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Lower Bound on Translative Covering Density of Tetrahedra

Yiming Li[1] · **Miao Fu**[2] · **Yuqin Zhang**[3]

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Abstract

In this paper, we present the first nontrivial lower bound on the translative covering density of tetrahedra. To this end, we show the lower bound, in any translative covering of tetrahedra, on the density relative to a given cube. The resulting lower bound on the translative covering density of tetrahedra is $1 + 1.227 \times 10^{-3}$.

Keywords Translative covering density · Tetrahedra · Cube

Mathematics Subject Classification 52C17 · 52B10 · 52C07

1 Introduction

More than 2,300 years ago, Aristotle claimed that *congruent regular tetrahedra can fill the whole space with neither gap nor overlap*. In modern terms, he claimed that regular tetrahedra of given size can form both a packing and a covering in $\mathbb{E}^3$ simultaneously. Unfortunately, this statement is wrong. Aristotle's mistake was discovered by Regiomontanus in the fifteenth century (see [16]). Then, two natural questions arose immediately: *What is the density of the densest tetrahedron packing and what is the*

Editor in Charge: János Pach

Yiming Li
xiaozhuang@tju.edu.cn

Miao Fu
miaofu@tju.edu.cn

Yuqin Zhang
yuqinzhang@tju.edu.cn

1 Center for Applied Mathematics, Tianjin University, Tianjin 300072, People's Republic of China

2 Systems Engineering Research Institute, CSSC Electronics Technology Co., Ltd, Beijing, People's Republic of China

3 School of Mathematics, Tianjin University, Tianjin 300354, People's Republic of China

density of the thinnest tetrahedron covering? In fact, the packing case was emphasized by Hilbert [12] as a part of his 18th problem. Since then, many scholars, including mathematicians, physicists, and chemical engineers have made contributions (mistakes as well) to tetrahedra packings. For the complicated history, we refer to [2, 16]. For packings and tetrahedron packings, we refer to [3, 11, 13, 19, 22, 25].

Covering is often regarded as a counterpart of packing. Let $\theta^c(\cdot)$, $\theta^t(\cdot)$ and $\theta^l(\cdot)$ denote the densities of the thinnest congruent covering, the thinnest translative covering and the thinnest lattice covering, respectively. Clearly,

$$1 \le \theta^c(\cdot) \le \theta^t(\cdot) \le \theta^l(\cdot).$$

In high dimensions, through the works of Bambah, Coxeter, Davenport, Erdős, Few, Watson, and in particular Rogers (see [18]), covering densities are much better understood than packing densities. For more details, we refer to [9, 17, 23]. Nevertheless, in low dimensions, little is known about covering compared to packing. For covering densities in the plane, we refer to [6, 14, 15, 20, 21, 24].

Now, we focus on results in $\mathbb{E}^3$. Let B denote the unit ball and let T denote the regular tetrahedron with unit edges. Except for the five types of parallelohedra which can translatively tile $\mathbb{E}^3$, the only known exact result is

$$\theta^l(B) = \frac{5\sqrt{5}\pi}{24},$$

which was discovered in 1954 by Bambah [1]. For tetrahedron coverings, several bounds have been achieved. In the lattice case,

$$\frac{2^{16}+1}{2^{16}} \le \theta^l(T) \le \frac{125}{63},$$

where the upper bound was discovered by Fiduccia et al. [7], Dougherty and Faber [5] and Forcade and Lamoreaux [8] in 1990s, and the lower bound was achieved by Xue and Zong [23] in 2018 by studying the volumes of generalized difference bodies. In 2022, Fu et al. [9] improved the lower bound to

$$\theta^l(T) \ge \frac{25}{18}.$$

In the congruent case, Conway and Torquato [4] obtained

$$\theta^c(T) \le \frac{9}{8}$$

by constructing a particular tetrahedron covering in 2006. It is surprising that, nothing nontrivial is known about $\theta^t(T)$ up to now.

