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Divisibility and coloring of some P_5-free graphs☆

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ABSTRACT

A P_5 is a path on 5 vertices, a banner is a graph obtained by adding a pendant edge to a vertex of a quadrilateral and a hammer is a graph obtained from a K_5 by deleting a banner as a partial subgraph. A graph G is perfect if $\chi(H) = \omega(H)$ for each induced subgraph H of G. We say that G admits a perfect division if $V(G)$ can be partitioned into two subsets A and B such that $G[A]$ is perfect and $\omega(G[B]) < \omega(G)$, and say that G admits a 2-division if $E(G) = \emptyset$ or $V(G)$ can be partitioned into two subsets A and B such that $\max\{\omega(G[A]), \omega(G[B])\} < \omega(G)$. Furthermore, G is perfectly divisible if each induced subgraph H of G admits a perfect division, and G is 2-divisible if each induced subgraph H admits a 2-division. In this paper, we show that each (P_5, banner)-free graph is perfectly divisible, and show that each (P_5, C_5, banner, hammer)-free graph G is $\omega^{\frac{3}{2}}(G)$-colorable. For every P_5-free graph G with $\alpha(G) \geq 3$, we show that G admits a 2-division if G is banner-free, and G is perfect if G is connected and $K_{1,3}$-free.

1. Introduction

All graphs considered in this paper are finite and simple. Let k be a positive integer. We use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. Let G be a graph. We use $\chi(G)$, $\omega(G)$ and $\alpha(G)$ to denote the *chromatic number*, *clique number* and *independent number* of G, respectively. A *path* (resp. *cycle*) on k vertex is denoted by P_k (resp. C_k). We say a graph G is bipartite, if G can be partitioned into two parts S and T such that every edge in G intersects both S and T nonempty. In particular, G is complete bipartite, if every vertex in S connects every vertex in T, and we denote G by $K_{s,t}$, where $|S| = s$ and $|T| = t$. For $x \in V(G)$ and $X \subseteq V(G)$, let $N(x)$ be the set of neighbors of x, let $N[x] = N(x) \cup \{x\}$, and let $N(X) = \bigcup_{v\in X} N(v)$. Let $M(x) = V(G) \setminus N[x]$, and let $M(X) = V(G) \setminus (N(X) \cup X)$. We say that X *dominates* G if $V(G) = X \cup N(X)$, say that x is *complete* (resp. *anticomplete*) to X, if $X \subseteq N(x)$ (resp. $X \subseteq M(x)$), and say that x is *mixed* to X if x is neither complete nor anticomplete to X.

Let X and Y be two subsets of $V(G)$. If each vertex of X is complete (resp. anticomplete) to Y, then we say that X is complete (resp. anticomplete) to Y. We say that X is mixed to Y if X is neither complete nor anticomplete to Y. Let $G[X]$ be the subgraph of G induced by X. We say that G induces H if G has an induced subgraph isomorphic to H, and say that G is *H-free* if G does not induce H. For a given family $\mathscr{H}$ of graphs, we say that G is *$\mathscr{H}$-free* if G is H-free for each member H of $\mathscr{H}$.

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A *hole* of G is an induced cycle of length at least 4. The complement of a hole is called an *antihole*. A hole (resp. an antihole) C is called an *odd hole* (resp. *odd antihole*) if C has odd number of vertices. A graph is *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. The famous *Strong Perfect Graph Theorem* states that a graph is perfect if and only if it induces neither an odd hole nor an odd antihole [7].

Let $k \geq 2$ be an integer. We say that G admits a *perfect division* if $V(G)$ can be partitioned into two subsets A and B such that $G[A]$ is perfect and $\omega(G[B]) < \omega(G)$, and say that G admits a *k-division* if either $E(G) = \emptyset$ or $V(G)$ can be partitioned into k subsets $V_1, V_2, \dots, V_k$ such that $\omega(G[V_i]) < \omega(G)$ for all $i \in [k]$. A graph G is *perfectly divisible* [14] if each induced subgraph H of G admits a perfect division, and G is *k-divisible* [15] if each induced subgraph H admits a k-division. By induction, it is easy to verify that each perfectly divisible graph G is $\binom{\omega(G)+1}{2}$-colorable, and each k-divisible graph G is $k^{\omega(G)-1}$-colorable.

Hoàng and McDiarmid [15] proved that every (odd hole, $K_{1,3}$)-free graph is 2-divisible, and they also proposed a conjecture, the Hoàng and McDiarmid Conjecture for short, claiming that a graph is 2-divisible if and only if it is odd hole free. The necessity of this conjecture is easy since each odd hole is not 2-divisible. Note that the 2-divisibility of G implies that $\chi(G) \leq 2^{\omega(G)-1}$. The trueness of Hoàng and McDiarmid Conjecture will determine a better upper bound on the chromatic number of odd hole free graphs than $\frac{2^{2^{\omega(G)+2}}}{48(\omega(G)+2)}$, the best known upper bound, due to Scott and Seymour [19].

Scott and Seymour mentioned a conjecture of Hoàng, which claims that $\chi(G) \leq \omega^2(G)$ for every odd hole free graph (see page 498 of [19]). Sivaraman [21] proposed a conjecture, that is a weakening version of Hoàng's conjecture, claiming that if G is a (hole of length at least 5)-free graph then $\chi(G) \leq \omega^2(G)$. It is easy to see that P_5-free graphs must be of (hole of length at least 6)-free. The currently best known upper bound to chromatic number of P_5-free graphs is due to Esperet et al. [12] who showed that $\chi(G) \leq 5 \cdot 3^{\omega(G)-3}$ if G is P_5-free with $\omega(G) \geq 3$. A still open conjecture of Choudum, Karthick and Shalu [5] claiming that there is a constant c such that $\chi(G) \leq c\omega^2(G)$ for all P_5-free graphs. There are quite a lot upper bounds to the chromatic number of P_5-free graphs by avoiding some further small graphs.

