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Antipaths in oriented graphs

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ABSTRACT

We show that for any natural number $k \geq 1$, any oriented graph D of minimum semidegree at least $(3k-2)/4$ contains an antidirected path of length k.
In fact, a slightly weaker condition on the semidegree sequence of D suffices, and as a consequence, we confirm a weakened antidirected path version of a conjecture of Addario-Berry, Havet, Linhares Sales, Thomassé and Reed.
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1. Introduction

In undirected graphs, a large minimum degree is very helpful for finding long paths. For instance, if we wish to ensure that an n-vertex graph G contains a path of length k (i.e., with k edges), a simple greedy embedding argument shows that it is enough to assume G has minimum degree at least k. Although in general, this bound on the minimum degree is best possible, there is room for improvement if k is large compared to n, or if G is assumed to be connected. Dirac [2] showed that a minimum degree of at least $n/2$ is enough to find a Hamilton cycle in an n-vertex graph, and with a minimum degree exceeding $(n-2)/2$ we can find a Hamilton path, i.e., a path of length $n-1$. With a similar proof, one can show that every connected graph on at least $k+1$ vertices that has minimum degree strictly greater than $(k-1)/2$ contains a path of length k.

It would be interesting to find extensions of these results to digraphs. We will focus on oriented graphs here (see Section 5 for some remarks on the general digraph case). We have to decide which parameter will play the role of the minimum degree, and the widespread notion of the *minimum semidegree* $\delta^0(D)$, which is defined as the minimum over all out-and in-degrees of all vertices of the oriented graph D, seems a natural choice. In the same way as in the undirected case, we can use a greedy embedding strategy to see that any oriented graph D with $\delta^0(D) \geq k$ must contain each orientation of the k-edge path. And as before, it seems reasonable to ask whether this bound can be lowered if the underlying graph G of D (i.e., the graph we obtain by omitting directions) has a sufficiently large connected component. Note that the condition $\delta^0(D) \geq k/2$ alone already implies that G has a connected component with at least $k+1$ vertices.

There are many results for Hamilton cycles in oriented graphs. As a Hamilton cycle of an n-vertex oriented graph D contains a path of length $n-1$, these results shed some light on our problem. In 1960, Ghoulia-Houri [4] proved a minimum

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semidegree of $n/2$ guarantees a directed Hamilton cycle in any n-vertex digraph D, and in the 1970's, Thomassen [14] asked for an analogous result for oriented graph, with a weaker condition on the minimum semidegree. Häggkvist [5] conjectured this to be $\delta^0(D) \geq (3n-4)/8$, which he showed to be best possible, and which, after previous results in [6,11], was confirmed by Keevash, Kühn and Osthus in [9] for all large oriented graphs. Häggkvist and Thomassen [7] conjectured that for all $\alpha > 0$, all sufficiently large oriented graphs D with $\delta^0(D) \geq (3/8+\alpha)n$ contain any orientation of a Hamilton cycle, and this was confirmed by Kelly [10]. In particular, it follows that for $3n/4 + o(n) \leq k < n$, every oriented graph on n vertices and of minimum semidegree at least $k/2$ contains each orientation of the k-edge path.

A corresponding result for oriented paths of length below $3n/4$ is still missing, except in the case of directed paths: Jackson [8] showed in 1981 that for every $\ell \in \mathbb{N}$, every oriented graph D with $\delta^0(D) \geq \ell$ contains the directed path on 2ℓ edges. In [12], the second author suggested that something similar might be true for all orientations of the k-edge path.

Conjecture 1. *[12] For each $k \in \mathbb{N}$, every oriented graph D with $\delta^0(D) > k/2$ contains each orientation of the path of length k.*

Conjecture 1 is sharp in the following sense. The bound on the minimum semidegree could not be lower than $k/2$, as one can see by considering the disjoint union of regular tournaments on k vertices, if k is odd. For *antidirected paths* (i.e., oriented paths that alternate edge directions), one can also consider the blow-up of a directed cycle of length ℓ, where each vertex v of C_ℓ is replaced by an independent set S_v of size $k/2$, and S_v, S_w span a complete bipartite graph whenever $vw \in E(C_\ell)$. Any largest antidirected path in this graph has length $k-1$, and the minimum semidegree of the graph is $k/2$.