In this paper, we prove the following result:

Theorem 1.1 *If $T + X$ is a translative covering of $\mathbb{E}^3$, then its density is at least $1 + \frac{\sqrt{2}}{1152}$. In other words, we have*

$$\theta^t(T) \geq 1 + \frac{\sqrt{2}}{1152} > 1 + 1.227 \times 10^{-3}.$$

2 Preliminaries

2.1 Proof Outline

Assume that $\mathbb{E}^3$ is covered by translates of the tetrahedron T with vertices $(1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1)$ and $(-1, -1, -1)$. In order to show that the density of this covering is more than $1 + 1.227 \times 10^{-3}$, it is sufficient to select a cube P and prove that no matter where a translate of this cube is placed, the covering density of the tetrahedra restricted to this 'cubical window' is more than $1 + 1.227 \times 10^{-3}$.

For technical purposes the cube P will have faces parallel to the coordinate planes. Here we say only that P will be large enough to contain every tetrahedron which overlaps another tetrahedron, which contains the cube's center. Select a member T' of the covering which contains the center of P. It turns out that for the proof we also need to identify a smaller polyhedron D' in cube P, so that already D' contains all such translates of T'.

Now consider a random translate of a P (together with D'). We will study the cluster of tetrahedra which overlap T'. We distinguish two cases:

Case 1. The cluster contains more than N tetrahedra (N will be specified later in the detailed proof).

Case 2. The cluster contains at most N tetrahedra.

Finally, let us explain how we get density bounds in the above two cases:

Conclusion in Case 1. In order to get a lower density bound we may assume that the region outside D' and inside P is single covered, and verify that if the total volume of the tetrahedra in the cluster is evenly spread out to cube P, like butter on a toast, the density will be more than $1 + 1.227 \times 10^{-3}$.

Conclusion in Case 2. In order to get a lower density bound we may assume that the entire cubical window P is single covered, except multiple covered parts of T'. These multiple covered parts are estimated. Finally, it is verified, that if the total volume of the parts overlapping tetrahedra T' is evenly spread out to cube P, like butter on a toast, the density will be more than $1 + 1.227 \times 10^{-3}$.

2.2 Geometric Lemmas

Let K be a convex body and define

$$D(K) = \{\mathbf{x} - \mathbf{y} : \mathbf{x}, \mathbf{y} \in K\}.$$

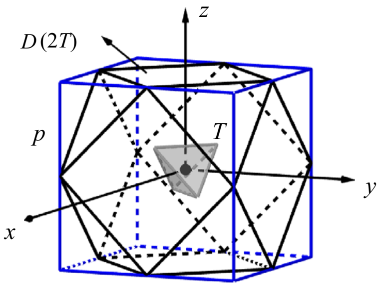

Fig. 1 The cuboctahedron $D(2T)$ contained in the cube P

Usually, we call $D(K)$ the *difference body* of K. Clearly $D(K)$ is a centrally symmetric convex set centered at the origin $\mathbf{o}$.

Example 2.1 (Groemer [10]). Let $2T$ denote the tetrahedron with vertices $(2, 2, -2)$, $(2, -2, 2)$, $(-2, 2, 2)$ and $(-2, -2, -2)$. It is well known that

$$D(2T) = \{(x, y, z) : \max\{|x|, |y|, |z|\} \le 4, |x| + |y| + |z| \le 8\},$$

i.e., a cuboctahedron with edge length $4\sqrt{2}$ and volume $\frac{1280}{3}$.

Without loss of generality, let P denote the cube (see Fig. 1) defined by

$$\{(x, y, z) : \max\{|x|, |y|, |z|\} \le 4\}. \tag{2.1}$$

Clearly,

$$D(2T) \subset P \text{ and } vol(P) = 512. \tag{2.2}$$

Lemma 2.2 *If* $\mathbf{o} \in T + \mathbf{x}$ *and* $(T + \mathbf{x}) \cap (T + \mathbf{y}) \ne \emptyset$, *then* $T + \mathbf{y} \subset D(2T)$.

