

5. Concluding remarks and open problems. At first glance, it is perhaps surprising that the bounds in Theorem 1.2 and Theorem 1.3 are so close to clean fractions of n, rather than having larger error terms. However, this turns out to be part of a pattern in the sense that for a large class of variants of the domination game, any bound of the form $cn + o(n)$ can always be replaced by the bound $cn + 1$.

To make this precise, we consider a variant of the total domination game that can be seen as a generalization of most other variants that have been studied. The *transversal game* was defined in [10] as a game played on a hypergraph H where the two players, Edge-hitter and Staller, alternately select a vertex from H, with the rule that each newly selected vertex must *hit*, i.e., be contained in, at least one edge that does not intersect the set of previously selected vertices. The game ends when the set of selected vertices becomes a transversal in H, i.e., every edge of H intersects the set of selected vertices. Edge-hitter aims to end the game in as few moves as possible, whereas Staller wants the game to last as long as possible. The *game transversal number* (respectively, *Staller-start game transversal number*) $\tau_g(H)$ (respectively, $\tau_g'(H)$) of H is the number of moves played if Edge-hitter (respectively, Staller) starts the game and both players play optimally.

LEMMA 5.1. *Let $\mathcal{C}$ be a set of hypergraphs closed under taking multiple disjoint copies of a hypergraph in $\mathcal{C}$, and suppose that there exists $c > 0$ such that for every hypergraph $H \in \mathcal{C}$ on n vertices, $\tau_g(H) \leq (c + o(1))n$. Then we also have, for every $H \in \mathcal{C}$ on n vertices, $\tau_g(H) \leq cn + 1$.*

Proof. For a given hypergraph H and a set of vertices A in H, let $T(H; A)$ be the *transversal game in H starting from A*, in which the players play the Edge-hitter-start transversal game starting with the set A already selected. Likewise, let $T'(H; A)$ be the analogous game in which Staller starts. We define $\tau_g(H; A)$ and $\tau_g'(H; A)$ to be the number of selected vertices excluding A at the end of the respective games $T(H; A)$ and $T'(H; A)$ if both players play optimally.

Note that for any H and A, we have $\tau_g(H; A) \leq \tau_g'(H; A) + 1$. Indeed, suppose that D has the legal move v in the game $T(H; A)$. If $\tau_g'(H; A \cup \{v\}) = 0$, then $\tau_g(H; A) = 1$ and there is nothing left to show. Otherwise, there must be a vertex w such that $\tau_g(H; A \cup \{v, w\}) \geq \tau_g(H; A) - 2$. But by a straightforward generalization of the *continuation principle* (see [23]), we have $\tau_g(H; A \cup \{v, w\}) \leq \tau_g(H; A \cup \{w\})$, which is in turn at most $\tau_g'(H; A) - 1$.

In order to arrive at a contradiction, assume that there exist $e > 1$ and $H \in \mathcal{C}$ on n vertices such that $\tau_g(H) > cn + e$. Let H^m be the graph consisting of m disjoint copies $H_1, \ldots, H_m$ of H. For a set $A \subset V(H^m)$, let $k(A)$ be the number of copies H_i, in which there is a playable vertex in the game $T(H_i; A \cap V(H_i))$. Note that $k(A)$ is decreasing in A. Abbreviating $\tau_g(H_i; A \cap V(H_i))$ to $\tau_g(H_i; A)$, we say that a set A is *good* if

$$\tau_g(H^m; A) \geq -k(A) + \sum_{j \in [m]} \tau_g(H_i; A).$$

If $\emptyset$ were good, we would have

$$\tau_g(H^m) = \tau_g(H^m; \emptyset) \geq m\tau_g(H; \emptyset) - m > cmn + (e - 1)m.$$

Since H^m has mn vertices, this contradicts our assumption if m is large enough. We may therefore assume that for some sufficiently large fixed m, $\emptyset$ is bad, and we consider a bad set A of maximal size in $V(H^m)$. Clearly, $\tau_g(H^m; A) \geq 2$ and there must be a vertex v such that for $A' = A \cup \{v\}$, we have $\tau_g'(H^m; A') = \tau_g(H^m; A) - 1$. Let $i \in [m]$ be such that $v \in V(H_i)$. If $\tau_g'(H_i; A') > 0$, then there must be a vertex