As noted above, Conjecture 1 is true for n-vertex oriented graphs and $k \geq 3n/4 + o(n)$ by the results of [10], and it is also true for directed paths [8]. It has been verified for all $k \leq 5$ [3]. Zárate-Guerén and the second author showed in [13] that an approximate version of Conjecture 1 holds for antidirected paths in large oriented graphs D, if k is linear in n.

We will focus here on a variant of Conjecture 1 for antidirected paths. We show, with a much easier proof than the one from [13], and for any $k \in \mathbb{N}^+$, that every oriented graph D with $\delta^0(D) \geq (3k-2)/4$ contains each antidirected path.[3] Actually, we will prove a slightly stronger statement. We define the *minimum pseudo-semidegree* $\bar{\delta}^0(D)$ of a digraph D as follows: $\bar{\delta}^0(D) = 0$ if D has no edges, and otherwise $\bar{\delta}^0(D)$ is the maximum $d \in \mathbb{N}$ such that each vertex in $V(D)$ has out-degree either 0 or $\geq d$, and in-degree either 0 or $\geq d$. Clearly $\bar{\delta}^0(D) \geq \delta^0(D)$ for each digraph D. Our main result is the following.

Theorem 2. *Let $k \in \mathbb{N}$ with $k \geq 3$ and let D be an oriented graph with $\bar{\delta}^0(D) \geq (3k-2)/4$. Then D contains each antidirected path of length k.*

Note that the case $k = 2$ needs to be excluded from our theorem, because the bound $\bar{\delta}^0(D) \geq (6-2)/4 = 1$ is below the bound from Conjecture 1 and not sufficient to guarantee an antipath of length two (as D could be a directed cycle).

In a similar vein as Conjecture 1, Addario-Berry, Havet, Linhares Sales, Thomassé and Reed conjectured the following in 2013.

Conjecture 3 *(Addario-Berry et al. [1]). Every digraph D with more than $(k-1)|V(D)|$ edges contains each antidirected tree with $k+1$ vertices.*

For symmetric digraphs, this conjecture is equivalent to the Erdős-Sós conjecture; and in oriented graphs, Conjecture 3 implies Burr's conjecture for antidirected trees (see [1] for details). Conjecture 3 is proved in [1] for trees of diameter at most 3, and an approximate version for large balanced antidirected trees in dense oriented graphs is given in [13].

It is also shown in [1, Theorem 17] that every digraph D with more than $4(m-1)|V(D)|$ edges contains each antidirected tree whose largest partition class has at most m vertices. This implies that every digraph D with more than $4(\lceil (k+1)/2 \rceil - 1)|V(D)|$ (that is, roughly $2k|V(D)|$) edges contains each antidirected k-edge path. We improve this bound to roughly $3k|V(D)|/2$.

Theorem 4. *For each $k \in \mathbb{N}^+$, every oriented graph D with more than $(3k-4)|V(D)|/2$ edges contains each antidirected path of length k.*

2. Notation

A *digraph* has directed edges, at most one for each direction between each pair of vertices u, v. For brevity we write *edge* instead of *directed edge*, and let uv denote an edge going from vertex u to vertex v. For the endvertices of such an edge, we say that v is an *out-neighbour* of u, and u is an *in-neighbour* of v. We write $d^-(v)$ and $d^+(v)$ for the *in-degree* and the *out-degree* of vertex v: this is the number of out-neighbours, or in-neighbours of v, respectively. As already mentioned in

[3] Note that if k is odd, there is only one antidirected path of length k (unless we specify a starting vertex). If k is even, there are two distinct antidirected paths.

the introduction, the *minimum semidegree* of a digraph D is $\delta^0(D) = \min\{d^-(v), d^+(v) : v \in V(D)\}$, and the *minimum pseudo-semidegree* $\bar{\delta}^0(D)$ of a digraph D is the minimum of $\min\{d^-(v) : v \in V(D), d^-(v) > 0\}$ and $\min\{d^+(v) : v \in V(D), d^+(v) > 0\}$, unless D has no edges, in which case $\bar{\delta}^0(D) = 0$.