Proof Since $\mathbf{o} \in T + \mathbf{x}$, $\mathbf{x} \in -T$. Then

$$T + \mathbf{x} \subset T - T = D(T).$$

Without loss of generality, suppose that $(T + \mathbf{x}) \cap (T + \mathbf{y}) = \mathbf{z}$. Since $\mathbf{z} \in T + \mathbf{x} \subset D(T)$ and $\mathbf{z} \in T + \mathbf{y}$, we have $\mathbf{y} \in -T + \mathbf{z} \subset -T + D(T)$ and therefore

$$T + \mathbf{y} \subset T - T + D(T) = D(2T).$$

The lemma is proved. □

Lemma 2.3 *If* $T \cap (T + \mathbf{x}) \ne \emptyset$, *then* $T + \mathbf{x} \subset 5T \cap -7T$.

Proof We claim that $-\frac{1}{3}\mathbf{u} \in T$ if $\mathbf{u} \in T$. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ denote the vertices of T. For any point $\mathbf{u} \in T$, we have

$$\mathbf{u} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4,$$

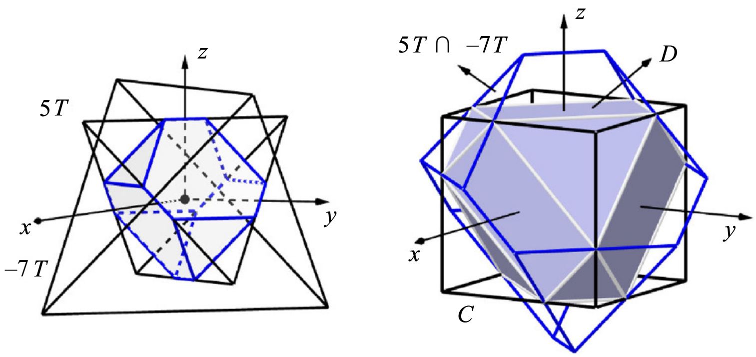

Fig. 2 $5T \cap -7T$ and D

where $\alpha_i \geq 0$ for all i and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$. Without loss of generality, suppose that $\alpha_4 = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Since $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{o}$, we have

$$\begin{aligned}-\frac{1}{3}\mathbf{u} &= -\frac{1}{3}\left(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4(-\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3)\right)\\ &= \frac{1}{3}(\alpha_4 - \alpha_1)\mathbf{v}_1 + \frac{1}{3}(\alpha_4 - \alpha_2)\mathbf{v}_2 + \frac{1}{3}(\alpha_4 - \alpha_3)\mathbf{v}_3.\end{aligned}$$

Since $\frac{1}{3}(\alpha_4 - \alpha_i) \geq 0$ for all i and the sum of them ≤ 1, combined with the convexity of T, we have $-\frac{1}{3}\mathbf{u} \in T$.

Since $T \cap (T + \mathbf{x}) \neq \varnothing$, there exist $\mathbf{u}_1, \mathbf{u}_2 \in T$ such that $\mathbf{u}_1 + \mathbf{x} = \mathbf{u}_2$. Then we have $-\frac{1}{3}\mathbf{u}_1 \in T$, $-\frac{1}{3}\mathbf{u}_2 \in T$. For any point $\mathbf{u} \in T$, we have $-\frac{1}{3}\mathbf{u} \in T$. By the convexity of T, we know that

$$\mathbf{u} + \mathbf{x} = \mathbf{u} + \mathbf{u}_2 - \mathbf{u}_1 = 5\left(\frac{1}{5}\mathbf{u} + \frac{1}{5}\mathbf{u}_2 + \frac{3}{5}\left(-\frac{1}{3}\mathbf{u}_1\right)\right) \in 5T,$$
$$\mathbf{u} + \mathbf{x} = \mathbf{u} + \mathbf{u}_2 - \mathbf{u}_1 = -7\left(\frac{3}{7}\left(-\frac{1}{3}\mathbf{u}\right) + \frac{3}{7}\left(-\frac{1}{3}\mathbf{u}_2\right) + \frac{1}{7}\mathbf{u}_1\right) \in -7T.$$