We use *diamond* to denote the graph obtained from K_4 by removing an edge, use *cricket* to denote the graph obtained from a $K_{1,4}$ by adding an edge between two pendant vertices of $K_{1,4}$. Let $v_1v_2v_3v_4v_5$ denote the path P_5 with vertices v_i for $i \in [5]$ and edges v_iv_{i+1} for $i \in [4]$. We call $P_5 + v_1v_3$ a *hammer*, call $P_5 + v_2v_4$ a *bull*, call $P_5 + v_1v_4$ a *banner*, call $P_5 + \{v_1v_4, v_1v_5\}$ a *house*, call $P_5 + \{v_1v_3, v_1v_4\}$ a *cochair*, call $P_5 + \{v_1v_4, v_2v_4\}$ a *dart*, and call $P_5 + \{v_1v_3, v_1v_4, v_1v_5\}$ a *gem*. A *gem*$^+$ is obtained from a gem by adding a vertex adjacent to its vertex of degree 4.

Fouquet et al. [13] proved that (P_5, house)-free graphs are perfectly divisible. Schiermeyer [18] proved that $\chi(G) \leq \omega^2(G)$ if G is (P_5, H)-free for $H \in$ {cricket, dart, diamond, gem, *gem*$^+$, $K_{1,3}$}. Hoàng [14] showed that every (odd holes, banner)-free graph is both 2-divisible and perfectly divisible. Chudnovsky and Sivaraman [8] showed that (P_5, bull)-free graphs and (odd hole, bull)-free graphs are both perfectly divisible, and (P_5, C_5)-free graphs are 2-divisible. Dong, Xu and Xu [10] proved that $(P_5, C_5, K_{2,3})$-free graphs are perfectly divisible and $\chi(G) \leq 2\omega^2(G) - \omega(G) - 3$ if G is $(P_5, K_{2,3})$-free with $\omega(G) \geq 2$. Improving the results of [3] and [5], Chudnovsky et al. [6] proved that $\chi(G) \leq \lceil \frac{5\omega(G)}{4} \rceil$ if G is (P_5, gem)-free. Let a 4-*wheel* be the graph obtained from a C_4 by adding a vertex complete to C_4, and let a *paraglider* be the graph obtained from a C_4 by adding a vertex joining to three vertices of C_4. Char and Karthick [4] showed that every (P_5, 4-wheel)-free graph G satisfies $\chi(G) \leq \frac{3\omega(G)}{2}$. Huang and Karthick [16] showed that every (P_5, paraglider)-free graph G satisfies $\chi(G) \leq \lceil \frac{3\omega(G)}{2} \rceil$. Very recently, Brause et al. [1] proved that every (P_5, banner)-free graph G is $\Theta(\frac{\omega^2(G)}{\log \omega(G)})$-colorable. Let x be a vertex of the complete graph K_5. Let K_5^- be the graph obtained from K_5 by removing an edge incident with x, let *HVN* be the graph obtained from K_5 by removing two edges incident with x, and let K_4^+ be the graph obtained from K_5 by removing three edges incident with x. Dong, Xu and Xu [11] proved that $\chi(G) \leq \max\{15, 2\omega(G)\}$ if G is (P_5, K_4^+)-free. Improving slightly a result of Malyshev [17], Xu [23] proved that $\chi(G) \leq \max\{\max\{16, \omega(G)+3\}, \omega(G)+1\}$ for all (P_5, *HVN*)-free graphs. Xu [24] proved also that $\chi(G) \leq \max\{13, \omega(G)+1\}$ for all (P_5, K_5^-)-free graphs.

In this paper, we show that each (P_5, banner)-free graph is perfectly divisible. We note that (P_5, banner)-free graphs are not necessarily 2-divisible, since the C_5 is a trivial counterexample. However, we can find a 2-division for (P_5, banner)-free graphs with independent number at least 3, and we can do even better with $(P_5, K_{1,3})$-free graphs.

Theorem 1.1. *Let G be a (P_5, banner)-free graph. Then, G is perfectly divisible, and G admits a 2-division if* $\alpha(G) \geq 3$.

As a corollary of Theorem 1.1, we have that $\chi(G) \leq \binom{\omega(G)+1}{2}$ for all (P_5, banner)-free graphs. We can do better on $(P_5, K_{1,3})$-free graphs with independent number at least 3.

Theorem 1.2. *If G is a connected $(P_5, K_{1,3})$-free graph with* $\alpha(G) \geq 3$, *then G is perfect.*

The restriction $\alpha(G) \geq 3$ in Theorems 1.1 and 1.2 are necessary, since C_5 is (P_5, banner)-free with independent number 2 but admits no 2-division, and all odd antiholes are $(P_5, K_{1,3})$-free with independent number 2 but imperfect.

Chudnovsky and Sivaraman [8] proved that (P_5, C_5)-free graphs are 2-divisible, and Scott, Seymour and Spirkl [20] proved that $\chi(G) \leq \omega(G)^{\log_2 \omega(G)}$ if G is P_5-free. Up to now, no polynomial binding function has been found even for (P_5, C_5)-graphs. Theorem 1.1 asserts that (P_5, banner)-free graphs are perfectly divisible, which provides us with an $O(\omega^2)$ binding function for such graphs. By a conclusion from [2] there is no linear binding function for (P_5, banner)-free graphs. Even for (P_5, C_5, banner)-free graphs, it seems difficult to get a binding function better than $O(\omega^2)$. We study (P_5, C_5, banner, hammer)-free graphs, and prove the following theorem.

Theorem 1.3. *Every (P_5, C_5, banner, hammer)-free graph G is $\omega^{\frac{3}{2}}(G)$-colorable.*

Before we begin our proofs, we list the following useful lemmas. A subset X of $V(G)$ is called a *homogeneous set* if $2 \leq |X| \leq |V(G)| - 1$ and every vertex in $V(G) \setminus X$ is either complete or anticomplete to X.

Lemma 1.1 (*Theorem 3.6, [8]*)**.** *If G is not perfectly divisible with minimum number of vertices, then G admits no homogeneous subset.*

A graph G is a 5-*ring* if its vertex set can be partitioned into sets $X_1, \ldots, X_5$ such that for $i \in [5]$, X_i is a stable set and xy is an edge for any $x \in X_i$, $y \in X_{i+1}$ with the subscript taken modulo 5.

