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Maximal matroids in weak order posets

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ABSTRACT

Let $\mathcal{X}$ be a family of subsets of a finite set E. A matroid on E is called an $\mathcal{X}$-matroid if each set in $\mathcal{X}$ is a circuit. We develop techniques for determining when there exists a unique maximal $\mathcal{X}$-matroid in the weak order poset of all $\mathcal{X}$-matroids on E and formulate a conjecture which would characterise the rank function of this unique maximal matroid when it exists. The conjecture suggests a new type of matroid rank function which extends the concept of weakly saturated sequences from extremal graph theory. We verify the conjecture for various families $\mathcal{X}$ and show that, if true, the conjecture could have important applications in such areas as combinatorial rigidity and low rank matrix completion.

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1. Introduction

1.1. Unique maximality problem and submodularity conjecture

Let $\mathcal{X}$ be a family of subsets of a finite set E. We will refer to any matroid on E in which each set in $\mathcal{X}$ is a circuit as an *$\mathcal{X}$-matroid on E*. The set of all $\mathcal{X}$-matroids on E forms a poset under the *weak order of matroids* in which, for two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ with the same groundset, we have $\mathcal{M}_1 \preceq \mathcal{M}_2$ if every independent set in $\mathcal{M}_1$ is independent in $\mathcal{M}_2$. The main problem addressed in this paper is to determine when this poset has a unique maximal element and to characterise this unique maximal matroid when it exists.

Our key tool is the following upper bound on the rank function of any $\mathcal{X}$-matroid on E from [8]. A *proper $\mathcal{X}$-sequence* is a sequence $\mathcal{S} = (X_1, X_2, \ldots, X_k)$ of sets in $\mathcal{X}$ such that $X_i \not\subseteq \bigcup_{j=1}^{i-1} X_j$ for all $i = 2, \ldots, k$. For $F \subset E$, let $\mathrm{val}(F, \mathcal{S}) = |F \cup (\bigcup_{i=1}^{k} X_i)| - k$.

Lemma 1.1 *([8, Lemma 3.3]). Suppose $\mathcal{M}$ is an $\mathcal{X}$-matroid on E and $F \subseteq E$. Then $r_{\mathcal{M}}(F) \leq \mathrm{val}(F, \mathcal{S})$ for any proper $\mathcal{X}$-sequence $\mathcal{S}$. Furthermore, if equality holds, then $r_{\mathcal{M}}(F - e) = r_{\mathcal{M}}(F) - 1$ for all $e \in F \setminus (\bigcup_{X \in \mathcal{S}} X)$ and $r_{\mathcal{M}}(F + e) = r_{\mathcal{M}}(F)$ for all $e \in \bigcup_{X \in \mathcal{S}} X$.*

We can use this lemma to derive a sufficient condition for the poset of all $\mathcal{X}$-matroids on E to have a unique maximal element. We need to consider a slightly larger poset. We say that a matroid $\mathcal{M}$ on E is *$\mathcal{X}$-cyclic* if each $X \in \mathcal{X}$ is a *cyclic set* in $\mathcal{M}$ i.e. for every $e \in X$, there is a circuit C of $\mathcal{M}$ with $e \in C \subseteq X$.

Lemma 1.2. *Let $\mathcal{X}$ be a family of subsets of a finite set E and define $\mathrm{val}_{\mathcal{X}} : 2^E \to \mathbb{Z}$ by*

$$\mathrm{val}_{\mathcal{X}}(F) = \min\{\mathrm{val}(F, \mathcal{S}) : \mathcal{S} \text{ is a proper } \mathcal{X}\text{-sequence}\} \qquad (F \subseteq E). \tag{1}$$

Suppose $\mathrm{val}_{\mathcal{X}}$ is a submodular set function on E. Then $\mathrm{val}_{\mathcal{X}}$ is the rank function of an $\mathcal{X}$-cyclic matroid $\mathcal{M}_{\mathcal{X}}$ on E. In addition, if the poset of all $\mathcal{X}$-matroids on E is nonempty, then $\mathcal{M}_{\mathcal{X}}$ is the unique maximal $\mathcal{X}$-matroid on E.

Proof. It is straightforward to check that $\mathrm{val}_{\mathcal{X}}$ is non-decreasing and satisfies $\mathrm{val}_{\mathcal{X}}(e) \leq 1$ for all $e \in E$. Since $\mathrm{val}_{\mathcal{X}}$ is also submodular, this implies that $\mathrm{val}_{\mathcal{X}}$ is the rank function of a matroid $\mathcal{M}_{\mathcal{X}}$. To see that $\mathcal{M}_{\mathcal{X}}$ is $\mathcal{X}$-cyclic, we choose $e \in X \in \mathcal{X}$ and let $\mathcal{S}$ be a proper $\mathcal{X}$-sequence such that $\mathrm{val}_{\mathcal{X}}(X - e) = \mathrm{val}(X - e, \mathcal{S})$. If $e \in \bigcup_{X_i \in \mathcal{S}} X_i$ then we have $\mathrm{val}_{\mathcal{X}}(X) \leq \mathrm{val}(X, \mathcal{S}) = \mathrm{val}(X - e, \mathcal{S}) = \mathrm{val}_{\mathcal{X}}(X - e)$. On the other hand, if $e \notin \bigcup_{X_i \in \mathcal{S}} X_i$ then we can extend $\mathcal{S}$ to a longer proper $\mathcal{X}$-sequence $\mathcal{S}'$ by adding X as the last element of $\mathcal{S}'$ and we will have $\mathrm{val}_{\mathcal{X}}(X) \leq \mathrm{val}(X, \mathcal{S}') = \mathrm{val}(X - e, \mathcal{S}) = \mathrm{val}_{\mathcal{X}}(X - e)$. In both cases equality must hold throughout since $\mathrm{val}_{\mathcal{X}}$ is non-decreasing. Since $\mathrm{val}_{\mathcal{X}}$ is the rank function of $\mathcal{M}_{\mathcal{X}}$, the equality $\mathrm{val}_{\mathcal{X}}(X) = \mathrm{val}_{\mathcal{X}}(X - e)$ implies that e belongs to a circuit of $\mathcal{M}_{\mathcal{X}}$ which is contained in X. Hence $\mathcal{M}_{\mathcal{X}}$ is $\mathcal{X}$-cyclic.

Lemma 1.1 implies that $\mathcal{M} \preceq \mathcal{M}_{\mathcal{X}}$ for every $\mathcal{X}$-matroid $\mathcal{M}$ on E. If there exists at least one $\mathcal{X}$-matroid on E, then this implies that each $X \in \mathcal{X}$ is a circuit in $\mathcal{M}_{\mathcal{X}}$ and that $\mathcal{M}_{\mathcal{X}}$ is the unique maximal $\mathcal{X}$-matroid on E. □

We conjecture that the converse to Lemma 1.2 is also true. The special case when $\mathcal{X}$ is the set of all non-spanning circuits of a matroid on E was previously given in [8].

Conjecture 1.3. *Let $\mathcal{X}$ be a family of subsets of a finite set E. Suppose there is at least one $\mathcal{X}$-matroid on E. Then the poset of all $\mathcal{X}$-matroids on E has a unique maximal element if and only if* $\mathrm{val}_{\mathcal{X}}$ *is a submodular set function on E.*[1]

We will develop techniques for determining when there exists a unique maximal $\mathcal{X}$-matroid in the weak order poset of all $\mathcal{X}$-matroids on E and use them to verify this conjecture for various families $\mathcal{X}$.

Conjecture 1.3 is motivated by the polynomial identity testing problem of symbolic determinants (or the Edmonds problem). In this problem, we are given a matrix A with entries in $\mathbb{Q}[x_1, \ldots, x_n]$, and we are asked to decide whether the rank of A over $\mathbb{Q}(x_1, \ldots, x_n)$ is at least a given number. The Schwarz-Zippel Lemma implies that the problem is in the class NP, but it is a long-standing open problem to show that it is also in co-NP. The following experimental approach may aid our understanding of this problem. We first test the linear independence/dependence of small sets of rows of A to obtain a family $\mathcal{X}$ of minimally dependent sets of rows. Then Lemma 1.1 tells us that we can use any $\mathcal{X}$-sequence to obtain a certificate that the rank of A is at most a specified value. In addition, if the "freest" matroid on the groundset E indexed by the rows of A in which each set in $\mathcal{X}$ is a circuit is uniquely determined, then Conjecture 1.3 would imply that its rank is $\mathrm{val}_{\mathcal{X}}$ and this function has the potential to be the rank function of the row matroid of A.

We are particularly interested in special cases of Edmond's problem which arise in the study of the rigidity of frameworks and the low rank completion of partially filled matrices If true, Conjecture 1.3 could have important applications in these areas. These applications will be discussed in Section 6 below.

1.2. Unique maximality problem on graphs

We will concentrate on the special case of Conjecture 1.3 when E is the edge set of a graph G and $\mathcal{X}$ is the family $\mathcal{H}_G$ of edge sets of all subgraphs of G which are isomorphic to some member of a given family $\mathcal{H}$ of graphs. To simplify terminology we say that a matroid $\mathcal{M}$ is a *$\mathcal{H}$-matroid on G* if it is an $\mathcal{H}_G$-matroid on $E(G)$. We will assume throughout that G contains at least one copy of each $H \in \mathcal{H}$ otherwise we can just

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[1] More generally, if we remove the hypothesis that there is at least one $\mathcal{X}$-matroid on E, then we conjecture that the poset of all $\mathcal{X}$-cyclic matroids on E has a unique maximal element if and only if $\mathrm{val}_{\mathcal{X}}$ is a submodular set function on E.

consider $\mathcal{H}\backslash\{H\}$. This implies that the edge sets of any two subgraphs of G which are isomorphic to the same subgraph of a graph $H \in \mathcal{H}$ will have the same rank in $\mathcal{M}$, but we do not require $\mathcal{M}$ to be completely *symmetric* i.e. we do not require that the edge sets of every pair of isomorphic subgraphs of G have the same rank.

We will simplify notation in the case when $\mathcal{H} = \{H\}$ and refer to a $\mathcal{H}$-matroid on G as a *H-matroid on G*. Two examples of K_3-matroids on K_n are the graphic matroid of K_n and the rank two uniform matroid on $E(K_n)$.

Chen, Sitharam and Vince previously considered the unique maximality problem for H-matroids on K_n for various graphs H. They announced at a workshop at BIRS in 2015, see [27], that there is a unique maximal K_5-matroid on K_n. Sitharam and Vince subsequently released a preprint [28] which claims to show that there is a unique maximal H-matroid on K_n for *all* graphs H. Unfortunately their claim is false. Pap [22] pointed out that the poset of C_5-matroids on K_n has two maximal elements. We will describe Pap's counterexample, and give other counterexamples to the Sitharam-Vince claim in Section 5.

Our interest in this topic was motivated by the work of Graver, Servatius, and Servatius [12,13] and Whiteley [31] on maximal abstract rigidity matroids, and that of Chan, Sitharam and Vince [27,28] on maximal H-matroids. In two joint papers with Clinch [7,8], we were able to confirm that there is a unique maximal K_5-matroid on K_n and, more importantly, give a good characterisation for the rank function of this matroid. The theory of matroid erections due to Crapo [9] is a key ingredient in our proof technique.

In this paper we will use results on matroid erection from [8] to construct a maximal element in the poset of all $\mathcal{X}$-matroids on a set E. We will show that this element is the unique maximal element in the poset of all $\mathcal{H}$-matroids on a graph G for various pairs $(\mathcal{H}, G)$, and verify that Conjecture 1.3 holds in each case.

1.3. Weakly saturated sequences

The function $\mathrm{val}_{\mathcal{X}}$ defined in (1) is related to the *weak saturation number* in extremal graph theory. Let $\mathcal{X}$ be a family of subsets of a finite set E, and $F_0 \subseteq E$. A proper $\mathcal{X}$-sequence $(X_1, X_2, \ldots, X_m)$ is said to be a *weakly $\mathcal{X}$-saturated sequence from F_0* if $|X_i \setminus (F_0 \cup \bigcup_{j<i} X_j)| = 1$ for all i with $1 \le i \le m$. We say that *E can be constructed by a weakly $\mathcal{X}$-saturated sequence $(X_1, X_2, \ldots, X_m)$ from F_0* if there is a weakly $\mathcal{X}$-saturated sequence $\mathcal{S}$ from F_0 with $E = F_0 \cup \bigcup_{i=1}^{m} X_i$. These sequences were first introduced by Bollobás [4], where he posed the problem of determining the size of a smallest set F_0 from which E can be constructed by a weakly $\mathcal{X}$-saturated sequence. The problem has subsequently been studied by several authors, typically in the case when E is the edge set of a complete k-uniform hypergraph or a complete bipartite graph, see for example [1,4,16,17,20,23,24]. We will see in Sections 3 and 4 that results on weakly $\mathcal{X}$-saturated sequences can sometimes be used to prove the unique maximality of an $\mathcal{X}$-matroid. However this approach is applicable only when the flats of the target matroid are easily

described. (The difficulty of deciding uniqueness when the structure of the flats is more complicated is illustrated by the matroids discussed in Section 6.)