Therefore, $T + \mathbf{x} \subset 5T \cap -7T$ if $T \cap (T + \mathbf{x}) \neq \varnothing$, the lemma is proved. □

Let C denote the cube defined by

$$\{(x, y, z) : \max\{|x|, |y|, |z|\} \leq 3\},$$

and let

$$D = 5T \cap -7T \cap C.$$

As shown in Fig. 2, we know that

$$vol(D) = vol(C) - 4 \cdot \frac{4}{3} - 4 \cdot \frac{32}{3} = 168. \tag{2.3}$$

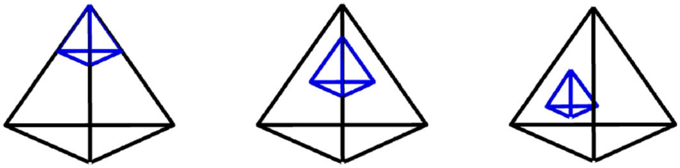

Fig. 3 Results for Corollary 2.6(2)

The following lemma holds:

Lemma 2.4 *If* $T \cap (T + \mathbf{x}) \neq \varnothing$*, then* $T + \mathbf{x} \subset D$.

Proof We know that T can be defined as the set of all points (x, y, z) where

$$\begin{aligned} x + y + z &\leq 1, \\ -x - y + z &\leq 1, \\ x - y - z &\leq 1, \\ -x + y - z &\leq 1, \end{aligned} \tag{2.4}$$

holds. Since $T \cap (T + \mathbf{x}) \neq \varnothing$, there exist $(x_1, y_1, z_1) \in T$ and $(x_2, y_2, z_2) \in T$ such that

$$\mathbf{x} = (x_2, y_2, z_2) - (x_1, y_1, z_1).$$

For any point $(x, y, z) \in T$, inserting (x, y, z) into the two inequalities in (2.4), (x_2, y_2, z_2) as well, inserting (x_1, y_1, z_1) the remaining two inequalities, and then adding all inequalities together, we have

$$|x + (x_2 - x_1)| \leq 3, \; |y + (y_2 - y_1)| \leq 3, \; |z + (z_2 - z_1)| \leq 3.$$

Therefore, $T + \mathbf{x} \subset C$ if $T \cap (T + \mathbf{x}) \neq \varnothing$. By Lemma 2.3, $T + \mathbf{x} \subset 5T \cap -7T \cap C$ if $T \cap (T + \mathbf{x}) \neq \varnothing$, the lemma is proved. □

Remark 2.5 In fact, by similar arguments it can be deduced that

$$\bigcup (T + \mathbf{x}) = D,$$

where $T \cap (T + \mathbf{x}) \neq \varnothing$. Then by Lemma 2.2 and (2.2),

$$\bigcup (T + \mathbf{y}) = \bigcup (D + \mathbf{x}) \subset D(2T) \subset P, \tag{2.5}$$

where $\mathbf{o} \in T + \mathbf{x}$ and $(T + \mathbf{x}) \cap (T + \mathbf{y}) \neq \varnothing$.

Since the common part of two translated regular tetrahedra is a smaller regular tetrahedra, there can be three different types of positions, see Fig. 3. Moreover, we have the following result.

Corollary 2.6 *If $T \cap (T+\mathbf{x}) \neq \emptyset$ and $\mathbf{x} \neq \mathbf{o}$, then the following holds:*

(1) *$T+\mathbf{x}$ intersects at most one vertex of T; If $T+\mathbf{x}$ intersects more than one edge (face) of T, then it must intersect the vertices (edges) of T.*
(2) *If $int(T+\mathbf{x})$ intersects one vertex (edge or face) of T, then $\alpha(T) \cap (T+\mathbf{x})$ is three (two or one) regular triangles with the same edge length.*