Lemma 1.2 (*Theorem 3.5, [22]*)**.** *A connected graph G is (P_5, K_3)-free if and only if G is either bipartite or a* 5*-ring.*

Proposition 1.1 (*Lemma 7.3, [14]*)**.** *Each graph with independent number at most* 2 *is perfectly divisible.*

We will prove Theorem 1.1 in Section 2, prove Theorem 1.2 in Section 3, and prove Theorem 1.3 in Section 4.

2. Proof of Theorem 1.1

Hoàng has proved the perfect divisibility and 2-divisibility of (odd hole, banner)-free graphs [14], and Chudnovsky and Sivaraman have proved the 2-divisibility of (P_5, C_5)-free graphs [8]. When consider the perfectly divisibility or 2-divisibility of (P_5, banner)-free graphs, we may assume that those graphs are connected and contain a hole of length 5.

Let G be a (P_5, banner)-free graph, and let $C = v_1v_2v_3v_4v_5v_1$ be the chordless cycle with vertices v_i for $i \in [5]$ and edges v_iv_{i+1} for $i \in [5]$ with the subscript taken modulo 5. If G is triangle-free, then $G = C$ by Lemma 1.2. So, we suppose that $\omega(G) \geq 3$.

Define A to be the set of all vertices of $N(C)$ of which each has exactly three or four consecutive neighbors in C, and define B to be the set of all vertices of $N(C)$ of which each is complete to C. It is easy to check that

$$N(C) = A \cup B, \tag{1}$$

as otherwise each $x \in N(C) \setminus (A \cup B)$ together with the cycle C will induce a P_5 or a banner.

By Proposition 1.1, we suppose that $\alpha(G) \geq 3$. Before proving Theorem 1.1, we first present some structural properties of G. Let B_1 be the set of all vertices of B of which each has a neighbor in $M(C)$, and let $B_2 = B \setminus B_1$.

Lemma 2.1. *If $M(C) \neq \emptyset$, then A is anticomplete to $M(C)$, and B_1 is complete to $A \cup B_2 \cup C$.*

Proof. Suppose that $M(C) \neq \emptyset$.

Suppose that A is not anticomplete to $M(C)$, and let $m \in M(C)$ and $n \in A$ with $mn \in E(G)$. If n has exactly three consecutive neighbors in C, we suppose, by symmetry, that $nv_1, nv_2, nv_3 \in E(G)$ and $nv_4, nv_5 \notin E(G)$, then $mnv_3v_4v_5$ is a P_5. If n has four consecutive neighbors in C, we may suppose that v_5 is the only non-neighbor of n on C, then $G[\{m, n, v_1, v_4, v_5\}]$ is a banner. Therefore, A is anticomplete to $M(C)$.

Since G is connected and $M(C) \neq \emptyset$, we see that $B_1 \neq \emptyset$. Let b_1 be a vertex of B_1, and let m be a neighbor of b_1 in $M(C)$. For each pair of $a \in A$ and $b_2 \in B_2$, we may suppose, by symmetry, that $c_1, c_3 \in N(a) \cap N(b_1)$, and so $ab_1 \in E(G)$ to avoid a banner on $\{a, b_1, c_1, c_3, m\}$, and $b_1b_2 \in E(G)$ to avoid a banner on $\{b_1, b_2, c_1, c_3, m\}$. Thus, B_1 is complete to $A \cup B_2 \cup C$. ■

Lemma 2.1 asserts that if $M(C) \neq \emptyset$, then $A \cup B_2 \cup C$ is a homogeneous subset of G.

Next, we consider the case that $M(C) = \emptyset$. If C has a vertex complete to $N(C)$, say v_1, then $(\{v_1, v_3, v_4, v_5\}, V(G) \setminus \{v_1, v_3, v_4, v_5\})$ is a perfect division (also a 2-division). Thus, we suppose that no vertex of C may be complete to $N(C)$.

Recall that each vertex in A has three or four consecutive neighbors on C and each vertex in B is complete to C. For each $i \in [5]$, we define $T_i = \{t : tv_i, tv_{i+1}, tv_{i+2} \in E(G),\ tv_{i+3}, tv_{i+4} \notin E(G)\}$, and $F_i = \{f : fv_i, fv_{i+1}, fv_{i+2}, fv_{i+3} \in E(G),\ fv_{i+4} \notin E(G)\}$ with the subscripts taken modulo 5. Then, $N(C) = B \cup (\cup_{i\in[5]}T_i) \cup (\cup_{i\in[5]}F_i)$.

Let t_1 and t_2 be two distinct vertices of T_i, and let f_1 and f_2 be two distinct vertices in F_i. Since none of $G[\{t_1, t_2, v_i, v_{i+2}, v_{i+4}\}]$ and $G[\{f_1, f_2, v_i, v_{i+2}, v_{i+4}\}]$ can be a banner, we see that both T_i and F_i are cliques. With a similar argument, one can verify that T_i is complete to $F_{i-1} \cup F_i$. Let t_3 be a vertex of T_{i+1}. Since $G[\{t_1, t_3, v_i, v_{i+2}, v_{i+4}\}]$ cannot be a P_5, we see that T_i is complete to T_{i+1}. Therefore, we have that

$$T_i \cup T_{i+1} \cup F_i \text{ is a clique.} \tag{2}$$

If $M(C) = \emptyset$, then

$$M(v_i) = F_{i+1} \cup T_{i+1} \cup T_{i+2} \cup \{v_{i+2}, v_{i+3}\},$$

and

$$N(v_i) = F_i \cup F_{i+2} \cup F_{i+3} \cup F_{i+4} \cup T_i \cup T_{i+3} \cup T_{i+4} \cup \{v_{i+1}, v_{i+4}\}.$$

By (2), we see that

$M(v_i)$ is a clique, (3)

and so $(M(v_i) \cup \{v_i\}, N(v_i))$ is a perfect-division. Therefore, we have

Lemma 2.2. *If $M(C) = \emptyset$, then G admits a perfect division.*

Now, we can prove Theorem 1.1. First, we show that each (P_5, banner)-free graph is perfectly divisible. Suppose to its contrary, and let G be a minimal (P_5, banner)-free non-perfectly divisible graph. Recall that Proposition 1.1 establishes the perfect divisibility of graphs with independence number no more than 2. We have $\alpha(G) \geq 3$.