The concept of $\mathcal{X}$-matroids was previously studied by Kalai [16] and Pikhurko [24] with the goal of constructing a maximum rank $\mathcal{X}$-matroid on E to obtain a lower bound on the size of a set F_0 from which E can be constructed by a weakly $\mathcal{X}$-saturated sequence. Our concern in this paper is different: we would like to gain a better understanding of the poset of all $\mathcal{X}$-matroids on a given finite set E by determining its maximal elements.

We next give an outline of the remaining sections in the paper. Section 2 uses the theory of matroid erections to construct a particular $\mathcal{X}$-matroid which will always be a maximal element in the poset of all $\mathcal{X}$-matroids. Sections 3 and 4 use the theories of weakly saturated sequences and submodular functions to show that this matroid is the unique maximal element for various families $\mathcal{X}$. Section 5 constructs examples in which the poset of all $\mathcal{X}$-matroids does not have a unique maximal element. Section 6 describes several families of $\mathcal{X}$-matroids arising in discrete geometry and matrix completion. We formulate conjectures which would characterise the rank functions of generic rigidity, birigidity and hyperconnectivity matroids, and obtain some partial results in support of these conjectures.

We close this introduction by listing notation used throughout this paper. Let $\mathcal{M}$ be a matroid on a finite set E. Its rank function and closure operator are denoted by $r_{\mathcal{M}}$ and $\mathrm{cl}_{\mathcal{M}}$, respectively. A set $F \subseteq E$ with $\mathrm{cl}_{\mathcal{M}}(F) = F$ is called a *flat*.

For a graph G, $V(G)$ and $E(G)$ denote its vertex set and its edge set, respectively. Let $N_G(v)$ be the set of neighbours of v in G. For $F \subseteq E(G)$, let $V(F)$ be the set of vertices incident to F and let $G[F]$ be the graph with vertex set $V(F)$ and edge set F. Let $d_F(v)$ be the number of edges in F incident to a vertex $v \in V(G)$, and let $N_F(v)$ be the set of neighbours of v in $G[F]$.

For disjoint sets X and Y, let $K(X)$ be the complete graph with vertex set X and $K(X;Y)$ be the complete bipartite graph with vertex partition (X, Y).

2. Maximal matroids and matroid elevations

Let $\mathcal{X}$ be a family of subsets of a finite set E. We first derive a sufficient condition for a given $\mathcal{X}$-matroid on E to be the unique maximal such matroid. We then use results from [8] to construct a maximal element in the poset of all $\mathcal{X}$-matroids on E (whenever this poset is non-empty).

2.1. A sufficient condition for unique maximality

Recall that a set F in a matroid $\mathcal{M}$ is *connected* if, for every pair of elements $e_1, e_2 \in F$, there exists a circuit C of $\mathcal{M}$ with $e_1, e_2 \in C \subseteq F$, and that F is a *connected component* of $\mathcal{M}$ if F is either a coloop of $\mathcal{M}$ or a maximal connected set in $\mathcal{M}$. It is well known that the set $\{F_1, F_2, \ldots, F_m\}$ of all connected components partitions the ground set of

$\mathcal{M}$ and that $\operatorname{rank}\mathcal{M} = \sum_{i=1}^{m} r_{\mathcal{M}}(F_i)$. In addition, F is connected in $\mathcal{M}$ if and only if $r_{\mathcal{M}}(F) < r_{\mathcal{M}}(F') + r_{\mathcal{M}}(F'')$ for all partitions $\{F', F''\}$ of F.

Lemma 2.1. *Let $\mathcal{X}$ be a family of subsets of a finite set E and $\mathcal{M}$ be a loopless $\mathcal{X}$-matroid on E. Suppose that, for every connected flat F of $\mathcal{M}$, there is a proper $\mathcal{X}$-sequence $\mathcal{S}$ with $r_{\mathcal{M}}(F) = \operatorname{val}(F, \mathcal{S})$. Then $\operatorname{val}_{\mathcal{X}} = r_{\mathcal{M}}$ and $\mathcal{M}$ is the unique maximal $\mathcal{X}$-matroid on E.*

Proof. Since $r_{\mathcal{M}} \leq \operatorname{val}_{\mathcal{X}}$ for all $\mathcal{X}$-matroids on E by Lemma 1.1, it will suffice to show that, for each $F \subseteq E$, there is a proper $\mathcal{X}$-sequence $\mathcal{S}$ such that $r_{\mathcal{M}}(F) = \operatorname{val}(F, \mathcal{S})$.

Suppose, for a contradiction, that this is false for some set F. We may assume that F has been chosen such that $r_{\mathcal{M}}(F)$ is as small as possible and, subject to this condition, $|F|$ is as large as possible. If F is not a flat then $r_{\mathcal{M}}(F + e) = r_{\mathcal{M}}(F)$ for some $e \in E \backslash F$ and we can now use the maximality of $|F|$ to deduce that there exists a proper $\mathcal{X}$-sequence $\mathcal{S}$ such that $r_{\mathcal{M}}(F + e) = \operatorname{val}(F + e, \mathcal{S})$. By Lemma 1.1 and $r_{\mathcal{M}}(F) = r_{\mathcal{M}}(F + e)$, $e \in \bigcup_{X \in \mathcal{S}} X$. Hence, $\operatorname{val}(F + e, \mathcal{S}) = \operatorname{val}(F, \mathcal{S}) = r_{\mathcal{M}}(F + e) = r_{\mathcal{M}}(F)$. This would contradict the choice of F. Hence F is a flat.

Suppose F is not connected. Then we have $r_{\mathcal{M}}(F) = r_{\mathcal{M}}(F_1) + r_{\mathcal{M}}(F_2)$ for some partition $\{F_1, F_2\}$ of F. Since $\mathcal{M}$ is loopless, F_i is a flat of $\mathcal{M}$ and $1 \leq r_{\mathcal{M}}(F_i) < r_{\mathcal{M}}(F)$ for both $i = 1, 2$. The choice of F now implies that there exists a proper $\mathcal{X}$-sequence $\mathcal{S}_i$ such that $r_{\mathcal{M}}(F_i) = \operatorname{val}(F_i, \mathcal{S}_i)$ for $i = 1, 2$. Since each F_i is a flat, we have $X_i \subseteq F_i$ for all $X_i \in \mathcal{S}_i$ by Lemma 1.1. This implies that the concatenation $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ is a proper $\mathcal{X}$-sequence and satisfies

$$\operatorname{val}(F, \mathcal{S}) = \operatorname{val}(F_1, \mathcal{S}_1) + \operatorname{val}(F_2, \mathcal{S}_2) = r_{\mathcal{M}}(F_1) + r_{\mathcal{M}}(F_2) = r_{\mathcal{M}}(F).$$

This contradicts the choice of F.

Hence F is a connected flat and we can use the hypothesis of the lemma to deduce that there is a proper $\mathcal{X}$-sequence $\mathcal{S}$ such that $r_{\mathcal{M}}(F) = \operatorname{val}(F, \mathcal{S})$, as required. □

2.2. Matroid elevations

The *truncation* of a matroid $\mathcal{M}_1 = (E, \mathcal{I}_1)$ of rank k is the matroid $\mathcal{M}_0 = (E, \mathcal{I}_0)$ of rank $k - 1$, where $\mathcal{I}_0 = \{I \in \mathcal{I}_1 : |I| \leq k - 1\}$. Crapo [9] defined *matroid erection* as the ‘inverse operation’ to truncation. So $\mathcal{M}_1$ is an *erection* of $\mathcal{M}_0$ if $\mathcal{M}_0$ is the truncation of $\mathcal{M}_1$. (For technical reasons we also consider $\mathcal{M}_0$ to be a *trivial erection* of itself.) Note that, although every matroid has a unique truncation, matroids may have several, or no, non-trivial erections.

Crapo [9] showed that the poset of all erections of a matroid $\mathcal{M}_0$ is actually a lattice. It is clear that the trivial erection of $\mathcal{M}_0$ is the unique minimal element in this lattice. Since this is a finite lattice, there also exists a unique maximal element which Crapo called the *free erection of* $\mathcal{M}_0$.

A *partial elevation* of $\mathcal{M}_0$ is any matroid $\mathcal{M}$ which can be constructed from $\mathcal{M}_0$ by a sequence of erections. A *(full) elevation* of $\mathcal{M}_0$ is a partial elevation $\mathcal{M}$ which has no non-trivial erection. The *free elevation* of $\mathcal{M}_0$ is the matroid we get from $\mathcal{M}_0$ by recursively constructing a sequence of free erections until we arrive at a matroid which has no non-trivial erection. The set of all partial elevations of $\mathcal{M}_0$ forms a poset $P(\mathcal{M}_0)$ under the weak order and $\mathcal{M}_0$ is its unique minimal element. Every maximal element of $P(\mathcal{M}_0)$ will have no non-trivial erection so will be a full elevation of $\mathcal{M}_0$. Given Crapo's result that the poset of all erections of $\mathcal{M}_0$ is a lattice, it is tempting to conjecture that $P(\mathcal{M}_0)$ will also be a lattice and that the free elevation of $\mathcal{M}_0$ will be its unique maximal element. But this is false: Brylawski gives a counterexample based on the Vamos matroid in [4] and we will construct another counterexample using $\mathcal{H}$-matroids on K_n in Section 5. The following weaker result is given in [8].

Lemma 2.2 *([8, Lemma 3.1]). Suppose that $\mathcal{M}_0$ is a matroid. Then the free elevation of $\mathcal{M}_0$ is a maximal element in the poset of all partial elevations of $\mathcal{M}_0$.*

Our next result extends Lemma 2.2 to $\mathcal{X}$-matroids.

Lemma 2.3. *Let E be a finite set, $\mathcal{X}$ be a family of subsets of E of size at most s, and $\mathcal{M}_0$ be an $\mathcal{X}$-matroid with rank s which is maximal in the poset of all $\mathcal{X}$-matroids on E with rank at most s. Then the free elevation of $\mathcal{M}_0$ is a maximal matroid in the poset of all $\mathcal{X}$-matroids on E.*

Proof. Let $\mathcal{M}$ be the free elevation of $\mathcal{M}_0$. Since $\mathcal{M}_0$ is an $\mathcal{X}$-matroid with rank s, every set in $\mathcal{X}$ is a non-spanning circuit of $\mathcal{M}_0$. This implies that every partial elevation of $\mathcal{M}_0$ is an $\mathcal{X}$-matroid. In particular, $\mathcal{M}$ is an $\mathcal{X}$-matroid.

Lemma 2.2 implies that $\mathcal{M}$ is a maximal element in the poset of all partial elevations of $\mathcal{M}_0$. Let $\mathcal{N}$ be an $\mathcal{X}$-matroid on E which is not a partial elevation of $\mathcal{M}_0$. Let $\mathcal{N}_0$ be the truncation of $\mathcal{N}$ to rank s if $\mathcal{N}$ has rank at least s, and otherwise let $\mathcal{N}_0 = \mathcal{N}$. Then $\mathcal{N}_0 \neq \mathcal{M}_0$. Since $\mathcal{M}_0$ is a maximal $\mathcal{X}$-matroid in the poset of all $\mathcal{X}$-matroids on E with rank at most s, $\mathcal{N}_0 \not\succ \mathcal{M}_0$ holds. Hence there exists $F \subseteq E$ with the properties that $|F| \leq s$, F is dependent in $\mathcal{N}_0$ and F is independent in $\mathcal{M}_0$. This implies that F is dependent in $\mathcal{N}$ and independent in $\mathcal{M}$ so $\mathcal{N} \not\succ \mathcal{M}$. Hence $\mathcal{M}$ remains as a maximal element in the poset of all $\mathcal{X}$-matroids on E. $\square$

Lemma 2.3 can be applied whenever there exists at least one $\mathcal{X}$-matroid $\mathcal{M}$ on E with rank at least s since we can truncate $\mathcal{M}$ to obtain an $\mathcal{X}$-matroid with rank s, and hence the poset of all $\mathcal{X}$-matroids on E with rank at most s will be non-empty.[2]

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[2] Note that in our main motivation, the Edmonds Problem, $\mathcal{X}$ will be a family of minimal row dependencies of a matrix A and hence the row matroid of A will be an $\mathcal{X}$-matroid.

