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Note

# Vertex degrees close to the average degree

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ABSTRACT

Let $G$ be a finite, simple, and undirected graph of order $n$ and average degree $d$. Up to terms of smaller order, we characterize the minimal intervals $I$ containing $d$ that are guaranteed to contain some vertex degree. In particular, for $d_+ \in \left(\sqrt{dn}, n-1\right]$, we show the existence of a vertex in $G$ of degree between $d_+ - \left(\frac{(d_+-d)n}{n-d_++\sqrt{d_+^2-dn}}\right)$ and $d_+$.



## 1. Introduction

An obvious observation concerning finite, simple, and undirected graphs is that every such graph $G$ with $n$ vertices and average degree $d$ has a vertex of degree at most $d$ as well as a vertex of degree at least $d$, that is, there are vertex degrees in the two intervals $[0, d]$ and $[d, n-1]$. In the present note we study minimal intervals $I$ containing $d$ for which every such graph $G$ necessarily contains some vertex $u$ whose degree $d_G(u)$ lies in $I$. Surely, which intervals have this property is implicit in characterizations of degree sequences such as the well-known Erdős-Gallai characterization [1,2]. Our goal here is to specify such intervals explicitly, and our motivation originally came from embedding problems that rely on special cases of the results presented here [3,4].

Our first result specifies a natural interval of length about half the order around the average degree that is guaranteed to contain some vertex degree. Some of the estimates in its proof, cf. (6) and (7) below, correspond to inequalities from the Erdős-Gallai characterization [1].

**Theorem 1.** *If $G$ is a graph with $n$ vertices and $m$ edges, then there is a vertex $u$ in $G$ with*

$$d - \frac{n-2}{2(n-1)}d \leq d_G(u) \leq d + \frac{n-2}{2(n-1)}\bar{d}, \tag{1}$$

*where $d = \frac{2m}{n}$ is the average degree of $G$ and $\bar{d} = n-1-d$ is the average degree of the complement $\overline{G}$ of $G$.*

*Furthermore, if $G$ has no vertex $u$ with $d - \frac{n-2}{2(n-1)}d < d_G(u) < d + \frac{n-2}{2(n-1)}\bar{d}$, then*

- *the vertex set of $G$ is the disjoint union of two sets $V_+$ and $V_-$,*
- *$V_+$ is a clique of order $\frac{dn}{n-1}$,*
- *$V_-$ is an independent set of order $\frac{\bar{d}n}{n-1}$,*

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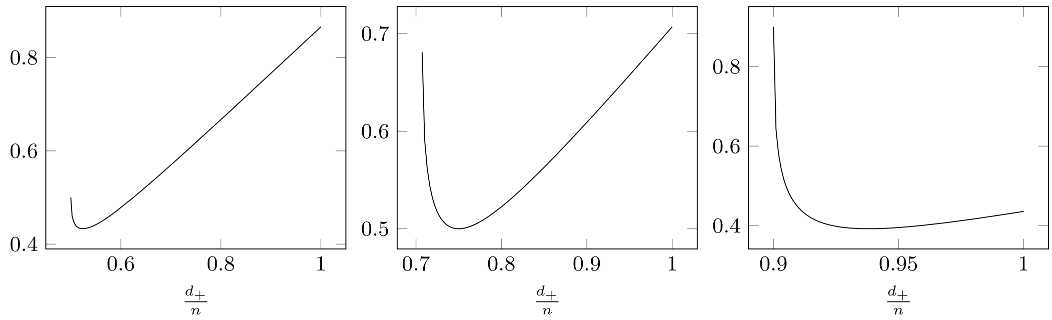


**Fig. 1.** The three plots show $\frac{\ell_{\min}(d_+)}{n}$ as a function of $\frac{d_+}{n} \in \left(\sqrt{\frac{d}{n}}, 1\right]$ for $\frac{d}{n} \in \{0.25, 0.5, 0.81\}$.