If $M(C) = \emptyset$, then for each $i \in [5]$, $(M(v_i) \cup \{v_i\}, N(v_i))$ is a perfect division of G by Lemma 2.2. If $M(C) \neq \emptyset$, then G admits a homogeneous set by Lemma 2.1, a contradiction to Lemma 1.1. Therefore, each (P_5, banner)-free graph is perfectly divisible.

To complete the proof of Theorem 1.1, we shall show that every (P_5, banner)-free graph G admits a 2-division, when $\alpha(G) \geq 3$. Before that, we need the following lemma.

Lemma 2.3. *Let $k \geq 2$ be an integer. If G is a minimal graph that admits no k-division, then G admits no homogeneous subset.*

Proof. Suppose that G admits a homogeneous subset and is a minimal graph such that G admits no k-division. Let H be a homogeneous subset belonging to G, $L = G[V(G) \setminus H]$.

Since G is a minimal counterexample, we have that both L and H admit a k-division. Suppose L has a k-division $(L_1, L_2, \ldots, L_k)$ and H has a k-division $(H_1, H_2, \ldots, H_k)$. Write $G_i = L_i \cup H_i$ for $i = [k]$. One can observe that $\omega(G) \geq \omega(G_i) \geq \max\{\omega(L_i), \omega(H_i)\}$. Let K be a maximum clique in G_i. Obviously, K will not entirely lie in L_i or H_i. Since H is complete to $N_G(H)$ and anticomplete to $L \setminus N_G(H)$, we have $K \cap H_i \neq \emptyset$, $K \cap N_G(H) \neq \emptyset$ and $K \cap (L \setminus N_G(H)) = \emptyset$. If $|K| = \omega(G)$, then $K \cap H_i$ must be a largest clique in H, a contradiction. ■

Suppose that $\alpha(G) \geq 3$. Notice that (odd holes, banner)-free graphs are 2-divisible (see [14]). By Lemmas 2.1 and 2.3, we may suppose that $M(C) = \emptyset$. For convenience, we use *co-triangle* to denote an independent set of size 3.

Since $M(v_i)$ is a clique for each $i \in [5]$ by (3), if there exists a v_i such that $M(v_i)$ is not a maximum clique of G, then $(N(v_i), M(v_i) \cup \{v_i\})$ is a 2-division. So, we suppose that

$M(v_i)$ is a maximum clique of G for each $i \in [5]$. (4)

We will complete the proof of Theorem 1.1 by showing

G contains no co-triangles. (5)

Suppose that (5) does not hold and let C_0 be a co-triangle of G. It is certain that $|C_0 \cap V(C)| \leq 2$.

Recall that $C = v_1v_2v_3v_4v_5v_1$. If $|C_0 \cap V(C)| = 2$, we suppose, by symmetry, that $C_0 = \{u, v_1, v_3\}$, then $u \notin (A \cup B)$, contradicting (1). If $|C_0 \cap V(C)| = 1$, we suppose that $C_0 = \{u_1, u_2, v_1\}$ (where $u_1, u_2 \notin C$), then $u_1, u_2 \in T_2 \cup T_3 \cup F_2$ as $M(C) = \emptyset$, contradicting (2) by taking $i = 2$. Therefore, C_0 contains no vertex of C.

Suppose that $C_0 = \{u_1, u_2, u_3\}$. By (2), we have the following possibilities, for some $i \in [5]$, on the locations of the vertices of C_0.

(a) $u_1 \in F_i$, $u_2 \in F_{i+1}$, and $u_3 \in F_{i+2} \cup T_{i+3}$.
(b) $u_1 \in F_i$, $u_2 \in F_{i+2}$, and $u_3 \in F_{i+3} \cup T_{i-1}$.
(c) $u_1 \in F_i \cup T_i$, $u_2 \in F_{i+1}$, and $u_3 \in B$.
(d) $u_1 \in F_i \cup T_i$, $u_2 \in F_{i+2} \cup T_{i+2}$, and $u_3 \in B$.
(e) $u_1 \in F_i \cup T_i$, $u_2, u_3 \in B$.
(f) $u_1 \in T_i$, $u_2 \in T_{i+2}$, and $u_3 \in F_{i+3}$.
(g) $C_0 \subseteq B$.

Since $G[\{u_1, u_2, u_3, v_i, v_{i+2}\}]$ is a banner in cases (a) to (c) and a P_5 in case (f), and $G[\{u_1, u_2, u_3, v_{i-1}, v_{i+2}\}]$ is a banner in cases (d) and (e), we turn to case (g).

Suppose that $C_0 \subseteq B$, and let v be a vertex in $T_i \cup F_i$. The vertex v exists, for otherwise $A = \emptyset$, and so C is a homogeneous set of G, a contradiction to Lemma 2.3. If v is anticomplete to C_0, then $G[\{v, v_{i+1}, u_1, u_2, v_{i+4}\}]$ is a banner. If v has exactly one neighbor in C_0, say $vu_1 \in E(G)$ by symmetry, then $G[\{v, v_{i+1}, u_2, u_3, v_{i+4}\}]$ is a banner. If v has exactly two neighbors in C_0, say $vu_3 \notin E(G)$ by symmetry, then $G[\{v, u_1, u_2, v_{i+4}, u_3\}]$ is a banner. So, we have that C_0 is complete to $T_i \cup F_i$. By symmetry between the pair (T_i, F_i) and the pair (T_{i+1}, F_i), one can verify easily that C_0 is complete to $T_{i+1} \cup F_i$. Therefore, C_0 is complete to $T_i \cup T_{i+1} \cup F_i$ if $C_0 \subseteq B$. Recall that $M(v_{i-1}) = F_i \cup T_i \cup T_{i+1} \cup \{v_{i+1}, v_{i+2}\}$. Since $u_1 \in B$, we see that $M(v_{i-1}) \cup \{u_1\}$ is a clique larger than $M(v_{i-1})$. This contradiction to (4) proves (5), and completes the proof of Theorem 1.1. ■

3. (P_5, $K_{1,3}$)-free graphs

We prove Theorem 1.2 in this section. Below lemma is very useful to our proof.