$w \in V(H_i)$ such that for $A'' = A' \cup \{w\}$, $\tau_g(H_i, A'') = \tau_g(H_i, A) - 2$. Since A'' is good and A is bad, we have

$$\tau_g(H^m; A'') \geq -k(A'') + \tau_g(H_i, A'') + \sum_{\substack{j \in [m] \\ j \neq i}} \tau_g(H_j; A)$$

$$\geq -k(A) - 2 + \sum_{j \in [m]} \tau_g(H_j; A)$$

$$> \tau_g(H^m; A) - 2,$$

but as $\tau_g(H^m; A'') \leq \tau'_g(H^m; A') - 1$, this contradicts $\tau'_g(H^m; A') = \tau_g(H^m; A) - 1$.

If we have instead that $\tau'_g(H_i; A') = 0$, then $k(A) = k(A') + 1$. Let $r \in [m]$ be such that $\tau'_g(H_r, A') > 0$. We know that $\tau'_g(H_r, A') \geq \tau_g(H_r, A') - 1$, and thus there must be a $w \in V(H_r)$ such that for $A'' = A' \cup \{w\}$, we have $\tau_g(H_r, A'') \geq \tau_g(H_r, A') - 2$. But since A is bad and A'' is good, we have

$$\tau_g(H^m; A) < -k(A) + \sum_{j \in [m]} \tau_g(H_j; A)$$

$$= -k(A') + \sum_{j \in [m]} \tau_g(H_j; A')$$

$$\leq -k(A'') + 2 + \sum_{j \in [m]} \tau_g(H_j; A'')$$

$$\leq \tau_g(H^m; A'') + 2,$$

contradicting again $\tau'_g(H^m; A') = \tau_g(H^m; A) - 1$. $\qquad\square$

We remark that the remaining error term of 1 can be removed if the bound $\tau_g(H) \leq (c + o(1))n$ can be achieved in a version of the transversal game where Staller is allowed to forfeit their move whenever they wish. To see how Lemma 5.1 relates to other variants of the domination game, we take the total domination game as an example. The open neighborhood hypergraph of a graph G is defined as the hypergraph H_G with vertex set $V(H_G) = V(G)$ and hyperedge set $E(H_G) = \{N_G(x) | x \in V(G)\}$ consisting of the open neighborhoods of vertices in G. As remarked in [10], the total domination game played on a graph G can be seen as a special instance of the transversal game being played on the open neighborhood hypergraph H_G of G. Since the class $\{H_G : G \text{ is a graph}\}$ is closed under taking disjoint copies, Lemma 5.1 also applies to the total domination game.

As mentioned earlier, Theorem 1.2 is sharp, as the example of disjoint copies of P_4 or P_8 shows. However, it is unclear whether Theorem 1.2 is asymptotically sharp if we add the condition that G is a connected graph, and Brešar, Henning, and Klavžar in [4] raised the following problem.

PROBLEM 5.2. *Determine all connected graphs satisfying $\gamma_{tg}(G) = \frac{3}{4}n$.*

To date, the largest graph G known to satisfy $\gamma_{tg}(G) = \frac{3}{4}n$ is the tree on 12 vertices shown in Figure 4, which was first presented in [4].

With respect to Theorem 1.3, we first remark that the strategy described in this paper cannot give a better bound than $\frac{5}{7}n$. Indeed, if G is the union of k copies of C_7, and D follows the strategy laid out in section 3, then it is easy to see that S has a strategy so that the game will last exactly $5k$ moves. However, even though $\gamma_{tg}(C_7) = 5$, for the disjoint union H_k of k copies of C_7, we have that $\gamma_{tg}(H_k) = 4k+1$, which provokes the question of whether Theorem 1.3 is asymptotically tight. More
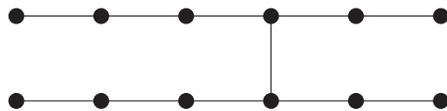
FIG. 4. *The largest known connected graph G for which $\gamma_{tg}(G) = \frac{3}{4}n$.*

generally, we ask the following question, which is analogous to the 1/2-conjecture made in [12].