- *every vertex in $V_+$ is adjacent to exactly half the vertices in $V_-$, and*
- *every vertex in $V_-$ is adjacent to exactly half the vertices in $V_+$.*

Theorem 1 gives a rather precise answer in a specific setting, where the interval depends on $n$ and $d$; allowing for the characterization of the corresponding extremal graphs. Our second result applies in a more general setting; it gives the precise answer up to terms of smaller order.

**Theorem 2.** *If $G$ is a graph of order $n$ and average degree $d$ with $0 < d < n-1$, and $d_+ \in \left(\sqrt{dn}, n-1\right]$, then there is a vertex $u$ in $G$ with*

$$d_+ - \frac{(d_+ - d)n}{n - d_+ + \sqrt{d_+^2 - dn}} \le d_G(u) \le d_+.$$

Let $d_-$ denote the lower bound for $d_G(u)$ specified in Theorem 2. For $d_+ \le \sqrt{dn}$, Lemma 3 below indicates that nothing really non-trivial can be said about $d_-$, more precisely, in this case, $d_-$ will be 0. Theorem 2 gives an estimate for $d_-$ that is best possible up to terms of smaller order; one can construct suitable almost extremal graphs approximating the values in (17) below. Applying Theorem 2 to the complement of $G$ yields a symmetric result, where one first specifies the lower bound $d_-$ for $d_G(u)$ and then determines the upper bound $d_+$ accordingly, that is, in this case, $d_+$ is considered as a function of $n$, $d$, and $d_-$. For given $n$ and $d$ as in Theorem 2, the length

$$\ell_{\min}(d_+) = d_+ - d_- = \frac{(d_+ - d)n}{n - d_+ + \sqrt{d_+^2 - dn}} = \left(\frac{\frac{d_+}{n} - \frac{d}{n}}{1 - \frac{d_+}{n} + \sqrt{\left(\frac{d_+}{n}\right)^2 - \frac{d}{n}}}\right) n$$

of the specified interval satisfies $\min\left\{\ell_{\min}(d_+) : d_+ \in \left(\sqrt{dn}, n-1\right]\right\} \le \frac{n}{2}$ for every $d \in (0, n-1)$. Up to terms of smaller order, Theorem 1 corresponds to the choice $\frac{d_+}{n} = \frac{n+d}{2n} = \frac{1}{2} + \frac{d}{2n}$, in which case $\ell_{\min}(d_+) = \frac{n}{2}$, that is, up to terms of smaller order, Theorem 2 implies Theorem 1. For $d \neq \frac{n}{2}$ and suitable choices of $d_+$, Theorem 2 guarantees a vertex degree within a smaller interval around $d$ than Theorem 1, cf. Fig. 1.

Applying our results repeatedly to a given graph, each time removing a vertex of degree close to the current average degree, yields several vertices whose degrees are guaranteed to lie in slowly changing intervals around the original average degree.

The proofs are given in the following section.

## 2. Proofs

We proceed to the proofs of our two results.

**Proof of Theorem 1.** Let $G$, $n$, $m$, $d$, and $\bar{d}$ be as in the statement. Since the statement is trivial for $d \in \{0, n-1\}$, we may assume that $0 < d < n-1$. Let

$$V_- = \left\{u \in V(G) : d_G(u) \le d - \frac{n-2}{2(n-1)}\bar{d}\right\} \quad \text{and} \quad V_+ = \left\{u \in V(G) : d_G(u) \ge d + \frac{n-2}{2(n-1)}\bar{d}\right\}.$$

In view of the desired statements, we may assume that $V(G)$ is the disjoint union of $V_+$ and $V_-$. Since $G$ has average degree $d$, the sets $V_+$ and $V_-$ are both non-empty, and, hence, we have $\frac{1}{n} \leq x \leq 1-\frac{1}{n}$ for $x=\frac{|V_+|}{n}$. The degree sum $\sum\limits_{u\in V_+} d_G(u)$ of the vertices in $V_+$ is at most the sum of the following two terms:

- Twice the number of edges in $G$ between vertices in $V_+$. This is at most $xn(xn-1)$ with equality if and only if $V_+$ is a clique.
- The number of edges in $G$ between $V_+$ and $V_-$. This is at most $\sum\limits_{u\in V_-} d_G(u)$ with equality if and only if $V_-$ is an independent set.