Lemma 3.1 (*Ben Rebea's Lemma, see* [9]). *Let G be a $K_{1,3}$-free graph which induces an odd antihole. If $\alpha(G) \geq 3$, then G contains an induced cycle of length* 5.

Proof of Theorem 1.2. Suppose to the contrary that Theorem 1.2 does not hold. Let G be an imperfect (P_5, $K_{1,3}$)-free graph with $\alpha(G) \geq 3$. Since G is P_5-free, G contains an odd antihole as induced subgraph. By Lemma 3.1, G contains a hole of length 5. Let $C = v_1v_2v_3v_4v_5v_1$ be a hole of length five in G.

We still define A to be the set of all vertices of $N(C)$ of which each has exactly three or four consecutive neighbors on C, define B to be the set of all vertices of $N(C)$ of which each is complete to C, and define T_i and F_i for each $i \in [5]$ in the same way as that of last section. Then, $A = (\cup_{i\in[5]}T_i) \cup (\cup_{i\in[5]}F_i)$, and $N(C) = A \cup B$ as G is certainly (P_5, banner)-free.

If $M(C) \neq \emptyset$, we may choose w to be a vertex in $M(C)$ that has a neighbor, say w', in $N(C)$, then $\{w, w'\}$ together with two nonadjacent neighbors of w' on C would induce a $K_{1,3}$. Therefore, $M(C) = \emptyset$.

Since $\alpha(G) \geq 3$, we may choose a stable set of size 3, say $S = \{u_1, u_2, u_3\}$. Note that each (P_5, $K_{1,3}$)-free graph must be (P_5, banner)-free. Since $M(C) = \emptyset$, both (2) and (3) still hold for each $i \in [5]$.

Since G is $K_{1,3}$-free, we see that $S \not\subseteq B$, and either $S \cap V(C) = \emptyset$ or $S \cap B = \emptyset$ as B is complete to C. If $|S \cap B| = 2$, we suppose by symmetry that $u_1, u_2 \in B$, and let v be a neighbor of u_3 on C. If $|S \cap B| = 1$, we suppose $u_1 \in B$, and let v be a common neighbor of u_2 and u_3 on C. In both cases, we have a $K_{1,3}$ induced by $S \cup \{v\}$. Therefore, we have $S \cap B = \emptyset$.

Recall that for each $i \in [5]$, both F_i and T_i are cliques by (2).

If $|S \cap (\cup_{i\in[5]}F_i)| \geq 2$, we suppose that $u_1, u_2 \in \cup_{i\in[5]}F_i$ by symmetry, then there exists $j \in [5]$ such that the common neighbors of u_1 and u_2 on C is either $\{v_j, v_{j+1}, v_{j+2}\}$ or $\{v_j, v_{j+1}, v_{j+3}\}$. Since u_3 has at least 3 neighbors on C, one can always find a vertex, say v, on C such that $G[S \cup \{v\}]$ is a $K_{1,3}$.

If $|S \cap (\cup_{i\in[5]}F_i)| = 1$, we may suppose that $u_1 \in F_1$ by symmetry, then $u_2, u_3 \in T_3 \cup T_4 \cup T_5 \cup \{v_5\}$ by (2), and so $|\{u_2, u_3\} \cap T_3| = 1 = |\{u_2, u_3\} \cap T_5|$, which implies an induced $P_5 = u_1v_3u_2v_5u_3$ or $u_1v_3u_3v_5u_2$.

So, we have that $S \cap (\cup_{i\in[5]}F_i) = \emptyset$ as well, and thus $S \subseteq \cup_{i\in[5]}T_i \cup V(C)$. But $G[\cup_{i\in[5]}T_i \cup V(C)]$ is a graph obtained by blowing up each vertex of a C_5 into a clique, which has independent number 2. This contradiction to $\alpha(G) \geq 3$ completes the proof of Theorem 1.2. ■

4. (P_5, C_5, banner, hammer)-free graphs

We prove Theorem 1.3 in this section.

Let G be a (P_5, C_5, banner, hammer)-free graph on n vertices. Following the *Strong Perfect Graph Theorem* [7], G is perfect if $n \leq 6$. If $n = 7$, G is imperfect if and only if G is an odd antihole, and $\chi(G) = 4 \leq \frac{4}{3}\omega(G) \leq \omega^{\frac{3}{2}}(G)$. If G is triangle-free, then G is bipartite by Lemma 1.2, and $\chi(G) = 2 \leq 2^{\frac{3}{2}}$. So, Theorem 1.3 holds for $n \leq 7$ or $\omega(G) \leq 2$.

Suppose that $n \geq 8$, $\omega(G) \geq 3$, and G is a counterexample to Theorem 1.3 with minimum n. We may assume that G is imperfect, and

$$\text{if } uv \notin E(G), \text{ then neither } N(u) \subseteq N(v) \text{ nor } N(v) \subseteq N(u) \text{ holds.} \tag{6}$$

Let $k \geq 7$ be an odd integer, and let C be an odd antihole of G with vertex set $\{c_1, c_2, \ldots, c_k\}$ such that c_i is adjacent to all vertices but c_{i-1} and c_{i+1} of C, here the subscripts are taken modulo k.

Let v be a vertex in $N(C)$. We call v an (i, j)-neighbor of C if v is complete to $\{c_i, c_{i+1}, \ldots, c_{i+j-1}\}$ and anticomplete to $\{c_{i-1}, c_{i+j}\}$. Especially, an $(i, 1)$-neighbor of C is a vertex adjacent to c_i but nonadjacent to c_{i-1} and c_{i+1}. To avoid a 5-hole $c_ic_{i+2}c_{i-1}c_{i+1}vc_i$, we see that

$$\text{no vertex of } N(C) \text{ can be an } (i, 2)\text{-neighbor of } C. \tag{7}$$

If v is an $(i, 1)$-neighbor of C for some i, then $vc_{i-2} \in E(G)$ to avoid a $P_5 = vc_ic_{i-2}c_{i+1}c_{i-1}$, and $vc_{i+2} \in E(G)$ to avoid a $P_5 = vc_ic_{i+2}c_{i-1}c_{i+1}$. Therefore, we have

$$vc_{i-2} \in E(G) \text{ and } vc_{i+2} \in E(G) \text{ for each } (i, 1)\text{-neighbor } v \text{ of } C, \tag{8}$$

and consequently, each vertex of $N(C)$ has at least two neighbors in C.