QUESTION 5.3. *What is the smallest constant $c > 0$ such that for any graph G with minimum degree at least 2 we have $\gamma_{tg}(G) \leq cn + 1$?*

Note that if G is a disjoint union of triangles, then $\gamma_{tg}(G) = 2n/3$. Thus, c must be at least $\frac{2}{3}$, and it seems plausible that $c = \frac{2}{3}$ is in fact the answer to Question 5.3. We remark further that by Lemma 5.1, it is sufficient to show the bound $\gamma_{tg}(G) \leq (c + o(1))n$.

The domination game itself was introduced by Brešar, Klavžar, and Rall in [5]. The difference between the standard and the total domination game is that a vertex selected by the players *dominates* itself, whereas is does not *totally* dominate itself. The *game domination number $\gamma_g(G)$* of G is defined as the number of vertices selected when Dominator starts the game and both players play optimally. One of the central topics related to the domination game is the 3/5-conjecture posed by Kinnersley, West, and Zamani [23].

CONJECTURE 5.4. *If G is an isolate-free graph of order n, then $\gamma_g(G) \leq \frac{3}{5}n$.*

A considerable amount of research has been done on the domination game and Conjecture 5.4 in particular [23, 6, 27, 24, 19, 16, 8, 12, 28]. There are also many other variants of the domination game, all of which are currently being studied, and we want to bring some of them to the attention of the reader. For instance, there are the connected [2, 9], fractional [15], and disjoint [14] domination games, the Z-, L-, LL-games [3], the maker-breaker domination game [17], and the maker-breaker total domination game [18]. For games played on hypergraphs, there is the aforementioned transversal game [10, 11], but also the domination game itself can be played on hypergraphs [13]. All these variants have been inspired by research on the domination game, but before the latter was first studied in 2010, there had already been research done on further optimization games that we want to reference due to their close similarity to the domination game. These are the oriented domination game [1], the competition-independence game, and the enclaveless game [25, 26]. We refer the reader to [4] for a general survey about domination games.

REFERENCES

[1] N. ALON, J. BALOGH, B. BOLLOBÁS, AND T. SZABÓ, *Game domination number*, Discrete Math., 256 (2002), pp. 23–33, https://doi.org/10.1016/S0012-365X(00)00358-7.
[2] M. BOROWIECKI, A. FIEDOROWICZ, AND E. SIDOROWICZ, *Connected domination game*, Appl. Anal. Discrete Math., 13 (2019), pp. 261–289, https://doi.org/10.2298/AADM171126020B.

[3] B. Brešar, C. Bujtás, T. Gologranc, S. Klavžar, G. Košmrlj, T. Marc, B. Patkós, Z. Tuza, and M. Vizer, *The variety of domination games*, Aequationes Math., 93 (2019), pp. 1085–1109, https://doi.org/10.1007/s00010-019-00661-w.

[4] B. Brešar, M. A. Henning, S. Klavžar, and D. F. Rall, *Domination Games Played on Graphs*, Springer, 2021, https://doi.org/10.1007/978-3-030-69087-8.

[5] B. Brešar, S. Klavžar, and D. F. Rall, *Domination game and an imagination strategy*, SIAM J. Discrete Math., 24 (2010), pp. 979–991, https://doi.org/10.1137/100786800.

[6] C. Bujtás, *Domination game on forests*, Discrete Math., 338 (2015), pp. 2220–2228, https://doi.org/10.1016/j.disc.2015.05.022.

[7] C. Bujtás, *On the game total domination number*, Graph. Combin., 34 (2018), pp. 415–425, https://doi.org/10.1007/s00373-018-1883-y.

[8] C. Bujtás, *General upper bound on the game domination number*, Discrete Appl. Math., 285 (2020), pp. 530–538, https://doi.org/10.1016/j.dam.2020.06.018.

[9] C. Bujtás, P. Dokyeesun, V. Iršič, and S. Klavžar, *Connected domination game played on Cartesian products*, Open Math., 17 (2019), pp. 1269–1280, https://doi.org/10.1515/math-2019-0111.