We close this subsection by stating a useful property of free elevations. We say that an $\mathcal{X}$-matroid $\mathcal{M}$ on a finite set E has the *$\mathcal{X}$-covering property* if every cyclic flat in $\mathcal{M}$ is the union of sets in $\mathcal{X}$.

Lemma 2.4 *([8, Lemma 3.8]). Let $\mathcal{M}_0$ be a matroid on a finite set E, $\mathcal{X}$ be a family of non-spanning circuits of $\mathcal{M}_0$ and $\mathcal{M}$ be the free elevation of $\mathcal{M}_0$. Suppose that each cyclic flat in $\mathcal{M}_0$ is the union of circuits in $\mathcal{X}$. Then each cyclic flat in $\mathcal{M}$ is the union of circuits in $\mathcal{X}$.*

2.3. Uniform $\mathcal{X}$-matroids

A family $\mathcal{X}$ of sets is *k-uniform* if each set in $\mathcal{X}$ has size k. Given a k-uniform family $\mathcal{X}$, the *$\mathcal{X}$-uniform system* $\mathcal{U}_{\mathcal{X}}$ is defined as the pair $(E, \mathcal{I}_{\mathcal{X}})$, where $E = \bigcup_{X \in \mathcal{X}} X$ and

$$\mathcal{I}_{\mathcal{X}} := \{F \subseteq E : |F| \leq k \text{ and } F \notin \mathcal{X}\}.$$

We first characterise when $\mathcal{U}_{\mathcal{X}}$ is a matroid. We say that the k-uniform family $\mathcal{X}$ is *union-stable* if, for any $X_1, X_2 \in \mathcal{X}$ and $e \in X_1 \cap X_2$, either $|(X_1 \cup X_2) - e| > k$ or $(X_1 \cup X_2) - e \in \mathcal{X}$.

Lemma 2.5. *Suppose that $\mathcal{X}$ is a k-uniform family. Then $\mathcal{U}_{\mathcal{X}}$ is a matroid if and only if $\mathcal{X}$ is union-stable.*

Proof. Let $\mathcal{C} = \mathcal{X} \cup \{C \subseteq E : |C| = k+1 \text{ and } X \not\subseteq C \text{ for all } X \in \mathcal{X}\}$. It is straightforward to check $\mathcal{U}_X$ is a matroid if and only if $\mathcal{C}$ satisfies the matroid circuit axioms and that the latter property holds if and only if $\mathcal{X}$ is union-stable. □

We will refer to the matroid $\mathcal{U}_{\mathcal{X}}$ given in Lemma 2.5 as the *uniform $\mathcal{X}$-matroid on* $E = \bigcup_{X \in \mathcal{X}} X$. The special case when $\mathcal{X}$ consists of all the subsets of E of size k is the *uniform matroid on E of rank $k-1$*, i.e. the matroid $\mathcal{U}_{k-1}(E)$ in which a set $F \subseteq E$ is independent if and only if $|F| \leq k-1$. Note that $\mathcal{U}_{\mathcal{X}}$ is a paving matroid and, when $\mathcal{U}_{\mathcal{X}} \neq \mathcal{U}_{k-1}(E)$, $\mathcal{X}$ is its set of non-spanning circuits.

Given an arbitrary k-uniform family $\mathcal{X}$, we can construct the *union-stable closure $\bar{\mathcal{X}}$ of $\mathcal{X}$* by first putting $\bar{\mathcal{X}} = \mathcal{X}$ and then recursively adding $(X_1 \cup X_2) - e$ to $\bar{\mathcal{X}}$ whenever $X_1, X_2 \in \bar{\mathcal{X}}$, $|X_1 \cup X_2| = k+1$ and $e \in X_1 \cap X_2$. It is straightforward to check that the resulting family $\bar{\mathcal{X}}$ is k-uniform and union-stable and that $\mathcal{U}_{\bar{\mathcal{X}}}$ is a maximal matroid in the poset of all $\mathcal{X}$-matroids on $E = \bigcup_{X \in \mathcal{X}} X$ with rank at most k. In addition, if $\mathcal{U}_{\bar{\mathcal{X}}} \neq \mathcal{U}_{k-1}(E)$, then $\mathcal{U}_{\bar{\mathcal{X}}}$ has rank k. We can now apply Lemma 2.3 to deduce:

Lemma 2.6. *Let $\mathcal{X}$ be a k-uniform family of sets and $E = \bigcup_{X \in \mathcal{X}} X$. Suppose that $\mathcal{U}_{\bar{\mathcal{X}}} \neq \mathcal{U}_{k-1}(E)$. Then the free elevation of $\mathcal{U}_{\bar{\mathcal{X}}}$ is a maximal $\mathcal{X}$-matroid on E.*

Note that if $\mathcal{U}_{\bar{\mathcal{X}}} = \mathcal{U}_{k-1}(E)$ then $\mathcal{U}_{\bar{\mathcal{X}}}$ is the unique maximal $\mathcal{X}$-matroid on E but the free-elevation of $\mathcal{U}_{\bar{\mathcal{X}}}$ is the *free matroid on* E i.e. the matroid in which every subset of E is independent.

Suppose that G and H are graphs with $|E(H)| = k$ and that every edge of G belongs to a subgraph which is isomorphic to H (we can reduce to this case by deleting all edges of G which do not belong to copies of H). Recall that $\{H\}_G$ denotes the k-uniform family containing all edge sets of copies of H in G. The graph H is said to be *union-stable* on G if $\{H\}_G$ is union-stable, i.e., for any two distinct copies H_1 and H_2 of H in G and any $e \in E(H_1) \cap E(H_2)$, either $H_1 \cup H_2 - e$ is isomorphic to H or $|E(H_1 \cup H_2 - e)| > k$. To simplify notation we denote the uniform $\{H\}_G$-matroid $\mathcal{U}_{\{H\}_G}$ by $\mathcal{U}_H(G)$ when H is union-stable. Examples of union-stable subgraphs of K_n are regular graphs of degree at least two, bipartite graphs of minumum degree at least two in which all vertices on the same side of the bipartition have the same degree and stars. Lemmas 2.3 and 2.5 immediately imply:

Lemma 2.7. *Suppose that G and H are graphs. Then $\mathcal{U}_H(G)$ is a matroid if and only if H is union-stable on G. Furthermore, if $\mathcal{U}_H(G)$ is a matroid, then its free elevation is a maximal H-matroid on G.*

3. Weakly saturated sequences

Let $\mathcal{X}$ be a family of subsets of a finite set E, and $F_0 \subseteq E$. Recall that a proper $\mathcal{X}$-sequence $(X_1, X_2, \ldots, X_m)$ is a weakly $\mathcal{X}$-saturated sequence from F_0 if $|X_i \setminus (F_0 \cup \bigcup_{j<i} X_j)| = 1$ for all i with $1 \leq i \leq m$. We say that a set $F \subseteq E$ can be constructed by a weakly $\mathcal{X}$-saturated sequence from F_0 if there is a weakly $\mathcal{X}$-saturated sequence $(X_1, X_2, \ldots, X_m)$ from F_0 with $F = F_0 \cup \bigcup_{i=1}^m X_i$. Note that if this is the case then we will have $\mathrm{val}(F, \mathcal{S}) = |F_0|$. We can combine this simple observation with Lemma 2.1 to give several examples of unique maximality.

Lemma 3.1. *Let $\mathcal{X}$ be a k-uniform family of sets. Suppose that $E = \bigcup_{X \in \mathcal{X}} X$ can be constructed by a weakly $\mathcal{X}$-saturated sequence from some $X_0 \in \mathcal{X}$. Then the rank $k-1$ uniform matroid $\mathcal{U}_{k-1}(E)$ is the unique maximal $\mathcal{X}$-matroid on E and its rank function is* $\mathrm{val}_{\mathcal{X}}$.

Proof. We denote $\mathcal{U} = \mathcal{U}_{k-1}(E)$. Note that $\mathcal{U}$ is an $\mathcal{X}$-matroid. Since $\mathcal{U}$ is uniform, E is the only connected flat in $\mathcal{U}$ and hence, by Lemma 2.1, it will suffice to show that there is a proper $\mathcal{X}$-sequence $\mathcal{S}$ such that $r_{\mathcal{U}}(E) = \mathrm{val}_{\mathcal{X}}(\mathcal{S}, E)$. By hypothesis, there is a weakly saturated $\mathcal{X}$-sequence $\mathcal{S}_0$ from X_0 to E. Let $\mathcal{S}$ be the proper $\mathcal{X}$-sequence obtained by inserting X_0 at the beginning of $\mathcal{S}_0$. Then

$$\mathrm{val}_{\mathcal{X}}(\mathcal{S}, E) = \mathrm{val}_{\mathcal{X}}(\mathcal{S}_0, E) - 1 = |X_0| - 1 = k - 1 = r_{\mathcal{U}}(E),$$

as required. □

The same proof technique can handle a slightly more complicated situation.

Lemma 3.2. *Let $\mathcal{X}$ be a k-uniform, union-stable family of sets. Suppose that $E = \bigcup_{X \in \mathcal{X}} X$ can be constructed by a weakly $\mathcal{X}$-saturated sequence from some $Y \subseteq E$ with $|Y| = k$ and $Y \notin \mathcal{X}$. Then $\mathcal{U}_{\mathcal{X}}$ is the unique maximal $\mathcal{X}$-matroid on E and its rank function is* $\mathrm{val}_{\mathcal{X}}$.

Proof. By Lemma 2.5, $\mathcal{U}_{\mathcal{X}}$ is an $\mathcal{X}$-matroid. By Lemma 2.1, it will suffice to show that there is a proper $\mathcal{X}$-sequence $\mathcal{S}$ such that $r_{\mathcal{U}_{\mathcal{X}}}(F) = \mathrm{val}_{\mathcal{X}}(\mathcal{S}, F)$ for every connected flat in $\mathcal{U}_{\mathcal{X}}$. Let F be a connected flat in $\mathcal{U}_{\mathcal{X}}$. Then the definition of $\mathcal{U}_{\mathcal{X}}$ implies that the rank of F is either k or $k-1$.

Suppose that the rank of F is k. Then we have $F = E$. By hypothesis, E can be constructed from Y by a weakly $\mathcal{X}$-saturated sequence $\mathcal{S}$. Then $r_{\mathcal{U}_{\mathcal{X}}}(E) = k = |Y| = \mathrm{val}(E, \mathcal{S})$ follows.

Hence we may assume that the rank of F is $k-1$. Then every subset of F of size k belongs to $\mathcal{X}$. We will use this fact to define a weakly $\mathcal{X}$-saturated sequence for F. Choose a set F_0 of $k-1$ elements in F, and let $X_e = F_0 \cup \{e\}$ for each $e \in F \setminus F_0$. Then each $X_e \in \mathcal{X}$, and $\{X_e : e \in F \setminus F_0\}$ (ordered arbitrarily) is a weakly $\mathcal{X}$-saturated sequence $\mathcal{S}'$ which constructs F from F_0. We have $\mathrm{val}(\mathcal{S}', F) = |F_0| = k - 1 = r_{\mathcal{U}_{\mathcal{X}}}(F)$ as required. □

Applications to matroids on graphs

Given graphs G and H and subgraphs $F_0, F \subseteq G$, we say that F *can be constructed by a weakly H-saturated sequence from F_0* if $E(F)$ can be constructed by a weakly $\{H\}_G$-saturated sequence from $E(F_0)$.

Lemma 3.3. *Let H_k be the vertex-disjoint union of k copies of K_2. Then K_n can be constructed by a weakly saturated H_k-sequence from any copy of H_k in K_n whenever $n \geq 2k+1$.*

Proof. Let $H = \{e_1, e_2, \ldots, e_k\}$ be a copy of H_k in K_n for some $n \geq 2k+1$. We show that K_n has a weakly saturated H_k-sequence starting from H by induction on n. Choose a vertex $v \in V(K_n) \setminus V(H)$.