In an *oriented graph*, for each pair of vertices u, v, at most one of the edges uv, vu is present. We say an oriented path or cycle has *length* k if it has k edges. An *antidirected path (antidirected cycle, antidirected tree)* is an oriented path (cycle, tree) where every vertex has either out-degree 0 or in-degree 0. We also write *antipath (anticycle, antitree)* for short. Note that each anticycle has even length. In particular, any anticycle in an oriented graph has length at least 4.

3. Proof of Theorem 2

We show Theorem 2 by combining three auxiliary lemmas, namely Lemmas 6, 7 and 8, which are stated and proved below. The proofs of these lemmas make use of different variants of a well-known argument that appears in the proof of Dirac's theorem. For convenience, we state this tool now, as Fact 5, and include its short proof for completeness.

Given a set F of edges in an undirected graph, we write $d_F(v, S)$ for the number of edges between a vertex v and a set S that belong to F.

Fact 5. *Let $m \in \mathbb{N}^+$, let $1 \le \ell \le m$ and let G be a graph. Let $X, Y \subseteq V(G)$, with $X = \{x_0, x_1, \dots, x_{m-1}\}$ and $Y = \{y_1, y_2, \dots, y_m\}$, and let $F_0, F_m \subseteq E(G)$. If $d_{F_0}(x_0, Y) + d_{F_m}(y_m, X) \ge m + \ell$, then there is an index i with $\ell \le i \le m$ such that $x_0 y_i \in F_0$ and $x_{i-\ell} y_m \in F_m$.*

Proof. Otherwise, $1_{x_0 y_i \in F_0} + 1_{x_{i-\ell} y_m \in F_m} \le 1$ for each $i = \ell, \dots, m$, and therefore,

$$m + \ell \le d_{F_0}(x_0, Y) + d_{F_m}(y_m, X) \le 2(\ell - 1) + \sum_{i=\ell}^{m} (1_{x_0 y_i \in F_0} + 1_{x_{i-\ell} y_m \in F_m}) \le m + \ell - 1,$$

a contradiction. □

The usual application of this argument in the proof of Dirac's theorem is setting $\ell = 1$, and, given a maximum length path $P = x_0 x_1 \dots x_m$, setting $y_i := x_i$ for $i = 1, \dots, m$, while letting F_0, F_m be the set of edges going from x_0 or x_m, respectively, to other vertices of P. Then the two edges $x_0 x_i$ and $x_{i-1} x_m$ given by Fact 5 are used to find a cycle in $V(P)$.

We are ready for the first auxiliary lemma.

Lemma 6. *Let $k \in \mathbb{N}$ and let D be an oriented graph of minimum pseudo-semidegree $\bar{\delta}^0(D) \ge k/2$. Let $P = v_0 v_1 \dots v_m$ be a longest antipath in D. If $m < k$ then m is odd.*

Proof. Assume otherwise, that is, suppose $m < k$ and m is even. We may assume that $m \neq 0$ (because $m = 0$ means that the longest antipath is trivial, implying that $\bar{\delta}^0(D) = 0$ and thus $k = 0$). Then m fulfils $3 \le m + 1 \le k$. By symmetry, we may assume that

$$v_0 v_1, v_m v_{m-1} \in E(D), \tag{1}$$

that is, the first edge of P is directed towards v_1, and that the last edge of P is directed towards v_{m-1}.

Note that by the maximality of P, all out-neighbours of v_0 lie on P, and the same is true for v_m. By (1), each of v_0, v_m has at least one out-neighbour, and therefore, by our assumption on the minimum pseudo-semidegree of D, each has at least $k/2 \ge (m+1)/2$ out-neighbours.

Let G be the underlying graph of D, and let $F_0, F_m \subseteq E(G)$ be the sets of all edges of G corresponding to edges of D that are leaving v_0 or leaving v_m, respectively. Then we can calculate

$$d_{F_0}(v_0, V(P - v_0)) + d_{F_m}(v_m, V(P - v_m)) \ge (m+1)/2 + (m+1)/2 = m + 1.$$

Let $x_i = y_i = v_i$ for $i = 1, \dots, m$. We now use Fact 5 with $\ell = 1$ in G to see that there is an index $i \in \{1, \dots, m\}$ such that $v_0 v_i \in F_0$ and $v_{i-1} v_m \in F_m$. So $v_0 v_i$ and $v_m v_{i-1}$ are edges (in these directions) in D.