[10] C. Bujtás, M. A. Henning, and Z. Tuza, *Transversal game on hypergraphs and the $\frac{3}{4}$-conjecture on the total domination game*, SIAM J. Discrete Math., 30 (2016), pp. 1830–1847, https://doi.org/10.1137/15M1049361.

[11] C. Bujtás, M. A. Henning, and Z. Tuza, *Bounds on the game transversal number in hypergraphs*, Eur. J. Combin., 59 (2017), pp. 34–50, https://doi.org/10.1016/j.ejc.2016.07.003.

[12] C. Bujtás, V. Iršič, and S. Klavžar, *1/2-conjectures on the domination game and claw-free graphs*, Eur. J. Combin., 101 (2022), 103467, https://doi.org/10.1016/j.ejc.2021.103467.

[13] C. Bujtás, B. Patkós, Z. Tuza, and M. Vizer, *Domination game on uniform hypergraphs*, Discrete Appl. Math., 258 (2019), pp. 65–75, https://doi.org/10.1016/j.dam.2018.11.013.

[14] C. Bujtás and Z. Tuza, *The disjoint domination game*, Discrete Math., 339 (2016), pp. 1985–1992, https://doi.org/10.1016/j.disc.2015.04.028.

[15] C. Bujtás and Z. Tuza, *Fractional domination game*, Electron. J. Combin., 26 (2019), P4–3, https://doi.org/10.37236/8730.

[16] C. Bujtás, *On the game domination number of graphs with given minimum degree*, Electron. J. Combin., 22 (2015), P3.29, https://doi.org/10.37236/4497.

[17] E. Duchêne, V. Gledel, A. Parreau, and G. Renault, *Maker–breaker domination game*, Discrete Math., 343 (2020), 111955, https://doi.org/10.1016/j.disc.2020.111955.

[18] V. Gledel, M. A. Henning, V. Iršič, and S. Klavžar, *Maker–breaker total domination game*, Discrete Appl. Math., 282 (2020), pp. 96–107, https://doi.org/10.1016/j.dam.2019.11.004.

[19] M. A. Henning and W. B. Kinnersley, *Domination game: A proof of the 3/5-conjecture for graphs with minimum degree at least two*, SIAM J. Discrete Math., 30 (2016), pp. 20–35, https://doi.org/10.1137/140976935.

[20] M. A. Henning, S. Klavžar, and D. F. Rall, *Total version of the domination game*, Graphs Combin., 31 (2015), pp. 1453–1462, https://doi.org/10.1007/s00373-014-1470-9.

[21] M. A. Henning, S. Klavžar, and D. F. Rall, *The 4/5 upper bound on the game total domination number*, Combinatorica, 37 (2017), pp. 223–251, https://doi.org/10.1007/s00493-015-3316-3.

[22] M. A. Henning and D. F. Rall, *Progress towards the total domination game $\frac{3}{4}$-conjecture*, Discrete Math., 339 (2016), pp. 2620–2627, https://doi.org/10.1016/j.disc.2016.05.014.

[23] W. B. Kinnersley, D. B. West, and R. Zamani, *Extremal problems for game domination number*, SIAM J. Discrete Math., 27 (2013), pp. 2090–2107, https://doi.org/10.1137/120884742.

[24] N. Marcus and D. Peleg, *The Domination Game: Proving the 3/5 Conjecture on Isolate-Free Forests*, preprint, arXiv:1603.01181, 2016.

[25] J. B. Phillips and P. J. Slater, *An introduction to graph competition independence and enclaveless parameters*, Graph Theory Notes N.Y., 41 (2001), pp. 37–41.

[26] J. B. Phillips and P. J. Slater, *Graph competition independence and enclaveless parameters*, Congr. Numer., 154 (2002), pp. 79–100.

[27] S. Schmidt, *The 3/5-conjecture for weakly $S(K_{1,3})$-free forests*, Discrete Math., 339 (2016), pp. 2767–2774, https://doi.org/10.1016/j.disc.2016.05.017.

[28] L. Versteegen, *A proof of the 3/5-conjecture in the domination game*, Eur. J. Combin., 122 (2024), 104034, https://doi.org/10.1016/j.ejc.2024.104034.