Suppose $n = 2k+1$. For each edge f from v to H we can choose a k-matching H^f containing f and $k-1$ edges of H. Then for each edge g of $(K_n - v) - E(H)$ we can choose a k-matching H^g containing g and $k-1$ edges of $H \cup \bigcup_{f \sim v} H^f$. Concatenating H with H^f for $f \sim v$ and then H^g for the remaining edges g gives a weakly saturated H_k-sequence which constructs K_{2k+1} from H.

Now suppose $n > 2k+1$. By induction, $K_n - v$ has a weakly saturated H_k-sequence $\mathcal{S}$ starting from H. For each edge f from v to H we can choose a k-matching H^f containing f and $k-1$ edges of $K_n - v$. Concatenating H with H^f gives the required weakly saturated H_k-sequence for K_n. □

Combining Lemmas 3.1 and 3.3, we immediately obtain:

Theorem 3.4. *Let H_k be the vertex-disjoint union of k copies of K_2. Then $\mathcal{U}_{k-1}(K_n)$ is the unique maximal H_k-matroid on K_n for all $n \geq 2k+1$, and val_{H_k} is its rank function.*

Let P_k denote the path with k edges, C_k the cycle with k edges and K_n^- the graph obtained from K_n by removing an edge. It is straightforward to show that K_n can be constructed by a weakly saturated P_k-sequence starting from a particular copy of P_k in K_n whenever $n \geq k+1$. Lemma 3.1 now gives:

Theorem 3.5. *$\mathcal{U}_{k-1}(K_n)$ is the unique maximal P_k-matroid on K_n for all $n \geq k+1$, and val_{P_k} is its rank function.*

Sitharam and Vince [28] showed that $\mathcal{U}_{K_{1,3}}(K_n)$ is the unique maximal $K_{1,3}$-matroid on K_n for all $n \geq 4$. Their result can be deduced from Lemma 3.2 since $K_{1,3}$ is union-stable and K_n can be constructed by a weakly $K_{1,3}$-saturated sequence starting from a copy of K_3. We may also deduce that $\mathrm{val}_{K_{1,3}}$ is the rank function of $\mathcal{U}_{K_{1,3}}(K_n)$.

We next derive an upper bound on the rank of all H-matroids on K_n which is linear in n for any fixed H. In the special case when H is a forest, our upper bound is independent of n and quadratic in the number of edges of H. This implication will be used in Section 5 to construct families of graphs $\mathcal{H}$ for which the poset of all $\mathcal{H}$-matroids on K_n has at least two maximal elements.

Lemma 3.6. *Suppose H is a graph with s vertices and minimum degree δ, and $\mathcal{M}$ is a H-matroid on K_n with $n \geq s-1$. Then the rank of $\mathcal{M}$ is at most $(\delta-1)(n-s+1)+\binom{s-1}{2}$.*

Proof. Let $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and put $V_i = \{v_1, v_2, \ldots, v_i\}$ and $E_i = E(K[V_i])$ for $1 \leq i \leq n$. Choose a base B_{s-1} of $E(K(V_{s-1}))$ in $\mathcal{M}$. Clearly $|B_{s-1}| \leq \binom{s-1}{2}$.

We inductively construct a base B_i of E_i in $\mathcal{M}$ from B_{s-1}, for $i = s, \ldots, n$. Suppose we have a base B_i of E_i, and let $B'_{i+1} = B_i \cup \{v_{i+1}v_1, \ldots, v_{i+1}v_{\delta-1}\}$. Then, for each j with $s \leq j \leq i$, $K(V_i) + \{v_{i+1}v_1, \ldots, v_{i+1}v_{\delta-1}, v_{i+1}v_j\}$ contains a graph isomorphic to H in which the degree of v_{i+1} is equal to δ. Hence B'_{i+1} spans E_i in $\mathcal{M}$. Let B_{i+1} be a base of B'_{i+1} obtained by extending B_i. Then B_{i+1} is obtained from B_i by adding at most $\delta - 1$ edges. This implies that B_n has size at most $(\delta-1)(n-s+1)+\binom{s-1}{2}$. The lemma now follows since B_n is a base of $\mathcal{M}$. □

We close this section by noting that Lemma 3.6 is tight when: $H = K_{1,2}$ with $\mathcal{M} = \mathcal{U}_1(K_n)$; $H = K_{1,3}$ with $\mathcal{M} = \mathcal{U}_{K_{1,3}}(K_n)$, $H = K_{d+2}$ for $d \geq 1$ with $\mathcal{M}$ equal to the d-dimensional rigidity matroid, see Section 6; $H = K_d^-$ for $d = 4, 5, 6$, see Theorem 4.3(c,d,e) below; $H = C_4$, see Theorem 4.6(a) below. It is not tight when: $H = P_k$ with $k \geq 3$ by Theorem 3.5; $H = C_k$ with $k \geq 5$ by a result of Borowiecki and Sidorwicz [5] which implies that the rank of any C_k-matroid on K_n is at most n with strict inequality when k is odd.

4. Matroids induced by submodular functions

In this section we use weakly saturated sequences and a matroid construction due to Edmonds to give more examples of unique maximal matroids.

Theorem 4.1 *(Edmonds [11]). Let E be a finite set and $f : 2^E \to \mathbb{Z}$ be a non-decreasing, submodular function. Put*

$$\mathcal{I}_f := \{F \subseteq E : |I| \le f(I) \textit{ for any } I \subseteq F \textit{ with } I \neq \emptyset\}.$$

Then $\mathcal{M}_f := (E, \mathcal{I}_f)$ is a matroid with rank function $\hat{f} : 2^E \to \mathbb{Z}$ given by

$$\hat{f}(F) := \min\left\{ |F_0| + \sum_{i=1}^{k} f(F_i) : F_0 \subseteq F \textit{ and } \{F_1, \ldots, F_k\} \textit{ is a partition of } F \backslash F_0 \right\}. \tag{2}$$

We refer to the matroid $\mathcal{M}_f$ given by Edmond's theorem as the *matroid induced by* f. Given a set $F \subseteq E$ it is straightforward to check that:

$$\begin{aligned} &F \text{ is a circuit in } \mathcal{M}_f \text{ if and only if } 0 \neq |F| = f(F) + 1 \text{ and} \\ &|F'| \le f(F') \text{ for all } F' \subseteq F \text{ with } \emptyset \neq F' \neq F; \end{aligned} \tag{3}$$

$$F \text{ is a flat in } \mathcal{M}_f \text{ if and only if } f(F + e) = f(F) + 1 \text{ for all } e \in E \backslash F. \tag{4}$$

For any graph G, the function $f_{a,b} : 2^{E(G)} \to \mathbb{Z}$ defined by $f_{a,b}(F) = a|V(F)| - b$ is submodular and non-decreasing for all $a, b \in \mathbb{Z}$ with $a \ge 0$ and hence induces a matroid $\mathcal{M}_{f_{a,b}}(G)$ on $E(G)$. These matroids are known as *count matroids*. It is well known that the cycle matroid of K_n is the count matroid $\mathcal{M}_{f_{1,1}}(K_n)$. Another well-known example is when $a = 2$ and $b = 3$, which gives the rigidity matroid of generic frameworks in $\mathbb{R}^2$. Sitharam and Vince [28] showed that $\mathcal{M}_{f_{1,1}}(K_n)$ and $\mathcal{M}_{f_{2,3}}(K_n)$ are, respectively, the unique maximal K_3-matroid and the unique maximal K_4-matroid on K_n. Slightly weaker versions of these results were previously obtained by Graver [12].

We will show that the maximality of both these matroids, as well as that of several other count matroids, follow easily from Lemma 2.1 and Theorem 4.1. We need the following observation on the connected flats of count matroids which follows immediately from (3) and (4).

Lemma 4.2. *Suppose $a, b \in \mathbb{Z}$ with $a \ge 0$ and $F \subseteq E(G)$ is a connected flat in $\mathcal{M}_{f_{a,b}}(G)$. Then $G[F]$ is the subgraph of G induced by $V(F)$ and $|F| \ge a|V(F)| - b + 1$.*

Recall that K_n^- denotes the graph obtained from K_n by removing an edge.

Theorem 4.3. *(a)* $\mathcal{M}_{f_{1,1}}(K_n)$ *is the unique maximal* K_3*-matroid on* K_n *and its rank function is* val_{K_3}.
(b) $\mathcal{M}_{f_{2,3}}(K_n)$ *is the unique maximal* K_4*-matroid on* K_n *and its rank function is* val_{K_4}.
(c) $\mathcal{M}_{f_{1,0}}(K_n)$ *is the unique maximal* K_4^-*-matroid on* K_n *and its rank function is* $\mathrm{val}_{K_4^-}$.
(d) $\mathcal{M}_{f_{2,2}}(K_n)$ *is the unique maximal* K_5^-*-matroid on* K_n *and its rank function is* $\mathrm{val}_{K_5^-}$.
(e) $\mathcal{M}_{f_{3,5}}(K_n)$ *is the unique maximal* K_6^-*-matroid on* K_n *and its rank function is* $\mathrm{val}_{K_6^-}$.

Proof. In each case $\mathcal{M}_{f_{a,b}}(K_n)$ is loopless and is an X-matroid on K_n for $X = K_3$, K_4, K_4^-, K_5^-, K_6^-, respectively, by (3). Lemmas 2.1 and 4.2 will now imply that $\mathcal{M}_{f_{a,b}}(K_n)$ is the unique maximal X-matroid on K_n once we have shown that, for every $K_m \subseteq K_n$ with $|E(K_m)| > am - b$, there is a proper X-sequence $\mathcal{S}$ with $r_{\mathcal{M}_{f_{a,b}}}(K_m) = \mathrm{val}(K_m, \mathcal{S})$. We will do this by finding a weakly saturated X-sequence which constructs K_m from a subgraph $G \subset K_m$ with $|E(G)| = am - b$. Let $V(K_m) = \{v_1, v_2, \ldots, v_m\}$.

In cases (a) and (b) we can use the well known fact that, for $m \geq d - 1$, K_m can be constructed by a weakly saturated K_d-sequence starting from the spanning subgraph G with $E(G) = \{v_iv_j : 1 \leq i < j \leq d-2\} \cup \{v_iv_j : 1 \leq i \leq d-2, d-1 \leq j \leq m\}$, then taking $d = 3, 4$ for cases (a), (b), respectively.

In cases (c), (d) and (e) we can use a result of Pikhurko [24] that, for $m \geq d$, K_m has a weakly saturated K_d^--sequence starting from a spanning subgraph with $(d-3)m - \binom{d-2}{2} + 1$ edges, and then taking $d = 4, 5, 6$ for cases (c), (d), (e), respectively. □

Lemma 2.1 can also be used to extend Theorem 4.3 to matroids on non-complete graphs. For example, if G is a chordal graph, then every connected flat of $\mathcal{M}_{f_{1,1}}(G)$ is a 2-connected chordal graph and we can use Lemma 2.1 and an appropriate weakly saturated K_3-sequence to deduce that $\mathcal{M}_{f_{1,1}}(G)$ is the unique maximal K_3-matroid on G and val_{K_3} is its rank function.

Our next result gives another example of uniqueness for matroids on non-complete graphs.

Theorem 4.4. *The matroid* $\mathcal{M}_{f_{1,0}}(K_{m,n})$ *is the unique maximal* $K_{2,3}$*-matroid on* $K_{m,n}$ *and* $\mathrm{val}_{K_{2,3}}$ *is its rank function.*

Proof. By (3) and (4), each copy of $K_{2,3}$ is a circuit in $\mathcal{M}_{f_{1,0}}(K_{m,n})$ and each connected flat is a copy of $K_{s,t}$ for some $s \geq 2, t \geq 3$. By the same argument as in the proof of Theorem 4.3, it will suffice to show that, for any $K_{s,t}$ with $s \geq 2, t \geq 3$, there is a weakly saturated $K_{2,3}$-sequence which constructs $K_{s,t}$ from a subgraph $G \subset K_{s,t}$ with $|E(G)| = s + t$. This follows easily by taking $V(G) = \{u_1, u_2, \ldots, u_s\} \cup \{w_1, w_2, \ldots, w_t\}$ and $E(G) = \{u_2w_2\} \cup \{u_1w_i : 1 \leq i \leq t\} \cup \{u_iw_1 : 2 \leq i \leq s\}$. □

In contrast to this result, we will see in Section 5 that there are two distinct maximal $K_{2,3}$-matroids on K_n.