3 Translative Coverings of Tetrahedra

To study $\theta^t(T)$, the most natural approach is localization. Assume that X is a discrete set of points in $\mathbb{E}^3$ such that $T+X$ is a translative covering of $\mathbb{E}^3$. Let P be the cube defined in (2.1). Then define

$$\theta(T, X, P) = \frac{\sum_{\mathbf{x}\in X} vol(P \cap (T+\mathbf{x}))}{vol(P)}. \tag{3.1}$$

Let $\mathfrak{X}$ denote the family of all such sets X. We call

$$\theta(T, P) = \min_{X\in\mathfrak{X}} \theta(T, X, P). \tag{3.2}$$

the covering density of T for P. Since P is a parallelohedron, clearly,

$$\theta^t(T) \geq \theta(T, P). \tag{3.3}$$

Proof of Theorem 1.1 $T+X$ is a translative covering of $\mathbb{E}^3$. Without loss of generality, we suppose that $\mathbf{o} \in T+\mathbf{x}_{m+1}$ and $T+\mathbf{x}_{m+1}$ is intersected by $T+\mathbf{x}_1, T+\mathbf{x}_2, \ldots, T+\mathbf{x}_m$. By (2.5), we have

$$\bigcup_{i=1}^{m+1} (T+\mathbf{x}_i) \subset D+\mathbf{x}_{m+1} \subset D(2T) \subset P. \tag{3.4}$$

We consider two cases.
Case 1. $m \geq 63$. By (2.2), (2.3) and (3.4),

$$\begin{aligned}\theta(T, X, P) &= \frac{\sum_{\mathbf{x}\in X} vol\,(P \cap (T+\mathbf{x}))}{vol(P)}\\ &\geq \frac{vol(P \setminus (D+\mathbf{x}_{m+1})) + \sum_{\mathbf{x}\in X} vol\,((D+\mathbf{x}_{m+1}) \cap (T+\mathbf{x}))}{vol(P)}\\ &\geq 1 - \frac{21}{64} + \frac{\sum_{i=1}^{m+1} vol\,((D+\mathbf{x}_{m+1}) \cap (T+\mathbf{x}_i))}{vol(P)}\end{aligned}$$

$$= \frac{43}{64} + (m+1) \cdot \frac{vol(T)}{vol(P)}$$
$$\geq \frac{193}{192}.$$

Case 2. $m \leq 62$. Let $\partial(T)$ denote the boundary of T. Then

$$\partial(T + \mathbf{x}_{m+1}) = \bigcup_{i=1}^{m} \left(\partial(T + \mathbf{x}_{m+1}) \cap (T + \mathbf{x}_i)\right). \tag{3.5}$$

Since $(T + \mathbf{x}_{m+1}) \cap (T + \mathbf{x}_i)$ is a single point or homothetic to T. Since m is finite, if $(T+\mathbf{x}_{m+1}) \cap (T+\mathbf{x}_1)$ is a single point $\mathbf{u}$, then there must be i such that $\mathbf{u} \in int(T+\mathbf{x}_i)$. So

$$\partial(T + \mathbf{x}_{m+1}) = \bigcup_{i=2}^{m} \left(\partial(T + \mathbf{x}_{m+1}) \cap (T + \mathbf{x}_i)\right),$$

and $m - 1 \leq 62$ still holds. Therefore, we suppose that

$$(T + \mathbf{x}_{m+1}) \cap (T + \mathbf{x}_i) = \lambda_i T + \mathbf{y}_i,\ 1 \leq i \leq m$$

holds for some suitable positive number λ_i and a point $\mathbf{y}_i$.