Let $B \subset N(C)$ be the set of all vertices complete to C, and let $A = N(C) \setminus B$. We first claim that

$$\text{each vertex of } A \text{ is an } (i, j)\text{-neighbor of } C \text{ for some } i \in [k] \text{ and } j \geq 3. \tag{9}$$

If it is not the case, we may suppose, without loss of generality, that $v \in A$ is a vertex such that $c_1v \in E(G)$, $c_2v \notin E(G)$ and $c_kv \notin E(G)$ by (7), then $c_3v \in E(G)$ and $c_{k-1}v \in E(G)$ by (8). Repeating this argument with odd integer $i \in \{1, 3, 5, \ldots, k-2\}$, since k is odd, we have that $c_kv \in E(G)$, contradicting our assumption that $c_kv \notin E(G)$. Therefore, (9) holds.

For each $j \in [k]$ and $c_jv \in E(G)$, since $\{v, c_j, c_{j+1}, c_{j+3}, c_{j+4}\}$ cannot induce a banner, we know that

$$\text{there will not exist } j \text{ such that } vc_j \in E(G) \text{ and } vc_{j+i} \notin E(G) \text{ for all } i \in [4]. \tag{10}$$

We consider two possibilities depending on $M(C) = \emptyset$ or not.

Case 1. First suppose that $M(C) \neq \emptyset$.

Let B_1 be the set of all vertices of B of which each has a neighbor in $M(C)$, and let $B_2 = B \setminus B_1$. We show that

$$A \text{ is anticomplete to } M(C), \text{ and } B_1 \text{ is complete to } A \cup B_2 \cup C. \tag{11}$$

The proof of (11) is almost the same as that of Lemma 2.1.

Let v be a vertex of A. Suppose that v has a neighbor, say w, in $M(C)$. By (9), we may suppose by symmetry that v is a $(1, j)$-neighbor for some $j \geq 3$. Then, $\{w, v, c_2, c_3, c_k\}$ induces a banner. So, A is anticomplete to $M(C)$.

Since G is connected, we see that B cannot be anticomplete to $M(C)$, and so $B_1 \neq \emptyset$. Let b_1 be a vertex of B_1, and let w be a neighbor of b_1 in $M(C)$. For each pair of $a \in A$ and $b_2 \in B_2$, we may suppose that, for some i, $c_i, c_{i+1} \in N(a) \cap N(b_1)$ by (9), and so $ab_1 \in E(G)$ to avoid a banner on $\{a, b_1, c_i, c_{i+1}, w\}$, and $b_1b_2 \in E(G)$ to avoid a banner on $\{b_1, b_2, c_i, c_{i+1}, w\}$. This proves (11).

Furthermore, we can also show that

$$B_1 \text{ is complete to } M(C), \tag{12}$$

as otherwise, let $w' \in N_{M(C)}(w) \setminus N(b_1)$, then $\{w', w, b_1, c_i, c_{i+2}\}$ induces a hammer.

Combine (11) and (12), we have that B_1 is complete to $G \setminus B_1$, and so $\omega(G) \leq \omega(B_1) + \omega(G \setminus B_1)$. Since G is a minimum counterexample to Theorem 1.3, we know that B_1 is $\omega^{\frac{3}{2}}(B_1)$-colorable and $G \setminus B_1$ is $\omega^{\frac{3}{2}}(G \setminus B_1)$-colorable. Then,

$$\begin{aligned}\chi(G) &\leq \chi(B_1) + \chi(G \setminus B_1)\\ &\leq \omega^{\frac{3}{2}}(B_1) + \omega^{\frac{3}{2}}(G \setminus B_1)\\ &\leq \omega^{\frac{3}{2}}(G),\end{aligned}$$

a contradiction.

Case 2. Now, we can suppose that $M(C') = \emptyset$ holds for any odd antihole C' in G. For $i \in [k]$, we denote the edge c_ic_{i+2} by $e_{i,i+2}$, call such an edge as a *main edge*, and define $M_{i,i+2}$ to be the set of all vertices which are anticomplete to $\{c_i, c_{i+2}\}$.

Let $A_i = N(c_i) \setminus N(c_{i+2})$, $B_i = N(c_{i+2}) \setminus N(c_i)$, and $D_{i,i+2} = N(c_i) \cap N(c_{i+2})$, where the summation of subindexes are taken modulo k. We partition $D_{i,i+2}$ into two subsets $D_{i,i+2,1}$ and $D_{i,i+2,2}$ such that each vertex in $D_{i,i+2,1}$ has a neighbor in $M_{i,i+2}$ and $D_{i,i+2,2} = D_{i,i+2} \setminus D_{i,i+2,1}$. Since C is an odd antihole, we have that $c_{i+3} \in A_i$, $c_{i-1} \in B_i$ and $c_{i+4}, c_{i-2} \in D_{i,i+2,1}$, which imply that $A_i, B_i, D_{i,i+2,1} \neq \emptyset$.