The *even cycle matroid* is the matroid $\mathcal{M}$ on $E(K_n)$, in which a set F is independent if and only if each connected component of the induced subgraph $K_n[F]$ contains at most one cycle, and this cycle is odd if it exists. The rank function of $\mathcal{M}$ is given by $r_{\mathcal{M}}(F) = |V(F)| - \beta(F)$, where $\beta(F)$ denotes the number of bipartite connected components in the graph $K_n[F]$. We can use this fact to define a modified version of count matroids.

For $a, b, c \in \mathbb{Z}$, define $g_{a,b,c} : 2^{E(K_n)} \to \mathbb{Z}$ by $g_{a,b,c}(F) = a|V(F)| - b\beta(F) - c$. Then $g_{a,b,c}$ is submodular and non-decreasing for all $a, b \in \mathbb{Z}$ with $a \geq b \geq 0$ since the functions $F \mapsto |V(F)|$ and $F \mapsto |V(F)| - \beta(F)$ are both submodular and non-decreasing. Hence $g_{a,b,c}$ induces a matroid $\mathcal{M}_{g_{a,b,c}}(K_n)$ on $E(K_n)$ whenever $a \geq b \geq 0$. We will give examples of families $\mathcal{H}$ for which $\mathcal{M}_{g_{a,b,c}}(K_n)$ is the unique maximal $\mathcal{H}$-matroid on K_n. We need the following observation on the connected flats of $\mathcal{M}_{g_{a,b,c}}(K_n)$ which follows immediately from (3) and (4).

Lemma 4.5. *Suppose $a, b, c \in \mathbb{Z}$ with $a \geq b \geq 0$, $c \geq 0$, and $F \subseteq E(K_n)$ is a connected flat in $\mathcal{M}_{g_{a,b,c}}(K_n)$. Then $K_n[F]$ is either a complete graph with $|F| \geq a|V(F)| - c + 1$ or a complete bipartite graph with $|F| \geq a|V(F)| - b - c + 1$.*

The hypothesis of Lemma 4.5 that $c \geq 0$ is needed to ensure that the circuits of $\mathcal{M}_{g_{a,b,c}}(K_n)$ induce connected subgraphs of K_n, which in turn implies that the same property holds for the connected flats of $\mathcal{M}_{g_{a,b,c}}(K_n)$. This is not true when $c \leq -1$, for example the disjoint union of two copies of C_4 is both a circuit and a connected flat in $\mathcal{M}_{g_{1,1,-1}}(K_n)$.

Theorem 4.6. *(a) The even cycle matroid $\mathcal{M}_{g_{1,1,0}}(K_n)$ is the unique maximal C_4-matroid on K_n and its rank function is val_{C_4}.*
(b) $\mathcal{M}_{g_{2,1,2}}$ is the unique maximal $\{K_5^-, K_{3,4}\}$-matroid on K_n and its rank function is $\mathrm{val}_{\{K_5^-, K_{3,4}\}}$.

Proof. In each case, $\mathcal{M}_{g_{a,b,c}}(K_n)$ is loopless and is an $\mathcal{X}$-matroid on K_n for $\mathcal{X} = \{C_4\}_{K_n}$ and $\mathcal{X} = \{K_5^-, K_{3,4}\}_{K_n}$, respectively, by (3). Lemmas 2.1 and 4.5 will now imply that $\mathcal{M}_{g_{a,b,c}}(K_n)$ is the unique maximal $\mathcal{X}$-matroid on K_n once we have shown that: for every $K_m \subseteq K_n$ with $|E(K_m)| \geq am - c + 1$, there is a weakly saturated $\mathcal{X}$-sequence which constructs K_m from a subgraph $G \subset K_m$ with $|E(G)| = am - c$; and for every $K_{s,t} \subseteq K_n$ with $|E(K_{s,t})| \geq am - b - c + 1$, there is a weakly saturated $\mathcal{X}$-sequence which constructs $K_{s,t}$ from a subgraph $G \subset K_{s,t}$ with $|E(G)| = am - b - c$. Let $V(K_m) = \{v_1, v_2, \dots, v_m\}$ and $V(K_{s,t}) = \{u_1, u_2, \dots, u_s\} \cup \{w_1, w_2, \dots, w_t\}$.
(a) For $m \geq 4$, K_m can be constructed by a weakly saturated C_4-sequence starting from a spanning subgraph G with $g_{1,1,0}(E(K_m)) = m$ edges by taking $E(G) = \{v_2v_3\} \cup \{v_1v_i : 2 \leq i \leq m\}$. For $s, t \geq 2$, $K_{s,t}$ can be constructed by a weakly saturated C_4-sequence starting from a spanning subgraph G with $g_{1,1,0}(E(K_{s,t})) = s + t - 1$ edges by taking $E(G) = \{u_1w_i : 1 \leq i \leq t\} \cup \{u_iw_1 : 1 \leq i \leq s\}$.
(b) For $m \geq 5$, K_m can be constructed by a weakly saturated K_5^--sequence starting from a spanning subgraph G with $g_{2,1,2}(E(K_m)) = 2m - 2$ edges by taking $E(G) =$

$\{v_1v_2, v_3v_4\} \cup \{v_iv_j : 1 \leq i \leq 2, 3 \leq j \leq m\}$. For $s \geq 3$ and $t \geq 4$, $K_{s,t}$ can be constructed by a weakly saturated $K_{3,4}$-sequence starting from a spanning subgraph G with $g_{2,1,2}(E(K_{s,t})) = 2(s+t) - 3$ edges by taking $E(G) = \{u_iw_j : 1 \leq i \leq 3, 1 \leq j \leq 3\} \cup \{u_iw_j : 1 \leq i \leq 2,\ 4 \leq j \leq t\} \cup \{u_iw_j : 4 \leq i \leq s,\ 1 \leq j \leq 2\}$. □

The matroid $\mathcal{M}_{g_{2,1,2}}(K_n)$ in Theorem 4.6(b) is the Dilworth truncation of the union of the graphic matroid and the even cycle matroid. It appears in the context of the rigidity of symmetric frameworks in $\mathbb{R}^2$, see for example [29].

The concept of count matroids has been extended to hypergraphs [23] and to group-labelled graphs [14]. The technique in this section can be adapted to both settings.

We close this section with a remark on the poset of all $\{K_4, K_{2,3}\}$-matroids on K_n. It is straightforward to check that $\mathcal{M}_{g_{1,1,-1}}(K_n)$ is a $\{K_4, K_{2,3}\}$-matroid on K_n. But we cannot show it is the unique maximal such matroid by using the same proof technique as Theorem 4.6 since Lemma 4.5 does not hold when $c < 0$. In fact, we will see in Theorem 5.4 below that there are at least two maximal $\{K_4, K_{2,3}\}$-matroids on K_n when n is sufficiently large.

5. Examples of non-uniqueness

We will give three examples of posets of $\mathcal{H}$-matroids on K_n in which there is not a unique maximal matroid. We will frequently use the following lemma on the free elevation of a symmetric matroid on K_n (where a matroid M on $E(K_n)$ is said to be *symmetric* if the edge sets of every pair of isomorphic subgraphs of K_n have the same rank in M). This result follows from the procedure for constructing the free erection of a matroid due to Duke [10], see for example [8, Algorithm 1].

Lemma 5.1. *If $\mathcal{M}_0$ is a symmetric matroid on K_n, then the free elevation of $\mathcal{M}_0$ is a symmetric matroid on K_n.*

Gyula Pap [22] observed that the cycle matroid of K_n and the uniform C_5-matroid on K_n are two distinct maximal C_5-matroids on K_n. We can use Lemma 2.7 to show that Pap's example extends to C_k for all $k \geq 5$.

Theorem 5.2. *There are two distinct maximal C_k-matroids on K_n for all $k \geq 5$ and $n \geq \binom{k-1}{2} + 2$.*

Proof. It is straightforward to check that C_k is union-stable. Hence $\mathcal{U}_{C_k}(K_n)$ is a matroid and the free elevation $\mathcal{M}$ of $\mathcal{U}_{C_k}(K_n)$ is a maximal C_k-matroid by Lemma 2.7. In addition, Lemma 5.1 implies that $\mathcal{M}$ is symmetric. We will show that $\mathcal{M}$ contains a circuit Z such that $K_n[Z]$ has minimum degree one. To see this, consider two distinct copies X and Y of C_k such that $X \cap Y$ forms a path of length $k - 2$. By the circuit elimination

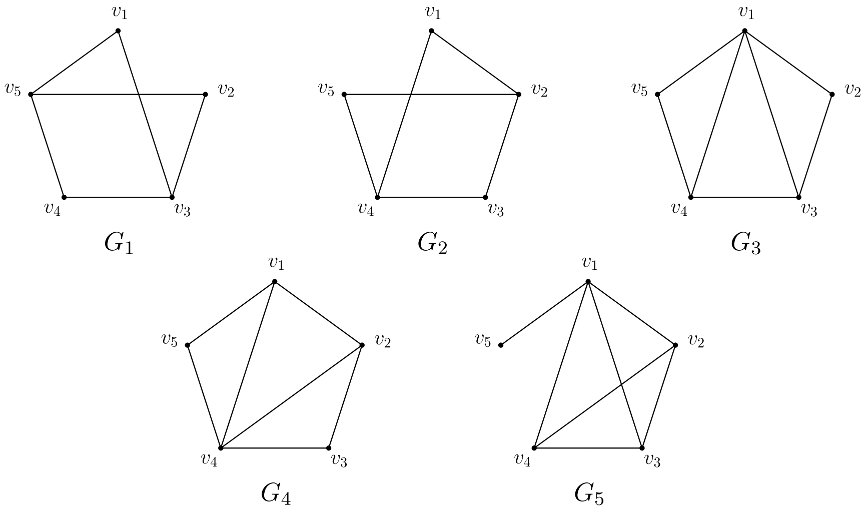

Fig. 1. The graphs $G_1, G_2, \ldots, G_5$ in the proof of Theorem 5.3.

axiom, $(X \cup Y) - e$ contains a circuit Z of $\mathcal{M}$ for any $e \in X \cap Y$. Since the copy of C_4 in $K_n[(X \cup Y) - e]$ is not a circuit in $\mathcal{U}_{C_k}(K_n)$, it cannot be a circuit in $\mathcal{M}$. Hence $(X \cup Y) - e$ contains a circuit Z such that $K_n[Z]$ has minimum degree one. We may now apply Lemma 3.6 with $H = Z$ to deduce that the rank of $\mathcal{M}$ is at most $\binom{k-1}{2}$.

The facts that $\mathcal{M}$ is a maximal C_k-matroid and the cycle matroid of K_n is a C_k-matroid of rank $n-1$, now imply that there are at least two maximal C_k-matroids on K_n whenever $n \geq \binom{k-1}{2} + 2$. $\square$

We saw in Theorem 4.4 that the poset of all $K_{2,3}$-matroids on $K_{m,n}$ has a unique maximal element. We next show that this statement becomes false if we change the ground set to $E(K_n)$.

Theorem 5.3. *There are two distinct maximal $K_{2,3}$-matroids on K_n for all $n \geq 7$.*

Proof. Since $K_{2,3}$ is union-stable, $\mathcal{U}_{K_{2,3}}(K_n)$ is a matroid by Lemma 2.7, and its free elevation $\mathcal{M}$ is a maximal $K_{2,3}$-matroid on K_n and is symmetric by Lemmas 2.6 and 5.1. We will show that $\mathcal{U}_{K_{2,3}}(K_n)$ has no non-trivial erection and hence $\mathcal{M} = \mathcal{U}_{K_{2,3}}(K_n)$.

We first show that $\mathcal{M}$ contains a circuit Z such that $K_n[Z]$ has minimum degree one. To see this consider the graphs G_1 and G_2 given in Fig. 1. Both are isomorphic to $K_{2,3}$, and hence, by the circuit elimination axiom, the edge set of $G_3 = (G_1 \cup G_2) - v_2v_5$, is dependent in $\mathcal{M}$. Since every set of six edges which does not induce a copy of $K_{2,3}$ is independent in $\mathcal{M}$, the edge set of G_3 is a circuit in $\mathcal{M}$. Since the graph G_4 in Fig. 1 is isomorphic to G_3, the circuit elimination axiom now implies the edge set of $G_5 = (G_3 \cup G_4) - v_4v_5$ is dependent in $\mathcal{M}$. Again, since every set of six edges which does not induce a copy $K_{2,3}$ is independent in $\mathcal{M}$, the edge set of G_5 is a circuit in $\mathcal{M}$.