We may assume that $v_i v_{i-1}$ is an edge (and thus $i \neq 1$), as otherwise we could reverse P, interchanging the roles of v_0 and v_m, and of v_i and v_{i-1}, and thus obtain the desired direction. So also $v_i v_{i+1}$ is an edge (unless $i = m$).

Consider the two antipaths

$$P' = v_i v_{i+1} \dots v_m v_{i-1} v_{i-2} \dots v_1 v_0$$

and

$$P'' = \begin{cases} v_i v_0 v_1 \dots v_{i-1} v_m v_{m-1} \dots v_{i+1} & \text{if } i \neq m \\ v_i v_0 v_1 \dots v_{i-1} & \text{if } i = m. \end{cases} \tag{2}$$

Both antipaths P' and P'' have the same vertex set as P (and hence maximum length) and start with vertex v_i. In P', the first edge is directed away from v_i, while in P'', the first edge is directed towards v_i. Therefore, and by the maximality of the antipath P, all in-neighbours and all out-neighbours of vertex v_i lie on P.

Also note that v_i has both in- and out-neighbours (namely, v_0 and v_{i-1}). So, since by assumption $\bar{\delta}^0(D) \geq k/2$, we know that v_i has at least $k/2$ in-neighbours and at least $k/2$ out-neighbours, and thus, $k = k/2 + k/2 \leq |V(P - v_i)| = m$, in contradiction to our assumption that $m < k$. $\square$

Our second auxiliary lemma turns the maximum length antipath into an anticycle on the same number of vertices. For this lemma (and only for this lemma) we need a minimum pseudo-semidegree of $\bar{\delta}^0(D) \geq (3k-2)/4$.

Lemma 7. *Let $k \in \mathbb{N}$, let D be an oriented graph of minimum pseudo-semidegree $\bar{\delta}^0(D) \geq (3k-2)/4$, and let m be the maximum length of an antipath in D. If $1 < m < k$, then D contains an anticycle of length $m+1$.*

Proof. Let $P = v_0v_1v_2 \dots v_m$ be an antipath of maximum length in D. Assume $1 < m < k$. In particular, $(3k-2)/4 \geq k/2$ and Lemma 6 implies that m is odd.

By symmetry, and since m is odd, we may assume that $v_0v_1, v_{m-1}v_m \in E(D)$, that is, the first edge is directed towards v_1, and the last edge is directed towards v_m. Observe that all edges on P are directed from their endvertex of even index towards their endvertex of odd index. Also observe that by maximality of P, all out-neighbours of v_0 and all in-neighbours of v_m lie on P.

Let G be the underlying graph of D. We set $x_i := v_{2i}$ and $y_{i+1} := v_{2i+1}$ for all $i = 0, \dots, m'-1$, where $m' = (m+1)/2$. Let $X = \{x_0, x_1, \dots, x_{m'-1}\}$ and $Y = \{y_1, y_2, \dots, y_{m'}\}$. Note that $X \cup Y$ is a partition of $V(P)$.

Next, define F_0 as the set of all edges of G that correspond to an edge of D leaving v_0 and ending at a vertex from Y. Further, let F_m be the set of all edges of G corresponding to edges of D that start at a vertex from X and end at v_m. As by assumption $k \geq m+1$, we know that

$$\begin{aligned} d_{F_0}(v_0, Y) + d_{F_m}(v_m, X) &= d^+(v_0) - |X \setminus \{v_0\}| + d^-(v_m) - |Y \setminus \{v_m\}| \\ &\geq (3k-2)/2 - (|V(P)| - 2) \\ &\geq (3m+1)/2 - m + 1 \\ &\geq m' + 1. \end{aligned}$$

Now, applying Fact 5 with $\ell = 1$ in G we find an index $i \in \{1, \dots, m'\}$ such that $x_0y_i = v_0v_{2i-1} \in F_0$ and $x_{i-1}y_{m'} = v_{2i-2}v_m \in F_m$. So,

$$v_0v_1v_2 \dots v_{2i-3}v_{2i-2}v_mv_{m-1} \dots v_{2i}v_{2i-1}v_0$$

is an anticycle of length $m+1$, which is as desired. $\square$

Our last auxiliary lemma uses the anticycle found in the previous lemma, and turns it into an antipath on more vertices.