We will show that, for each $i \in [k]$,

$$A_i \cup B_i \cup M_{i,i+2} \text{ is a clique.} \tag{13}$$

By symmetry, we may take $e = e_{1,3}$ as an example. Denote $M = M_{1,3}$, $D = D_{1,3}$, $D_1 = D_{1,3,1}$ and $D_2 = D_{1,3,2}$ for simplicity. Since c_2 is anticomplete to e, we have that $c_2 \in M$. Let a_1 be a vertex in A_1, and b_1 a vertex in B_1. If $a_1c_2 \notin E(G)$, then $a_1c_1c_3c_kc_2$ is an induced P_5 or banner, a contradiction. This shows that $a_1c_2 \in E(G)$. Similarly, we have that $b_1c_2 \in E(G)$. Therefore,

$$c_2 \text{ is complete to } A_1 \cup B_1.$$

If $N_M(a_1) \neq N_M(b_1)$, we may suppose $m \in N_M(a_1) \setminus N_M(b_1)$, then $\{m, a_1, b_1, c_1, c_3\}$ will induce a P_5 or a banner. So, $N_M(a_1) = N_M(b_1)$. Since G is C_5-free, we have that $a_1b_1 \in E(G)$, and thus

$$A_1 \text{ is complete to } B_1.$$

Let $a_1' \in A_1$. Since $G[\{a_1, a_1', c_1, c_2, c_3\}]$ is not a banner, we know that $a_1a_1' \in E(G)$, which implies that A_1 is a clique. Similarly, we have that B_1 is also a clique. So,

$$A_1 \cup B_1 \text{ is a clique.}$$

Let a be a vertex in $A_1 \cup B_1 \cup D_1$ and M' be a component of $G[M]$ such that a has a neighbor in M'. If a is not complete to M', then there exists an edge $w_1w_2 \in E(M')$ such that $\{a, w_1, w_2\}$ forms a P_3 and thus $\{c_1, c_3, a, w_1, w_2\}$ is a P_5 or a hammer. Therefore,

$$A_1 \cup B_1 \cup D_1 \text{ is complete to } M'.$$

We will show further that

$$wc_2 \in E(G) \text{ for each vertex } w \in M. \tag{14}$$

Suppose that $wc_2 \notin E(G)$ for some $w \in M$. Then, $wc_1, wc_2, wc_3 \notin E(G)$. Since $M(C) = \emptyset$, there exists i such that $wc_i \in E(G)$. Suppose such an i makes $\min\{|i-1|, |i-3|\}$ minimum under taking index modulo k. Then, it is not hard to verify $wc_4, wc_k \in E(G)$ by (10) and hence $\{w, c_2, c_3, c_4, c_5\}$ will induce a P_5 or a banner. Therefore, (14) holds, and thus $G[M]$ is connected, which implies that $A_1 \cup B_1 \cup D_1$ is complete to M. By symmetry, we have that

$$A_i \cup B_i \cup D_{i,i+2,1} \text{ is complete to } M_{i,i+2} \text{ for each } i \in [k]. \tag{15}$$

To prove (13), it remains to show that

M is a clique. (16)

If it is not the case, then let w_1, w_2 be two nonadjacent vertices in M. Since $E(G) \cap \{w_1c_1, w_1c_3, w_2c_1, w_2c_3\} = \emptyset$, we have $\{w_1c_4, w_1c_k, w_2c_4, w_2c_k\} \subseteq E(G)$ by (8). For $v \in \{w_1, w_2\}$, as $\{v, c_2, c_3, c_t, c_{t+1}\}$ will not induce a banner, we have that

there will not exist t with $5 \le t \le k-1$ such that $vc_t, vc_{t+1} \notin E(G)$. (17)

If $\{w_1, w_2\}$ is not anticomplete $\{c_5, c_{k-1}\}$, we may suppose $w_1c_5 \in E(G)$, then $\{w_1, w_2, c_3, c_4, c_5\}$ will induce a banner or a P_5, a contradiction. So, $\{w_1c_5, w_1c_{k-1}, w_2c_5, w_2c_{k-1}\} \cap E(G) = \emptyset$, which implies that $\{w_1c_6, w_2c_6, w_1c_{k-2}, w_2c_{k-2}\} \subseteq E(G)$ by (8) and (17). Consider iteratively the subsets $\{w_1, w_2, c_{1+2t}, c_{2+2t}, c_{3+2t}\}$ for $1 \le t \le \frac{k-7}{2}$. With almost the same arguments as above, one can verify, by (8) and (17), that $\{w_1c_{1+2t}, w_2c_{1+2t}\} \cap E(G) = \emptyset$ and $\{w_1c_{2+2t}, w_2c_{2+2t}\} \subseteq E(G)$ for $1 \le t \le \frac{k-5}{2}$. This shows that $\{w_1c_{k-4}, w_2c_{k-4}\} \cap E(G) = \emptyset$ and $\{w_1c_{k-3}, w_2c_{k-3}\} \subseteq E(G)$. This contradicts (7), as $\{w_1c_{k-2}, w_2c_{k-2}\} \subseteq E(G)$ and $\{w_1c_{k-1}, w_2c_{k-1}\} \cap E(G) = \emptyset$. Therefore, (16) follows and so does (13).

Let $d \in D_2$. Since $G[\{d, c_1, c_2, c_3, a_1\}]$ and $G[\{d, c_1, c_2, c_3, b_1\}]$ cannot be hammers, we have that $a_1d \in E(G)$ and $b_1d \in E(G)$. By symmetry, we have that

$D_{i,i+2,2}$ is complete to $A_i \cup B_i$ for each $i \in [k]$. (18)

Now, we consider the main edge $e_{k,2}$. Since $D_{1,3,1}$ is complete to $M_{1,3}$ by (15), we have that $D_{1,3,1} \subseteq N(c_2)$. Since $D_{1,3,2}$ is complete to $A_1 \cup B_1$ by (18), we have that $D_{1,3,2} \subseteq N(c_k)$. Notice that $A_1 \cup B_1 \cup M_{1,3} \setminus \{c_k, c_2\} \subseteq N(c_2) \cap N(c_k)$ by (13). We have that $A_k = N(c_k) \setminus N(c_2) = D_{1,3,2} \cup \{c_3\}$, and thus $D_{1,3,2}$ is a clique as A_k is a clique by (13). With the similar argument, we can show that $M_{k,2} = \{c_1\}$. By symmetry, we have that, for each $i \in [k]$,

$N(c_i) \setminus N(c_{i+2}) = D_{i+1,i+3,2} \cup \{c_{i+3}\}$, which is a clique, and $M_{i,i+2} = \{c_{i+1}\}$. (19)

Recall that $D_1 = D_{1,3,1}$ and $D_2 = D_{1,3,2}$. Let $t \ge 0$, and let $D_2 = \{d_1, d_2, \ldots, d_t\}$. By (19), we have that $D_2 \cup \{c_1, c_3\}$ is a clique.