This implies that $\mathcal{M}$ is a G_5-matroid on K_n and hence, by Lemma 3.6, the rank of $\mathcal{M}$ is at most 6. Since $\mathcal{U}_{K_{2,3}}(K_n)$ has rank 6, this gives $\mathcal{M} = \mathcal{U}_{K_{2,3}}(K_n)$, and hence $\mathcal{U}_{K_{2,3}}(K_n)$ is a maximal $K_{2,3}$-matroid on K_n. Since the bicircular matroid $\mathcal{M}_{f_{1,0}}$ is a $K_{2,3}$-matroid on K_n of rank n, we have at least two maximal $K_{2,3}$-matroids on K_n whenever $n \geq 7$. □

Our final example of this section shows that the unique maximality property may not hold even if we restrict our attention to the poset of all partial elevations of a given $\mathcal{H}$-matroid on K_n (and hence provides another example, in addition to that given by Brylawski [6], which shows that the free elevation may not be the unique maximal matroid in the poset of all partial elevations of a given matroid). Note that the matroids described in Theorem 5.2 and 5.3 do not give such an example.

Theorem 5.4. *There are two distinct maximal matroids in the poset of all partial elevations of the uniform $\{K_4, K_{2,3}\}$-matroid $\mathcal{U}_{\{K_4,K_{2,3}\}}(K_n)$ whenever $n \geq 36$.*

Proof. Let $\mathcal{X} = \{K_4, K_{2,3}\}_{K_n}$. Since $\mathcal{X}$ is 6-uniform and union-stable, $\mathcal{U}_{\mathcal{X}}$ is a matroid. Hence the free elevation $\mathcal{M}$ of $\mathcal{U}_{\mathcal{X}}$ is a symmetric $\mathcal{X}$-matroid on K_n by Lemma 5.1. We will show that $\mathcal{M}$ has bounded rank.

Claim 5.5. *The rank of $\mathcal{M}$ is at most* 36.

Proof. Let D_1 be the edge set of the union of a vertex-disjoint 3-cycle and 4-cycle in K_n and D_2 be the edge set of the union of two vertex-disjoint 4-cycles in K_n. We split the proof into three cases.

Case 1: D_1 is dependent in $\mathcal{M}$. Since $|D_1| = 7$ and neither K_4 nor $K_{2,3}$ is contained in $K_n[D_1]$, every proper subset of D_1 is independent in $\mathcal{M}$, and hence D_1 is a circuit in $\mathcal{M}$. By Lemma 2.4, the closure $\mathrm{cl}_{\mathcal{M}}(D_1)$ of D_1 is the union of copies of K_4 and $K_{2,3}$. Hence $\mathrm{cl}_{\mathcal{M}}(D_1) \neq D_1$ and, for each $e \in \mathrm{cl}_{\mathcal{M}}(D_1) \setminus D_1$, there exists a circuit C with $e \in C \subsetneq D_1 + e$. Since $K_n[D_1 + e]$ cannot contain K_4 or $K_{2,3}$ and $|D_1 + e| = 8$ we have $|C| = 7$. Observe that any 7-element subset of $D_1 + e$ containing e has a vertex of degree one. We may now use Lemma 3.6 and the fact that $|V(C)| \leq 9$ to deduce that $\mathcal{M}$ has rank at most $\binom{8}{2} = 28$.

Case 2: D_1 is independent in $\mathcal{M}$ and D_2 is dependent in $\mathcal{M}$. If some proper subset C of D_2 is a circuit in $\mathcal{M}$ then $K_n[C]$ would contain a vertex of degree one and we could again use Lemma 3.6 and the fact that $|V(C)| \leq 8$ to deduce that $\mathcal{M}$ has rank at most $\binom{7}{2} = 21$. Hence we may assume that D_2 is a circuit.

By Lemma 2.4, $\mathrm{cl}_{\mathcal{M}}(D_2)$ is the union of copies of K_4 and $K_{2,3}$. Hence $\mathrm{cl}_{\mathcal{M}}(D_2) \neq D_2$ and, for each $e \in \mathrm{cl}_{\mathcal{M}}(D_2) \setminus D_2$, there exists a circuit C' with $e \in C' \subsetneq D_2 + e$. Since $K_n[D_2 + e]$ cannot contain K_4 or $K_{2,3}$, and $|D_2 + e| = 9$, we have $7 \leq |C'| \leq 8$. Observe that every subset of $D_2 + e$ of size 7 or 8 which contains e and is distinct from D_1, has

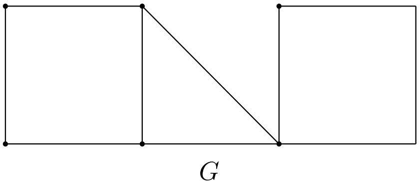

Fig. 2. The graph G in the proof of Theorem 5.4.

a vertex of degree one. Hence, we may use Lemma 3.6 and the fact that $|V(C')| \leq 10$ to deduce that $\mathcal{M}$ has rank at most $\binom{9}{2} = 36$.

Case 3: D_1, D_2 are independent in $\mathcal{M}$. By Theorem 4.4, if $F \subseteq E(K_n)$ induces a bipartite subgraph in K_n, then F has rank at most $|V(F)|$ in any $K_{2,3}$-matroid. We can use this fact, to compute the rank of the edge set D_3 of the graph G in Fig. 2, in $\mathcal{M}$. Let B be a base of D_3 which contains the independent subset D_2. Since $D_2 + e$ induces a bipartite graph with $|V(D_2+e)|+1$ edges for each $e \in D_3 \backslash D_2$, D_2+e is dependent. Hence $B = D_2$ and $r_{\mathcal{M}}(D_3) = |D_2| = 8$.

On the other hand the edge set D_4 of the 5-cycle with a chord in G is independent in $\mathcal{M}$ (since D_4 is independent in $\mathcal{U}_{\mathcal{X}}$). Since $r_{\mathcal{M}}(D_3) = 8$, this implies that $D_3 - e$ contains a circuit C of $\mathcal{M}$ with $1 \leq |C \backslash D_4| \leq 3$. Then $K_n[C]$ has a vertex of degree 1, and we may again use Lemma 3.6 and the fact that $|V(C)| \leq 8$ to deduce that $\mathcal{M}$ has rank at most $\binom{7}{2} = 21$. □

We can now complete the proof of Theorem 5.4 by observing that the matroid $\mathcal{M}_{g_{1,1,-1}}(K_n)$ from Section 4 is a partial elevation of $\mathcal{U}_{\mathcal{X}}$ and has rank $n+1$. Since $\mathcal{M}$ is a maximal partial elevation of $\mathcal{U}_{\mathcal{X}}$ by Lemma 2.2 and has rank at most 36 by Claim 5.5, $\mathcal{M}$ is not the unique maximal partial elevation of $\mathcal{U}_{\mathcal{X}}$ for all $n \geq 36$. This completes the proof. □

The above proof also implies that there are two distinct maximal $\{K_4, K_{2,3}\}$-matroids on K_n for all $n \geq 36$. The only modification we need is to use Lemma 2.6 in the final paragraph of the proof to deduce that $\mathcal{M}$ is a maximal $\{K_4, K_{2,3}\}$-matroid on K_n.

6. Matroids from rigidity, hyperconnectivity and matrix completion

6.1. Rigidity matroids and cofactor matroids

Given a generic realisation $p : V(K_n) \to \mathbb{R}^d$, we would like to know when a subgraph $G \subset K_n$ is *d-rigid* i.e., every continuous motion of the vertices of (G, p) which preserves the distances between adjacent pairs of vertices must preserve the distances between all pairs of vertices. The (edge sets of the) minimal d-rigid spanning subgraphs of K_n are the bases of a matroid $\mathcal{R}_d(K_n)$ which is referred to as the *d-dimensional generic rigidity matroid.* It is well known that $\mathcal{R}_d(K_n)$ is a K_{d+2}-matroid on K_n and $\mathcal{R}_1(K_n)$ is the cycle matroid of K_n. Pollaczek-Geiringer [25] and subsequently Laman [19] showed that $\mathcal{R}_2(K_n) = \mathcal{M}_{f_{2,3}}(K_n)$. Characterising $\mathcal{R}_d(K_n)$ for $d \geq 3$ is an important open problem in discrete geometry.

Graver [12] suggested we may get a better understanding of $\mathcal{R}_d(K_n)$ by studying the poset of all *abstract d-rigidity matroids on K_n*. This can be defined, using a result of Nguyen [21], as the poset of all K_{d+2}-matroids on K_n of rank $dn - \binom{d+1}{2}$. Graver conjectured that $\mathcal{R}_d(K_n)$ is the unique maximal element in this poset and verified his conjecture for the cases when $d = 1, 2$. The same proofs yield the slightly stronger results given in Theorem 4.3(a) and (b) of this paper.

Whiteley [31] showed that Graver's conjecture is false when $d \geq 4$ using the cofactor matroid $\mathcal{C}_{d-1}^{d-2}(K_n)$ from the theory of bivariate splines. He first showed that $\mathcal{C}_{d-1}^{d-2}(K_n)$ is an abstract d-rigidity matroid for all $d \geq 1$ and that $E(K_{d+2,d+2})$ is independent in $\mathcal{C}_{d-1}^{d-2}(K_n)$ whenever $d \geq 4$. He then applied a result of Bolker and Roth [3] that $E(K_{d+2,d+2})$ is dependent in $\mathcal{R}_d(K_n)$ to deduce that $\mathcal{R}_d(K_n) \not\succeq \mathcal{C}_{d-1}^{d-2}(K_n)$ for all $d \geq 4$ and $n \geq 2d + 4$.

Whiteley [31] made the revised conjecture that $\mathcal{C}_{d-1}^{d-2}(K_n)$ is the unique maximal element in the poset of all abstract d-rigidity matroids on K_n for all $d \geq 1$. This holds by Graver's result when $d = 1, 2$ since we have $\mathcal{R}_d(K_n) = \mathcal{C}_{d-1}^{d-2}(K_n)$. We recently verified the case $d = 3$ of Whiteley's conjecture in joint work with Clinch.

Theorem 6.1 *([7]). The cofactor matroid $\mathcal{C}_2^1(K_n)$ is the unique maximal K_5-matroid on K_n and its rank function is* val_{K_5}.

We propose the following extensions of Whiteley and Graver's conjectures.

Conjecture 6.2. $\mathcal{C}_{d-1}^{d-2}(K_n)$ *is the unique maximal K_{d+2}-matroid on K_n for all $d \geq 1$ and its rank function is* $\mathrm{val}_{K_{d+2}}$.

Conjecture 6.3. $\mathcal{R}_n^d(K_n)$ *is the unique maximal $\{K_{d+2}, K_{d+2,d+2}\}$-matroid on K_n for all $d \geq 3$ and its rank function is* $\mathrm{val}_{\{K_{d+2},K_{d+2,d+2}\}}$.

6.2. Birigidity and rooted $K_{s,t}$-matroids on $K_{m,n}$

Let H be a bipartite graph with bipartition (A, B) and $K_{m,n}$ be a copy of the complete bipartite graph with bipartition (U, W) where $|U| = m$ and $|W| = n$. We say that a subgraph H' of $K_{m,n}$ is a *rooted copy of H* in $K_{m,n}$ if there is an isomorphism θ from H to H' with $\theta(A) \subseteq U$ and $\theta(B) \subseteq W$. Let $\{H\}_{K_{m,n}}^*$ be the set of all rooted-copies of H in $K_{m,n}$. A matroid $\mathcal{M}$ on $K_{m,n}$ is said to be a *rooted H-matroid* if it is a $\{H\}_{K_{m,n}}^*$-matroid. Note that the given ordered bipartition (A, B) of H plays a significant role in this definition - we do not require that an isomorphic image $\theta(H)$ of H in $K_{m,n}$ is a circuit in $\mathcal{M}$ when $\theta(A) \not\subseteq U$. On the other hand, if H has an automorphism which maps A onto B, then we will get the same matroid for each ordering of the bipartition of H and this matroid will be equal to the (unrooted) H-matroid on $K_{m,n}$.