This implies

$$\begin{aligned} dn &= \sum_{u\in V_+} d_G(u) + \sum_{u\in V_-} d_G(u) \\ &\leq xn(xn-1) + 2\sum_{u\in V_-} d_G(u) \qquad (2) \\ &\leq xn(xn-1) + 2\underbrace{(1-x)n}_{=|V_-|}\underbrace{\left(d-\frac{n-2}{2(n-1)}d\right)}_{\geq d_G(u)\ for\ u\in V_-}. \qquad (3) \end{aligned}$$

Applying exactly the same counting argument to $\overline{G}$ implies

$$\begin{aligned} (n-1-d)n &= \sum_{u\in V_-} d_{\overline{G}}(u) + \sum_{u\in V_+} d_{\overline{G}}(u) \\ &\leq (1-x)n((1-x)n-1) + 2\sum_{u\in V_+}(n-1-d_G(u)) \qquad (4) \\ &\leq (1-x)n((1-x)n-1) + 2xn\underbrace{\left(n-1-d-\frac{n-2}{2(n-1)}\overline{d}\right)}_{\geq d_{\overline{G}}(u)\ for\ u\in V_+}. \qquad (5) \end{aligned}$$

By (3) and (5), we have

$$\begin{aligned} f_1(x) &:= x(xn-1) + 2(1-x)\left(d-\frac{n-2}{2(n-1)}d\right) - d = \frac{(xn-1)(x(n-1)-d)}{n-1} \geq 0 \text{ and} \qquad (6) \\ f_2(x) &:= (1-x)((1-x)n-1) + 2x\left(n-1-d-\frac{n-2}{2(n-1)}\overline{d}\right) - (n-1-d) \\ &= \frac{(x(n-1)-d)((x-1)n+1)}{n-1} \geq 0. \qquad (7) \end{aligned}$$

Recall that $x \in \left[\frac{1}{n}, 1-\frac{1}{n}\right]$. Since $f_1\left(\frac{1}{n}\right)=0$, $f_2\left(1-\frac{1}{n}\right)=0$, $f_1\left(\frac{d}{n-1}\right)=f_2\left(\frac{d}{n-1}\right)=0$, and $f_1(x)$ as well as $f_2(x)$ are both strictly convex, we have $\min\{f_1(x), f_2(x)\}<0$ for $x\in\left[\frac{1}{n},1-\frac{1}{n}\right]\setminus\left\{\frac{1}{n},\frac{d}{n-1},1-\frac{1}{n}\right\}$. By (6) and (7), this implies

$$x \in \left\{\frac{1}{n}, \frac{d}{n-1}, 1-\frac{1}{n}\right\} \qquad \text{and} \qquad \min\left\{f_1(x), f_2(x)\right\} = 0.$$

In order to show the existence of a vertex $u$ in $G$ that satisfies (1), we suppose, for a contradiction, that no vertex in $V_+$ has degree $d+\frac{n-2}{2(n-1)}\overline{d}$ and that no vertex in $V_-$ has degree $d-\frac{n-2}{2(n-1)}d$. Since $V_+$ and $V_-$ are both non-empty, the two inequalities (3) and (5) become strict, which implies the contradiction $\min\{f_1(x), f_2(x)\}>0$. Hence, some vertex $u$ in $G$ satisfies (1).