Firstly, let $\mathbf{v}$ be a vertex of $T + \mathbf{x}_{m+1}$. If $\mathbf{v} \in \lambda_1 T + \mathbf{y}_1$ and $\mathbf{v} \in \lambda_2 T + \mathbf{y}_2$, then we must have $\lambda_1 T + \mathbf{y}_1 \subset \lambda_2 T + \mathbf{y}_2$ or $\lambda_2 T + \mathbf{y}_2 \subset \lambda_1 T + \mathbf{y}_1$, say $\lambda_1 T + \mathbf{y}_1 \subset \lambda_2 T + \mathbf{y}_2$. So

$$\partial(T + \mathbf{x}_{m+1}) = \bigcup_{i=2}^{m} \left(\partial(T + \mathbf{x}_{m+1}) \cap (T + \mathbf{x}_i)\right),$$

and $m - 1 \leq 62$ still holds. Therefore, we suppose that each vertex of $T + \mathbf{x}_{m+1}$ is covered exactly once. By Corollary 2.6(1), these vertices are covered by four different elements in $\{T + \mathbf{x}_1, T + \mathbf{x}_2, \ldots, T + \mathbf{x}_m\}$, say

$$T + \mathbf{x}_1, T + \mathbf{x}_2, T + \mathbf{x}_3, T + \mathbf{x}_4.$$

It follows from Corollary 2.6(2) that the total number of regular triangles obtained by intersecting $T + \mathbf{x}_1, T + \mathbf{x}_2, T + \mathbf{x}_3, T + \mathbf{x}_4$ with $\partial(T + \mathbf{x}_{m+1})$ is 12, and the sum of the areas of these regular triangles is

$$3 \cdot \frac{\sqrt{3}}{4} \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2\right).$$

Secondly, let e_j denote the edge of $T + \mathbf{x}_{m+1}$ and let E_j denote the set consisting of all elements in $\{T + \mathbf{x}_5, T + \mathbf{x}_6, \ldots, T + \mathbf{x}_m\}$ that intersect e_j, where $1 \leq j \leq 6$. Since they do not intersect the vertices of $T + \mathbf{x}_{m+1}$, it follows from Corollary 2.6(1) that

$E_i \cap E_j = \varnothing$ if $i \neq j$. Thus we have $|E_1 \cup E_2 \cup \cdots \cup E_6| = |E_1| + |E_2| + \cdots + |E_6|$. Without loss of generality, we suppose that

$$E_1 \cup E_2 \cup \cdots \cup E_6 = \{T + \mathbf{x}_5, T + \mathbf{x}_6, \ldots, T + \mathbf{x}_{|E_1|+\cdots+|E_6|+4}\}.$$

It follows from Corollary 2.6(2) that the total number of regular triangles obtained by intersecting $\bigcup E_j$ with $\partial(T + \mathbf{x}_{m+1})$ is

$$2\left(|E_1| + |E_2| + \cdots + |E_6|\right),$$

and the sum of the areas of these regular triangles is

$$2 \cdot \frac{\sqrt{3}}{4}\left(\lambda_5^2 + \lambda_6^2 + \cdots + \lambda_{|E_1|+\cdots+|E_6|+4}^2\right).$$

Finally, let f_j denote the face of $T + \mathbf{x}_{m+1}$ and let F_j denote the set consisting of all elements in $\{T + \mathbf{x}_{|E_1|+\cdots+|E_6|+5}, T + \mathbf{x}_{|E_1|+\cdots+|E_6|+6}, \ldots, T + \mathbf{x}_m\}$ that intersect f_j, where $1 \leq j \leq 4$. Since they do not intersect the edges of $T + \mathbf{x}_{m+1}$, it follows from Corollary 2.6(1) that $F_i \cap F_j = \varnothing$ if $i \neq j$. Thus we have

$$\left|\bigcup_{i=|E_1|+\cdots+|E_6|+5}^{m} (T + \mathbf{x}_i)\right| = \left|\bigcup_{j=1}^{4} F_j\right| = \sum_{j=1}^{4} |F_j|,$$

which implies that

$$m = 4 + \sum_{j=1}^{6} |E_j| + \sum_{j=1}^{4} |F_j|. \tag{3.6}$$

It follows from Corollary 2.6(2) that the total number of regular triangles obtained by intersecting $\bigcup F_j$ with $\partial(T + \mathbf{x}_{m+1})$ is

$$|F_1| + |F_2| + |F_3| + |F_4|,$$

and the sum of the areas of these regular triangles is

$$\frac{\sqrt{3}}{4}\left(\lambda_{|E_1|+\cdots+|E_6|+5}^2 + \lambda_{|E_1|+\cdots+|E_6|+6}^2 + \cdots + \lambda_m^2\right).$$