6.2.1. Birigidity matroids

As a primary example of matroids on complete bipartite graphs, we shall introduce the birigidity matroids of Kalai, Nevo, and Novik [18].

Let $G = (U \cup W, E)$ be a bipartite graph with $m = |U|$ and $n = |W|$, $p : U \to \mathbb{R}^k$, and $q : W \to \mathbb{R}^\ell$. We assume that the vertices of U and W are ordered as $u_1, u_2, \dots, u_m$ and $v_1, v_2, \dots, v_n$, respectively. We define the *(k, ℓ)-rigidity matrix* of (G, p, q), denoted by $R^{k,\ell}(G, p, q)$, to be the matrix of size $|E| \times (\ell m + kn)$ in which each vertex in U labels a set of ℓ consecutive columns from the first ℓm columns, each vertex in W labels a set of k consecutive columns from the last kn columns, each row is associated with an edge, and the row labelled by the edge $e = u_i w_j$ is

$$\begin{array}{c}\\ {\scriptstyle e=u_iw_j}\end{array}\begin{array}{c} \begin{array}{ccccc} & {\scriptstyle u_i} & & {\scriptstyle w_j} & \end{array} \\ \begin{bmatrix} 0 \dots 0 & q(w_j) & 0 \dots 0 & p(u_i) & 0 \dots 0 \end{bmatrix} \end{array}.$$

The generic (k, ℓ)-rigidity matroid $\mathcal{R}^{k,\ell}_{m,n}$ is the row matroid of $R^{k,\ell}(K_{m,n}, p, q)$ for any generic p and q. It can be checked that the rank of $\mathcal{R}^{k,\ell}_{m,n}$ is equal to $\ell m + kn - k\ell$, from which it follows that $K_{k+1,\ell+1}$ is a circuit and $\mathcal{R}^{k,\ell}_{m,n}$ is a rooted $K_{k+1,\ell+1}$-matroid.

As pointed out in [18], $\mathcal{R}^{k,\ell}_{m,n}$ coincides with the picture lifting matroids extensively studied by Whiteley [30] when $\min\{k, \ell\} = 1$. We will show that this matroid is the unique maximal rooted $K_{k+1,\ell+1}$-matroid in this case.

Theorem 6.4. *$\mathcal{R}^{k,1}_{m,n}$ is the unique maximal rooted $K_{k+1,2}$-matroid on $K_{m,n}$.*

Proof. Whiteley [30] showed that the picture lifting matroid is the matroid induced by the submodular, non-decreasing function $h : 2^{E(K_{m,n})} \to \mathbb{Z}$ defined by

$$h(F) := |U(F)| + k|W(F)| - k \quad (F \subseteq E(K_{m,n})),$$

where $U(F)$ and $W(F)$ denote the sets of vertices in U and W, respectively, that are incident to F. Since every connected flat in $\mathcal{M}_h(K_{m,n})$ is a complete bipartite graph $K_{m',n'}$ for some $m' \geq 1$ and $n' \geq 2$, we may deduce the theorem from Lemma 2.1 by showing that $K_{m',n'}$ can be constructed by a weakly saturated, rooted $K_{k+1,2}$-sequence from a subgraph G with $m' + kn' - k$ edges. Such a sequence is easily obtained by taking

$$E(G) = \{u_i w_1 : 1 \leq i \leq m'\} \cup \{u_i w_j : 1 \leq i \leq k \text{ and } 2 \leq j \leq n'\}. \quad \square$$

We refer the reader to [1] for more details on weakly saturated, rooted $K_{s,t}$-sequences in $K_{m,n}$.

Lemma 2.1 also tells us that the rank function of $\mathcal{R}^{k,1}_{m,n}$ is determined by proper, rooted $K_{k+1,2}$-sequences. We conjecture that this extends to $\mathcal{R}^{k,\ell}_{m,n}$ for all $k, \ell \geq 1$.

Conjecture 6.5. *$\mathcal{R}^{k,\ell}_{m,n}$ is the unique maximal rooted $K_{k+1,\ell+1}$-matroid on $K_{m,n}$ and the rank of any $F \subseteq E(K_{n,m})$ is given by*

$$r(F) = \min\{\mathrm{val}(F, \mathcal{S}) : \mathcal{S} \textit{ is a proper, rooted } K_{k+1,\ell+1}\textit{-sequence in } K_{m,n}\}.$$

The special case of this conjecture for $\mathcal{R}^{2,2}_{m,n}$ is equivalent to a conjecture on the rank function of $\mathcal{R}^{2,2}_{m,m}$ given in [15, Section 8]. Bernstein [2] gave an NP-type combinatorial characterization for independence in $\mathcal{R}^{2,2}_{m,n}$, but no co-NP-type characterization is known. The special case $k = \ell = 2$ of Conjecture 6.5 would provide such a certificate but even this special case seems challenging. Some evidence in support of the conjecture is given in Theorem 6.15 and the discussion after Theorem 6.15.

6.3. *Hyperconnectivity matroids, matrix completion and $\{K_d, K_{s,t}\}$-matroids on K_n*

Let $p : V(K_n) \to \mathbb{R}^d$ be a generic map. We assume that the vertices of K_n are ordered as $v_1, v_2, \ldots, v_n$. Kalai [16] defined the *d-hyperconnectivity matroid*, $\mathcal{H}^d_n$, to be the row matroid of the matrix of size $\binom{n}{2} \times dn$ in which each vertex of K_n labels a set of d consecutive columns, each row is labelled by an edge of K_n, and the row labelled by the edge $e = v_iv_j$ with $i < j$ is

$$\begin{array}{c} \\ e{=}v_iv_j \end{array} \begin{array}{c} \qquad\quad v_i \qquad\qquad\quad v_j \\ \begin{bmatrix} 0\ldots0 & p(v_j) & 0\ldots0 & -p(v_i) & 0\ldots0 \end{bmatrix}. \end{array} \tag{5}$$

He showed that, when $n \geq 2d + 2$, this matroid is a $\{K_{d+2}, K_{d+1,d+1}\}$-matroid of rank $dn - \binom{d+1}{2}$.

As a variant of $\mathcal{H}^d_n$, Kalai [16] also introduced the matroid $\mathcal{I}^d_n$, which is the row matroid of the $(\binom{n}{2} \times dn)$-matrix with rows

$$\begin{array}{c} \\ e{=}v_iv_j \end{array} \begin{array}{c} \qquad\quad v_i \qquad\qquad v_j \\ \begin{bmatrix} 0\ldots0 & p(v_j) & 0\ldots0 & p(v_i) & 0\ldots0 \end{bmatrix} \end{array}$$

instead of (5). He showed that, when $n \geq 2d + 2$, $\mathcal{I}^d_n$ is a $K_{d+1,d+1}$-matroid on K_n of rank $dn - \binom{d}{2}$. In the special case when $d = 2$, this rank constraint implies that $\mathcal{I}^2_n$ is a $\{K_5, K_{3,3}\}$-matroid.

The matroids $\mathcal{H}^d_n$ and $\mathcal{I}^d_n$ arise naturally in the context of the rank d completion problem for partially filled $n \times n$ matrices which are skew-symmetric and symmetric, respectively, see [4,26]. The restriction of either $\mathcal{I}^d_n$ or $\mathcal{H}^d_n$ to the complete bipartite graph $K_{m,n}$ is the birigidity matroid $\mathcal{R}^{d,d}_{m,n}$, and this matroid arises in the context of the rank d completion problem for partially filled $m \times n$ matrices, see [26].

When $d = 1$, $\mathcal{H}^1_n$ is the cycle matroid (and hence is the unique maximal $\{K_3, K_{2,2}\}$-matroid on K_n by Theorem 4.3(a)) and $\mathcal{I}^1_n$ is the even cycle matroid (and hence is the unique maximal $K_{2,2}$-matroid on K_n by Theorem 4.6(a)).

We conjecture that each of $\mathcal{H}^d_n$ and $\mathcal{I}^2_n$ is the unique maximal matroid in its respective poset.

Conjecture 6.6. *(a) $\mathcal{H}_n^d$ is the unique maximal $\{K_{d+2}, K_{d+1,d+1}\}$-matroid on K_n for all $d \geq 1$ and its rank function is* $\mathrm{val}_{\{K_{d+2},K_{d+1,d+1}\}}$.
(b) $\mathcal{I}_n^2$ is the unique maximal $\{K_5, K_{3,3}\}$-matroid on K_n and its rank function is $\mathrm{val}_{\{K_5,K_{3,3}\}}$.

6.4. $\{K_4, K_{3,3}\}$-matroids on K_n

The smallest unsolved case of Conjecture 6.6(a) is when $d = 2$. More generally, a better understanding of the poset of all $\{K_4, K_{3,3}\}$-matroids would have applications in such areas as the rank two completion of partially filled skew-symmetric matrices [4] and partially-filled rectangular matrices [9], and the rigidity of 2-dimensional frameworks whose points lie on a conic. In this section, we will describe how the combinatorial operations of 0-extension and diamond splitting can be used to attack Conjecture 6.6(a) and indicate what is missing to complete the proof.

Given a graph G, the *0-extension operation* constructs a new graph by adding a new vertex v_0 and two edges v_0v_1 and v_0v_2 with distinct $v_1, v_2 \in V(G)$. We say that a matroid $\mathcal{M}$ on K_n has the *0-extension property* if the 0-extension operation preserves independence in $\mathcal{M}$, i.e. $E(G')$ is independent if $E(G)$ is independent and G' is obtained from G by a 0-extension operation for all $G, G' \subseteq K_n$.

Given a vertex v_1 of a graph G, the *diamond splitting operation* at v_1 (with respect to a fixed partition $\{U_0, U^*, U_1\}$ of $N_G(v_1)$ with $|U^*| = 2$) removes the edges between v_1 and the vertices in U_0, adds a new vertex v_0, and adds new edges v_0u for all $u \in U_0 \cup U^*$. We say that a matroid $\mathcal{M}$ on K_n has the *diamond splitting property* if the diamond splitting operation preserves independence in $\mathcal{M}$.

We can use standard techniques from rigidity theory to prove the following, see [18, Lemmas 3.7, 3.8] or [15, Lemmas 2.3, 4.5].

Lemma 6.7. *$\mathcal{H}_n^2$ has the 0-extension property and the diamond splitting property.*

Let $R_9(\mathcal{H}_n^2)$ be the matroid on K_n obtained by truncating $\mathcal{H}_n^2$ to rank 9. Our next two lemmas imply that the free elevation of $R_9(\mathcal{H}_n^2)$ is a maximal $\{K_4, K_{3,3}\}$-matroid and has the $\{K_4, K_{3,3}\}$-covering property (and hence is equal to $\mathcal{H}_n^2$ if Conjecture 6.6(a) is true). Let W_n denote the wheel on n vertices.

Lemma 6.8. *Suppose $n \geq 6$. Then:*
(a) the edge set of every copy of W_5 in K_n is dependent in every K_4-matroid on K_n;
(b) every circuit of $\mathcal{H}_n^2$ of rank at most 8 induces a copy of K_4, W_5 or $K_{3,3}$.