We proceed to the proof of the second part of the statement. Note that the hypothesis that $G$ has no vertex $u$ with $d-\frac{n-2}{2(n-1)}d < d_G(u) < d+\frac{n-2}{2(n-1)}\overline{d}$ is equivalent to our assumption that $V(G)$ is the disjoint union of $V_+$ and $V_-$. If $V_+$ is not a clique, then

- twice the number of edges in $G$ between vertices in $V_+$ is strictly less than $xn(xn-1)$, which implies that (2) becomes a strict inequality, and

- the number of edges in $\overline{G}$ between $V_+$ and $V_-$ is strictly less than $\sum\limits_{u\in V_+} d_{\overline{G}}(u)$, which implies that (4) becomes a strict inequality.

Symmetrically, if $V_-$ is not an independent set, then (2) and (4) become strict inequalities. Therefore, if $V_+$ is not a clique or $V_-$ is not an independent set, then (2) and (4) become strict inequalities, and we obtain the contradiction $\min\{f_1(x), f_2(x)\} > 0$. Similarly, if some vertex in $V_+$ has a degree strictly larger than $d + \frac{n-2}{2(n-1)}\overline{d}$ and some vertex in $V_-$ has a degree strictly smaller than $d - \frac{n-2}{2(n-1)}d$, then (3) and (5) become strict inequalities, and again we obtain the contradiction $\min\{f_1(x), f_2(x)\} > 0$. If $V_+$ is a clique, $V_-$ is an independent set, all vertices in $V_-$ have degree exactly $d - \frac{n-2}{2(n-1)}d$, and some vertex in $V_+$ has degree strictly larger than $d + \frac{n-2}{2(n-1)}\overline{d}$, then

$$\begin{aligned} dn &= \sum_{u\in V_+} d_G(u) + \sum_{u\in V_-} d_G(u) \\ &> \left(d + \frac{n-2}{2(n-1)}\overline{d}\right) xn + \left(d - \frac{n-2}{2(n-1)}d\right)(1-x)n \\ &= \frac{(n-2)n}{2}x + \frac{n^2 d}{2(n-1)} \end{aligned}$$

implies the contradiction $x < \frac{d}{n-1}$. Hence, by symmetry between $G$ and $\overline{G}$, we may assume that $V_+$ is a clique, $V_-$ is an independent set, all vertices in $V_-$ have degree exactly $d - \frac{n-2}{2(n-1)}d$, and all vertices in $V_+$ have degree exactly $d + \frac{n-2}{2(n-1)}\overline{d}$. As above, this implies $x = \frac{d}{n-1}$, and it follows easily that every vertex in $V_+$ is adjacent to exactly half the vertices in $V_-$ and that every vertex in $V_-$ is adjacent to exactly half the vertices in $V_+$. This completes the proof. $\square$

The interval

$$\left[d - \frac{n-2}{2(n-1)}d, d + \frac{n-2}{2(n-1)}\overline{d}\right]$$

from Theorem 1 always has the same length $\frac{n-2}{2}$ but it naturally shifts with changes of $d$. Now, for a given value of $d_+$ from $[d, n-1]$, we consider the largest $d_-$ from $[0, d]$ such that the interval $I = [d_-, d_+]$ around $d$ necessarily contains some vertex degree, that is, we specify the upper end $d_+$ of $I$ and consider the lower end $d_-$ of $I$ as a function of $n$, $d$, and $d_+$. Quite naturally, if $d_+$ is close enough to $d$, then $d_-$ will have to be 0, which leads to the obvious observation at the very beginning of this note.

Now, let $n$ and $m$ be integers with $0 < m < \binom{n}{2}$, and let $d = \frac{2m}{n}$.

For $d_+ \in (d, n-1)$, let

$$d_- = d_-(n, d, d_+)$$

be the smallest possible value of $d_- \in [0, d]$ such that there is some graph $G$ of order $n$ and average degree $d$ whose vertex set $V(G)$ is partitioned into the two sets

$$V_- = \{u \in V(G) : d_G(u) \le d_-\} \qquad \text{and} \qquad V_+ = \{u \in V(G) : d_G(u) \ge d_+\}.$$

Clearly, the choices of $d_-$ and $G$ imply that some vertex in $G$ has degree $d_-$, in particular, $d_-$ is an integer.