Combining the preceding results, we conclude that the total number of regular triangles obtained by intersecting $T + \mathbf{x}_1, T + \mathbf{x}_2, \ldots, T + \mathbf{x}_m$ with $\partial(T + \mathbf{x}_{m+1})$ is

$$t = 12 + 2\sum_{j=1}^{6} |E_j| + \sum_{j=1}^{4} |F_j| \leq 2m + 4 \leq 128, \tag{3.7}$$

and the sum of the areas of these regular triangles is

$$S = \frac{\sqrt{3}}{4}\left(3\sum_{i=1}^{4}\lambda_i{}^2 + 2\sum_{i=5}^{|E_1|+\cdots+|E_6|+4}\lambda_i{}^2 + \sum_{i=|E_1|+\cdots+|E_6|+5}^{m}\lambda_i{}^2\right).$$

From (3.5), we know that

$$S \geq 8\sqrt{3}. \tag{3.8}$$

According to Power-Mean Inequality, (3.6) and (3.7), we have

$$\left(\left(3\sum_{i=1}^{4}\lambda_i{}^3 + 2\sum_{i=5}^{|E_1|+\cdots+|E_6|+4}\lambda_i{}^3 + \sum_{i=|E_1|+\cdots+|E_6|+5}^{m}\lambda_i{}^3\right)/t\right)^{\frac{1}{3}} \geq \left(S/\frac{\sqrt{3}}{4}t\right)^{\frac{1}{2}},$$

where $\lambda_i > 0$. Then it follows from (3.8) that

$$\sum_{i=1}^{m}\lambda_i{}^3 \geq \frac{128\sqrt{2t}}{3t} \geq \frac{16}{3}.$$

Therefore, in this case, we have

$$\begin{aligned}
\theta(T, X, P) &= \frac{\sum_{\mathbf{x}\in X} vol\,(P \cap (T + \mathbf{x}))}{vol(P)} \\
&= 1 + \frac{\sum_{\mathbf{x}_i,\mathbf{x}_j\in X, i<j} vol\,\big(P \cap (T + \mathbf{x}_i) \cap (T + \mathbf{x}_j)\big)}{vol(P)} \\
&\geq 1 + \frac{\sum_{1\leq i<j\leq m+1} vol\,\big((T + \mathbf{x}_i) \cap (T + \mathbf{x}_j)\big)}{vol(P)} \\
&\geq 1 + \frac{\sum_{i=1}^{m} vol\,((T + \mathbf{x}_i) \cap (T + \mathbf{x}_{m+1}))}{vol(P)} \\
&= 1 + \frac{\frac{\sqrt{2}}{12}(\lambda_1^3 + \lambda_2^3 + \cdots + \lambda_m^3)}{512} \\
&\geq 1 + \frac{\sqrt{2}}{1152}.
\end{aligned}$$

As a conclusion of these two cases, we have

$$\theta(T, X, P) \geq 1 + \frac{\sqrt{2}}{1152} > 1 + 1.227 \times 10^{-3}.$$

According to equations (3.1), (3.2) and (3.3), we obtain

$$\theta^t(T) \geq \theta(T, P) = \min_{X \in \mathfrak{X}} \theta(T, X, P) \geq 1 + \frac{\sqrt{2}}{1152} > 1 + 1.227 \times 10^{-3},$$

and the theorem is proved. □

Remark 3.1 By covering the structure with asymptotic $\delta^l(D(2T)) = \frac{45}{49}$ (see [13]) instead of $\frac{vol(D(2T))}{vol(P)} = \frac{5}{6}$, we can slightly improve the lower bound in Theorem 1.1. However, since the improvement is not essential, its proof is not included here.

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