Proof. (a) This follows from the circuit elimination axiom applied to two copies of K_4 with a common triangle.
(b) Let $G = (V, C)$ be the graph induced by a circuit C of $\mathcal{H}_n^2$ of rank at most 8. Then $|C| \leq 2|V| - 2$ with strict inequality when G is bipartite. In addition G has minimum

degree three since $\mathcal{H}_n^2$ has the 0-extension property. Since $|C| \leq 9$ this implies that $4 \leq |V| \leq 6$. If $|V| = 4$ then $G = K_4$, so we may assume $5 \leq |V| \leq 6$. Let v be a vertex of degree three in G. If $|V| = 5$ then $G - v = K_4^-$ (since G has minimum degree three and $E(G - v)$ is independent in $\mathcal{H}_n^2$) and we have $G = W_5$. Hence we may assume that $|V| = 6$. Since G has minimum degree three and $|C| \leq 9$, G is 3-regular. This implies that G is either $K_{3,3}$ or the triangular prism. Since the triangular prism can be obtained from W_5 by first deleting an edge and then applying the diamond split operation to the vertex of degree four, it's edge set is independent in $\mathcal{H}_n^2$. Hence $G = K_{3,3}$. □

Lemma 6.9. *Suppose $n \geq 6$. Then the free elevation of $R_9(\mathcal{H}_n^2)$ is a maximal $\{K_4, K_{3,3}\}$-matroid on K_n and has the $\{K_4, K_{3,3}\}$-covering property.*

Proof. The assertion that free elevation of $R_9(\mathcal{H}_n^2)$ is a maximal $\{K_4, K_{3,3}\}$-matroid on K_n follows from Lemmas 2.3 and 6.8. Since $\mathcal{H}_n^2$ has rank $2n - 3$, the $\mathcal{H}_n^2$-closure of every copy of W_5 in K_n is a complete subgraph on five vertices. Lemma 6.8 now implies that every cyclic flat of $R_9(\mathcal{H}_n^2)$ is a copy of $K_{3,3}$ or K_t for some $4 \leq t \leq n$. Hence $R_9(\mathcal{H}_n^2)$ has the $\{K_4, K_{3,3}\}$-covering property. Lemma 2.4 now implies that the $\{K_4, K_{3,3}\}$-covering property also holds for the free elevation of $R_9(\mathcal{H}_n^2)$. □

So far we have seen that $\mathcal{H}_n^2$ has the 0-extension property and the diamond splitting property (Lemma 6.7) and the free elevation of $R_9(\mathcal{H}_n^2)$ has the $\{K_4, K_{3,3}\}$-covering property (Lemma 6.9). Our main result of this section, Theorem 6.12 below, shows that, if some $\{K_4, K_{3,3}\}$-matroid on K_n has all of these three properties, then it will be the unique maximal $\{K_4, K_{3,3}\}$-matroid on K_n. To see this we need two rather technical lemmas.

Lemma 6.10. *[8, Lemma 5.5] Let $\mathcal{M}$ be a matroid defined on the edge set of a graph G. Suppose that $G[C]$ is 2-connected for every circuit C in $\mathcal{M}$. Then, for every connected set X in $\mathcal{M}$,*

$$\sum_{v \in V(X)} \min\{d_B(v) : B \textit{ is a basis of } X\} \leq 2(r_{\mathcal{M}}(X) - 1) - |V(X)|.$$

Lemma 6.11. *Let $\mathcal{M}$ be a K_4-matroid on K_n with the 0-extension property. Then, every circuit in $\mathcal{M}$ induces a 2-connected subgraph of K_n.*

Proof. Suppose, for a contradiction, that some circuit C in $\mathcal{M}$ does not induce a 2-connected subgraph of K_n.

We first consider the case when C is connected. Then C can be partitioned into two sets X and Y such that $|V(X) \cap V(Y)| = 1$. Let K be the edge set of the complete graph on $V(Y)$. Since $\mathcal{M}$ is a K_4-matroid, Theorem 4.3(b) gives $r_{\mathcal{M}}(K) \leq 2|V(Y)| - 3$. The fact that $X \cup Y$ is a circuit now gives $r_{\mathcal{M}}(X \cup K) \leq r_{\mathcal{M}}(X) + r_{\mathcal{M}}(K) - 1 \leq |X| + 2|V(Y)| - 4$.

We may construct an independent subset of $X \cup K$ by extending the independent set X using 0-extensions. Let e be an edge in K incident to the vertex in $V(X) \cap V(Y)$. Then $X + e$ is independent by the 0-extension property. Repeatedly applying the 0-extension operation, we can extend $X + e$ to an independent set B of size $|X| + 1 + 2(|V(Y)| - 2) = |X| + 2|V(Y)| - 3$ by adding edges in K. This contradicts the fact that the rank of $X \cup K$ is at most $|X| + 2|V(Y)| - 4$.

The case when C is not connected can be proved similarly. □

We can now prove our main results on $\{K_4, K_{3,3}\}$-matroids.

Theorem 6.12. *Let $\mathcal{X} = \{K_4, K_{3,3}\}_{K_n}$ and let $\mathcal{M}$ be an $\mathcal{X}$-matroid on K_n that has the 0-extension property, the diamond splitting property, and the $\mathcal{X}$-covering property. Then $\mathcal{M}$ is the unique maximal $\mathcal{X}$-matroid on K_n and its rank function is* $\mathrm{val}_{\mathcal{X}}$.

Proof. By Lemma 2.1, it suffices to show that, for each connected flat F of $\mathcal{M}$, there is a proper $\mathcal{X}$-sequence $\mathcal{S}$ such that $r_{\mathcal{M}}(F) = \mathrm{val}(F, \mathcal{S})$. We prove this by induction on the rank of F.

Since $\mathcal{M}$ has the 0-extension property, Lemma 6.11 implies that every circuit in $\mathcal{M}$ induces a 2-connected subgraph of K_n. Since $\mathcal{M}$ is a K_4-matroid, $r_{\mathcal{M}}(F) \leq 2|V(F)| - 3$ and Lemma 6.10 now implies that there exists a base B of F and a vertex $v \in V(B)$ such that $d_B(v) \leq 2$. Let F_v and B_v be the set of edges in F and B, respectively, which are not incident to v. We first show that

$$F_v \text{ is a flat in } \mathcal{M},\ d_B(v) = 2 \text{ and } r_{\mathcal{M}}(F_v) = r_{\mathcal{M}}(F) - 2. \tag{6}$$

To verify (6) we first note that, since $\mathcal{M}$ has the 0-extension property, every circuit in $\mathcal{M}$ has minimum degree at least three. Since $d_B(v) \leq 2$, this implies that $\mathrm{cl}_{\mathcal{M}}(B_v) = F_v$ and hence F_v is a flat in $\mathcal{M}$. In addition, since F is connected in $\mathcal{M}$, we have $d_F(v) \geq 3$. The facts that $d_B(v) \leq 2$ and $\mathcal{M}$ has the 0-extension property, now give $|B| = r_{\mathcal{M}}(F) \geq r_{\mathcal{M}}(F_v) + 2 = |B_v| + 2 \geq |B|$. Hence equality holds throughout and (6) holds.

Claim 6.13. *Let $x, y \in N_F(v)$ and $z \in V(F_v)$ be three distinct vertices. Suppose that $xz, yz \in F_v$. Then $uz \in F_v$ for all $u \in N_F(v) \setminus \{z\}$.*

Proof. Suppose, for a contradiction, that $uz \notin F_v$ for some $u \in N_F(v) \setminus \{z\}$. Since F_v is a flat, $B_v + uz$ is independent. We may construct $B' = B_v \cup \{vx, vy, vu\}$ from $B_v + uv$ by applying the diamond splitting operation to z in such a way that the new vertex v has degree three and is adjacent to x, y, u. Then B' is contained in F and is independent in $\mathcal{M}$. Since $|B'| > |B|$, this contradicts the fact that B is a base of F. □

We may apply induction to each connected component of F_v in $\mathcal{M}$ to obtain a proper $\mathcal{X}$-sequence $\mathcal{S}' = (X_1, X_2 \ldots, X_t)$ such that $r_{\mathcal{M}}(F_v) = \mathrm{val}(F_v, \mathcal{S}')$. Since $\mathcal{M}$ has the $\mathcal{X}$-covering property, F is the union of copies of K_4 and $K_{3,3}$. Let $N_F(v) = \{u_1, u_2, \ldots, u_k\}$.

Suppose that some edge of F which is incident to v is contained in a copy K_4 in F. Relabelling if necessary, we may suppose that the complete graph $K(v, u_1, u_2, u_3)$ satisfies $K(v, u_1, u_2, u_3) \subseteq F$. We will show that $K(v, u_1, u_2, \ldots, u_k) \subseteq F$. For each $i = 4, 5, \ldots, k$, we may apply Claim 6.13 with $x = u_1, y = u_2, z = u_3, u = u_i$ to deduce that $u_i u_1, u_i u_2 \in F$. Since $\mathcal{M}$ has the 0-extension property, this implies that F contains an independent set of size $2|N_F(v)| - 3$ on $N_F(v)$. Since F is a flat and every $A \subseteq E(K_n)$ has rank at most $2|V(A)| - 3$, this implies that $K(v, u_1, u_2, \ldots, u_k) \subseteq F$. Let X_{t+i} be a copy of K_4 on $\{v, u_i, u_{i+1}, u_{i+2}\}$ for $i = 1, \ldots, k-2$, and let $\mathcal{S} = (X_1, \ldots, X_t, X_{t+1}, \ldots, X_{t+i+2})$ be obtained by appending $(X_{t+1}, \ldots, X_{t+k-2})$ to $\mathcal{S}'$. Then we have $\mathrm{val}(F, \mathcal{S}) = \mathrm{val}(F_v, \mathcal{S}') + 2 = r_{\mathcal{M}}(F_v) + 2 = r_{\mathcal{M}}(F)$, as required.

It remains to consider the case when no edge of F incident to v is contained in a copy of K_4 in F. Then every edge in F which is incident to v is contained in a copy of $K_{3,3}$. Relabelling if necessary we may suppose that the complete bipartite graph $K(v, w_1, w_2; u_1, u_2, u_3)$ is contained in F. Then $w_i \notin N_F(v)$ for $i = 1, 2$, since otherwise the facts that F is a flat and $\mathcal{M}$ is a K_4-matroid would imply that $K(v, u_1, u_2, w_i) \subseteq F$. By Claim 6.13, F contains $u_i w_1$ and $u_i w_2$ for all $u_i \in N_F(v)$. Hence, F contains the complete bipartite graph $K(N_F(v); \{w_1, w_2\})$. Let $X_{t+i} = K(v, w_1, w_2; u_i, u_{i+1}, u_{i+2})$ for $i = 1, \ldots, k-2$, and let $\mathcal{S} = (X_1, \ldots, X_t, X_{t+1}, \ldots, X_{t+i+2})$ be obtained by appending $(X_{t+1}, \ldots, X_{t+k-2})$ to $\mathcal{S}'$. Then we have $\mathrm{val}(F, \mathcal{S}) = \mathrm{val}(F_v, \mathcal{S}') + 2 = r_{\mathcal{M}}(F_v) + 2 = r_{\mathcal{M}}(F)$, as required.

This completes the proof of Theorem 6.12. □

Corollary 6.14. *Let* $\mathcal{X} = \{K_4, K_{3,3}\}_{K_n}$.
(a) Suppose $\mathcal{H}_n^2$ *is the unique maximal* $\mathcal{X}$*-matroid on* K_n*. Then it is the free elevation of* $R_9(\mathcal{H}_n^2)$ *and its rank function is* $\mathrm{val}_{\mathcal{X}}$.
(b) Suppose the free elevation of $R_9(\mathcal{H}_n^2)$ *has the 0-extension and diamond splitting properties. Then it is the unique maximal* $\mathcal{X}$*-matroid on* K_n *and its rank function is* $\mathrm{val}_{\mathcal{X}}$.

Proof. (a) Lemma 2.6 and the unique maximality of $\mathcal{H}_n^2$ imply that $\mathcal{H}_n^2$ is the free elevation of $R_9(\mathcal{H}_n^2)$ and hence has the $\mathcal{X}$-covering property by Lemma 2.4. In addition, $\mathcal{H}_n^2$ has the 0-extension and diamond splitting properties by Lemma 6.7. Theorem 6.12 now implies that the rank function of $\mathcal{H}_n^2$ is $\mathrm{val}_{\mathcal{X}}$.
(b) The free elevation of $R_9(\mathcal{H}_n^2)$ has the $\mathcal{X}$-covering property by Lemma 2.4. We can now apply Theorem 6.12 to deduce that it is the unique maximal $\mathcal{X}$-matroid on K_n and its rank function is $\mathrm{val}_{\mathcal{X}}$. □

We can prove the following bipartite counterpart of Theorem 6.12 by restricting the argument to complete bipartite graphs.

Theorem 6.15. *Let* $\mathcal{X} = \{K_{3,3}\}_{K_{m,n}}$ *and let* $\mathcal{M}$ *be an* $\mathcal{X}$*-matroid on* $K_{m,n}$ *that has the 0-extension property, the diamond splitting property, and the* $\mathcal{X}$*-covering property. Then* $\mathcal{M}$ *is the unique maximal* $\mathcal{X}$*-matroid on* $K_{m,n}$ *and its rank function is* $\mathrm{val}_{\mathcal{X}}$.

As in the case of $\{K_4, K_{3,3}\}$-matroids, $\mathcal{R}^{2,2}_{m,n}$ satisfies the 0-extension property and the diamond splitting property whereas the free elevation of the rank nine truncation of $\mathcal{R}^{2,2}_{m,n}$ has the $K_{3,3}$-covering property.

Data availability

No data was used for the research described in the article.

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