Lemma 8. *Let $k \in \mathbb{N}$ and let D be an oriented graph of minimum pseudo-semidegree $\bar{\delta}^0(D) > k/2$, and let C be an anticycle of length $m+1$ in D. If $m < k$, then D has an antipath of length $m+1$.*

Proof. Let $C = v_0v_1v_2 \dots v_mv_0$. By symmetry, we may assume that $v_0v_1, v_0v_m \in E(D)$. Observe that we may assume the following for all $i = 0, 1, \dots, m$:

If i is even, then all out-neighbours of v_i lie on C; and (3)

if i is odd, then all in-neighbours of v_i lie on C, (4)

as any such out- or in-neighbour could be added to C to obtain an antipath of length $m+1$ in D, and then we would be done.

In particular, (3) and (4) imply that all out-neighbours of v_0 and all in-neighbours of v_m lie on the anticycle C. Set $x_i = y_i = v_i$ for $i = 0, \dots, m$, and consider the underlying graph G of D. Let F_0 comprise of all edges of G corresponding to edges of D that start at v_0 and end on C. Let F_m contain all edges of G corresponding to edges of D that start on C and end at v_m. Since $\bar{\delta}^0(D) > k/2$, and $k \geq m+1$ by assumption, we have that

$$d_{F_0}(v_0, \{v_1 \dots, v_m\}) + d_{F_m}(v_m, \{v_0 \dots, v_{m-1}\}) \geq k+1 \geq m+2.$$

Now, we use Fact 5 with $\ell = 2$ to see that there is an index $i \in \{2, \dots, m\}$ such that v_0v_i and $v_{i-2}v_m$ are edges of D, in these directions.

Let us first assume that i is even, that is, $v_{i-2}v_{i-1}, v_iv_{i-1} \in E(D)$. Since v_i has both in-and out-neighbours in D (for example v_0 and v_{i-1}), and because of our assumption on the minimum pseudo-semidegree, we know that v_i has at least $k/2$ in-neighbours and $k/2$ out-neighbours. By (3), all out-neighbours of v_i belong to $V(C)$. So at most $|V(C - v_i)| - k/2 = m - k/2 \leq k/2 - 1$ vertices of C are in-neighbours of v_i, which means that v_i has an in-neighbour $x \in V(D) \setminus V(C)$.

Since $\bar{\delta}^0(D) > k/2$, and since x has an out-neighbour (namely v_i), we know that x has at least $(k+1)/2 > (m+1)/2$ out-neighbours in D. These cannot all lie in $V(C)$, as otherwise one of them would be a vertex v_j with j odd, a contradiction to (4). Thus vertex x has an out-neighbour y that does not lie on C.

Consider the antipath

$$P = yxv_iv_0v_1 \dots v_{i-3}v_{i-2}v_mv_{m-1}v_{m-2} \dots v_{i+1}.$$

As $x, y \in V(P) \setminus V(C)$, and $V(C) \setminus V(P) = \{v_{i-1}\}$, the antipath P has length $m+1$, which is as desired.

If i is odd, we can find, in a similar way as above, vertices x and y such that

$$yxv_{i-2}v_mv_{m-1} \dots v_iv_0v_1 \dots v_{i-3}$$

is an antipath of length $m+1$. This finishes the proof. □

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Let m be the length of a longest antipath P of D. It is easy to see that $m \geq 2$.

First assume $m < k$. Then by Lemma 7, D has an anticycle of length $m+1$, and therefore, by Lemma 8 (which can be used since $(3k-2)/4 > k/2$ for $k \geq 3$), D has an antipath of length $m+1$, a contradiction to the choice of m.

So $m \geq k$. Let P' be an antipath of length k. If $m > k$, or if $m = k$ and k is odd, then P contains P' (possibly reverting P). So we can assume $m = k$ and k is even. Now, we can apply Lemma 6 with $k' = 3k/2 - 1$, because $\delta^0(D) \geq (3k-2)/4$ implies that $\delta^0(D) \geq k'/2$, and furthermore, $m = k < k'$ since $k > 2$. So m is odd, a contradiction. □

4. Proof of Theorem 4

We will start by proving a lemma that allows us to rewrite the condition on the edge density as a condition on the minimum pseudo-semidegree. This lemma also appears in [13], but for completeness, we include its short proof here.