This definition implies that

- every graph of order $n$ and average degree $d$ has a vertex of degree in $[d_-, d_+]$, while
- some such graph has no vertex of degree in $(d_-, d_+)$,

that is, $[d_-, d_+]$ is indeed a minimal interval with the desired property. Our goal is to estimate $d_- = d_-(n, d, d_+)$.

Let $G$ be as above.

Similarly as in the proof of Theorem 1, the condition $d_+ > d$ implies that $V_-$ and $V_+$ are both non-empty.

For the average degrees $\overline{d}_\pm = \frac{1}{|V_\pm|}\sum\limits_{u\in V_\pm} d_G(u)$ within $V_-$ and $V_+$, respectively, we obtain

$$0 \le \overline{d}_- \le d_- \le d < d_+ \le \overline{d}_+ \le n-1. \tag{8}$$

For $x = \frac{|V_+|}{n} \in (0,1)$, we have

$$(1-x)n\overline{d}_- + xn\overline{d}_+ = dn. \tag{9}$$

Since the number of edges between $V_-$ and $V_+$ is at least $\left(\overline{d}_+ - xn + 1\right)xn$, we have the following Erdős-Gallai type inequality

$$(1-x)n\overline{d}_- \geq \left(\overline{d}_+ - xn + 1\right)xn. \tag{10}$$

It follows that

$$d_-(n, d, d_+) \geq \mathrm{OPT}(P) \tag{11}$$

for the following optimization problem $(P)$:

$$(P) \qquad \begin{aligned} &\min d_- \\ &s.th.\ (1-x)\overline{d}_- + x\overline{d}_+ = d && (12)\\ &\qquad (1-x)\overline{d}_- \geq \left(\overline{d}_+ - xn\right)x && (13)\\ &\qquad 0 \leq \overline{d}_- \leq d_- && (14)\\ &\qquad d_+ \leq \overline{d}_+ \leq n && (15)\\ &\qquad 0 \leq x \leq 1 && (16)\\ &\qquad d_-, \overline{d}_-, \overline{d}_+, x \in \mathbb{R}. \end{aligned}$$

Note that $(P)$ is obtained by relaxing some of the conditions in (8), (9), and (10) as well as the integrality of $d_-$ and the rationality of $\overline{d}_-$, $\overline{d}_+$, and $x$. Since $(P)$ is a minimization problem, these relaxations do not affect the validity of (11).

Note that for the entire rest of the paper, the variables $d_-$, $\overline{d}_-$, $\overline{d}_+$, and $x$ are considered to be real numbers as in $(P)$, and the values of $n$, $d$, and $d_+$ are considered fixed parameters for $(P)$.

**Lemma 3.** *In the above setting,*

$$\mathrm{OPT}(P) = \begin{cases} 0 & \text{, if } d_+ \leq \sqrt{dn} \\ d_+ - \frac{(d_+-d)n}{n-d_++\sqrt{d_+^2-dn}} & \text{, if } d_+ > \sqrt{dn}. \end{cases}$$

**Proof.** Note that $\mathrm{OPT}(P) \geq 0$ by (14).

If $d_+ \leq \sqrt{dn}$, then $\left(d_-, \overline{d}_-, \overline{d}_+, x\right) = \left(0, 0, \sqrt{dn}, \sqrt{\frac{d}{n}}\right)$ is an optimal solution of $(P)$.

Now, let $d_+ > \sqrt{dn}$.