Lemma 9. *Let $\ell \in \mathbb{N}$. If a digraph D has more than $\ell|V(D)|$ edges, then it contains a digraph D' with $\bar{\delta}^0(D') \geq (\ell+1)/2$.*

Proof. Note that the vertices of D have, on average, in-degree greater than ℓ and out-degree greater than ℓ. Consider the following folklore construction of an auxiliary bipartite graph B associated to D: first, divide each vertex $v \in V(D)$ into two vertices v_{in} and v_{out}, letting v_{in} be adjacent to all edges ending at v, and letting v_{out} be adjacent to all edges starting at v; second, omit all directions on edges.

Then the average degree of B is greater than ℓ, and a standard argument shows that B has a non-empty subgraph B' of minimum degree exceeding $\ell/2$ (for this, it suffices to successively delete vertices of degree $\leq \ell/2$ and to calculate that we have not deleted the entire graph). Translating B' back to the digraph setting, we see that D has a subdigraph D' with minimum pseudo-semidegree exceeding $\ell/2$. □

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Use Lemma 9 to find a subdigraph D' of D with $\bar{\delta}^0(D') \geq ((3k-4)/2+1)/2 = (3k-2)/4$. As a subdigraph of D, also D' is an oriented graph. So Theorem 2 can be applied to find each antipath of length k. □

5. Final remarks and open problems

A lower bound in Theorem 2. We believe the bound in Theorem 2 is not best possible. We think the lower bound $\delta^0(D) > k/2$ from Conjecture 1 should be closer to the truth, although we have not been able to improve our result in that direction. We remark that if one could improve the bound from Lemma 7, then, following all steps of our proof, one would automatically obtain an improved bound for Theorem 2.

Other orientations of the path. In Conjecture 1, all possible orientations of the k-edge path are considered. As a weakening of the conjecture, we could ask the following.

Problem 10. Does Theorem 2 hold for other types of oriented paths?

The fact that Problem 10 holds for the extreme opposites of possible orientations of paths – antipaths and directed paths [8] – may induce some hope that Problem 10 is true, whether or not Conjecture 1 holds.

Antitrees. It seems natural to replace anti-paths with anti-trees in Conjecture 1. Together with Zárate-Guerén the second author shows in [13] the following result, where an anti-tree is balanced if it has as many vertices of out-degree 0 as vertices of in-degree 0.

Theorem 11. ***[13]*** *For all $\epsilon, c > 0$ there is an n_0 such that for all $n \geq n_0$ and all $k \geq \epsilon n$, every oriented graph D with $\delta^0(D) > (1/2+\epsilon)k$ contains each balanced anti-tree T with k edges and with maximum degree at most $c \log n$.*

The proof of Theorem 11 uses digraph regularity, but it might be possible to find a simpler proof and/or drop either the approximation or the additional condition on the balancedness and the maximum degree of T if we only look for specific anti-trees.

Problem 12. Does every oriented graph D with $\delta^0(D) > k/2$ (or $\delta^0(D) > (1/2+\epsilon)k$, or $\delta^0(D) > 3k/4$) contain each anti-tree T with k edges, if we add some additional restriction on T (e.g. T is a caterpillar, spider, has small diameter,...)?

Digraphs. Analogous questions can be asked for digraphs. Observe that as for oriented graphs, a greedy embedding argument gives that $\delta^0(D) \geq k$ is enough to guarantee a copy of any oriented k-edge tree T in a digraph D. So it seems natural to ask whether this bound can be lowered. However, in contrast to the situation in oriented graphs, it will now be necessary to add an additional condition on the order of the largest connected component of the underlying graph of D, as the minimum semidegree condition alone is not sufficient to ensure that $|V(D)| \geq |V(T)|$ (since D could be the union of complete digraphs of order $\delta^0(D) + 1$). For instance, one could ask whether every digraph D with $\delta^0(D) > k/2$ (or some other bound) having a component of size at least $k + 1$ contains each oriented k-edge path. For this question and further comments see [12].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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