Since $\left(d_-, \overline{d}_-, \overline{d}_+, x\right)$ equal to

$$\left(d_+ - \frac{(d_+ - d)n}{n - d_+ + \sqrt{d_+^2 - dn}}, d_+ - \frac{(d_+ - d)n}{n - d_+ + \sqrt{d_+^2 - dn}}, d_+, \frac{1}{n}\left(d_+ - \sqrt{d_+^2 - dn}\right)\right) \tag{17}$$

is a feasible solution of $(P)$, we have

$$\mathrm{OPT}(P) \leq d_+ - \frac{(d_+ - d)n}{n - d_+ + \sqrt{d_+^2 - dn}} < d.$$

In fact, it will turn out that (17) is an optimal solution.

Let $\left(d_-, \overline{d}_-, \overline{d}_+, x\right)$ be an optimal solution of $(P)$. In particular, $d_- < d$. In view of the objective function, (14) implies $d_- = \overline{d}_-$. Since $\overline{d}_- < d < \overline{d}_+$, (12) and (16) imply $0 < x < 1$. If $d_- = 0$, then (13) implies $\overline{d}_+ \leq xn$, and, using (12) and (15), this implies the contradiction $d_+ \leq \overline{d}_+ \leq \sqrt{dn}$. Hence, $d_- > 0$. If (13) would not be satisfied with equality, then decreasing $d_-$ and $\overline{d}_-$ both by some sufficiently small $\epsilon > 0$, and increasing $x$ by $\delta = \frac{(1-x)\epsilon}{\overline{d}_+ + \epsilon - \overline{d}_-} \leq \frac{\epsilon}{d_+ - d}$ leads to a better feasible solution, which is a contradiction. Hence, (13) is satisfied with equality. Together with (12), this implies $d - x\overline{d}_+ = (\overline{d}_+ - xn)x$. Solving this for $x$ yields

$$x \in \left\{\frac{1}{n}\left(\overline{d}_+ + \sqrt{\overline{d}_+^2 - dn}\right), \frac{1}{n}\left(\overline{d}_+ - \sqrt{\overline{d}_+^2 - dn}\right)\right\}.$$

By (13) satisfied with equality, we have $\overline{d}_+ \geq xn$, and, hence, $x = \frac{1}{n}\left(\overline{d}_+ - \sqrt{\overline{d}_+^2 - dn}\right)$.

Now, substituting this value for $x$, (12) can be solved for $d_-$, which yields

$$\begin{aligned} d_- &= \overline{d}_+ - \frac{\left(\overline{d}_+ - d\right)n}{n - \overline{d}_+ + \sqrt{\overline{d}_+^2 - dn}} \\ &= n\left(z - \frac{z - z_0}{1 - z + \sqrt{z^2 - z_0}}\right) \text{ for } z = \frac{\overline{d}_+}{n} \text{ and } z_0 = \frac{d}{n}. \end{aligned}$$

Since $d_+ > \sqrt{dn}$, we have $z \geq \frac{d_+}{n} > \sqrt{z_0}$.

For the function,

$$f : \left(\sqrt{z_0}, 1\right] \to \mathbb{R} : z \mapsto z - \frac{z - z_0}{1 - z + \sqrt{z^2 - z_0}},$$

we obtain

$$f'(z) = \frac{(1-z)\left(2z^2 - z_0 - 2z\sqrt{z^2 - z_0}\right)}{\sqrt{z^2 - z_0}\left(1 - z + \sqrt{z^2 - z_0}\right)^2}.$$

Since $f'(z) \geq 0$ for $z \in \left(\sqrt{z_0}, 1\right]$, the smallest possible value of $d_- = nf(z)$ with $z \in \left[\frac{d_+}{n}, 1\right]$ is achieved for $z = \frac{d_+}{n}$, which implies

$$d_- = d_+ - \frac{(d_+ - d)n}{n - d_+ + \sqrt{d_+^2 - dn}}.$$

Altogether, it follows that $\left(d_-, \overline{d}_-, \overline{d}_+, x\right)$ is exactly as in (17), which completes the proof. $\square$

Now, (11) and Lemma 3 imply Theorem 2.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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