For a subset $Z \subset V(G)$ and a vertex $x \in V(G)$, let $M_Z(x)$ be the set of vertices of Z which are not adjacent to x. For $i \in [t]$, let $U_i = M_{D_1}(d_i)$, which is the set of non-neighbors of d_i in D_1. By (18), we have that $U_i = M_{A_1\cup B_1\cup D_1}(d_i)$. We will prove that

$\bigcup_{v\in D_2\cup\{c_1,c_3\}} M_{A_1\cup B_1\cup D_1}(v)$ is a clique. (20)

To prove (20), we first prove that

$M_{A_1\cup B_1\cup D_1}(v)$ is a nonempty clique for each vertex $v \in D_2 \cup \{c_1, c_3\}$. (21)

Since $M_{A_1\cup B_1\cup D_1}(c_1) = B_1$ and $M_{A_1\cup B_1\cup D_1}(c_3) = A_1$, which are both cliques by (13), we only need to verify that (21) holds for the vertices in D_2. If $U_i = \emptyset$ for some i, then $N(c_1) \subseteq N(d_i)$ by (18), contradicting (6). Therefore, $U_i \ne \emptyset$ for all $i \in [t]$. If there exists an $i \in [t]$ and two nonadjacent vertices $u_i, u_i' \in U_i$, then $G[\{d_i, u_i, u_i', c_1, c_2\}]$ is a banner, a contradiction. So, U_i is a clique for all $i \in [t]$, and thus (21) holds.

If $U_i \cap U_j \ne \emptyset$ for some $1 \le i < j \le t$, then there exists $u \in U_i \cap U_j$ such that $d_iu \notin E(G)$ and $d_ju \notin E(G)$, which implies $G[\{d_i, d_j, u, c_1, c_2\}]$ is a hammer. Therefore,

$U_1, U_2, \ldots, U_t$ are pairwisely disjoint,

and consequently, $A_1, B_1, U_1, U_2, \ldots, U_t$ are pairwisely disjoint.

Let $a_1 \in A_1$ and $b_1 \in B_1$. For integers $1 \le i < i' \le t$, let $u_i \in U_i$ and $u_{i'} \in U_{i'}$. From (18), we have that $a_1d_i, b_1d_i, a_1d_{i'}, b_1d_{i'} \in E(G)$. If $u_iu_{i'} \notin E(G)$, then $G[\{d_i, d_{i'}, u_i, u_{i'}, c_2\}]$ is a C_5. So, $u_iu_{i'} \in E(G)$. If $a_1u_i \notin E(G)$, then $G[\{d_i, c_3, u_i, a_1, c_2\}]$ is a C_5. So, $a_1u_i \in E(G)$. Similarly, we have that $b_1u_i \in E(G)$. Therefore, $U_i, U_{i'}, A_1, B_1$ are pairwisely complete for $1 \le i < i' \le t$. By (21), we have (20) holds.

If $\omega(G \setminus \{c_2\}) < \omega(G)$, then

$$\begin{aligned}\chi(G) &\le \chi(G \setminus \{c_2\}) + 1\\ &\le (\omega(G) - 1)^{\frac{3}{2}} + 1\\ &< \omega^{\frac{3}{2}}(G).\end{aligned}$$

So, $\omega(G \setminus \{c_2\}) = \omega(G)$. Let $W_0 = \{c_1, c_3\} \cup D_2$ and $W_1 = A_1 \cup B_1 \cup U_1 \cup U_2 \cup \cdots \cup U_t$. By (19) and (20), we have W_0 and W_1 are cliques. By (21), we have $\omega(W_0) \le \omega(W_1)$. Since c_2 is complete to $A_1 \cup B_1 \cup D_1$, $A_1 \cup B_1 \cup D_1$ contains no maximum cliques, which means both $G \setminus (W_0 \cup \{c_2\})$ and $G[W_1]$ contain no maximum cliques. Therefore, $\omega(G \setminus (W_0 \cup \{c_2\})) < \omega(G)$ and $\omega(W_1) < \omega(G)$.

If $\omega(W_1) \geq \omega^2(W_0)$, then

$$\begin{aligned}\chi(G) &\leq \chi(G \setminus (W_0 \cup \{c_2\})) + \chi(W_0 \cup \{c_2\}) \\ &\leq \omega^{\frac{3}{2}}(G \setminus (W_0 \cup \{c_2\})) + \omega(W_0) \\ &\leq (\omega(G) - 1)^{\frac{3}{2}} + \omega^{\frac{1}{2}}(W_1) \\ &\leq (\omega(G) - 1)^{\frac{3}{2}} + (\omega(G) - 1)^{\frac{1}{2}} \\ &\leq (\omega(G) - 1)^{\frac{1}{2}} \cdot \omega(G) \\ &\leq \omega^{\frac{3}{2}}(G).\end{aligned}$$

Suppose that $\omega(W_1) < \omega^2(W_0)$. Let $d_0 \in W_0$ be a vertex such that its corresponding nonadjacent clique $U_0 \subseteq W_1$ has minimum size among $A_1, B_1, U_1, U_2, \ldots, U_t$. We have that $\omega(W_0) \cdot \omega(U_0) = (t+2)\omega(U_0) \leq \omega(A_1) + \omega(B_1) + \sum_{i=1}^{t} \omega(U_i)\omega(W_1) < \min\{\omega^2(W_0), \omega(G)\}$, which implies that $\omega(U_0) < \omega(W_0)$ and $\omega(U_0) < (\omega(G)-1)^{\frac{1}{2}}$. Since d_0 is complete to $G \setminus (U_0 \cup \{c_2, d_0\})$, we have $\omega(G \setminus (U_0 \cup \{c_2, d_0\})) \leq \omega(G) - 1$ and then

$$\begin{aligned}\chi(G) &\leq \chi(G \setminus (U_0 \cup \{c_2, d_0\})) + \chi(U_0 \cup \{d_0\}) + \chi(\{c_2\}) \\ &\leq \omega^{\frac{3}{2}}(G \setminus (U_0 \cup \{c_2, d_0\})) + \omega(U_0) + 1 \\ &\leq (\omega(G) - 1)^{\frac{3}{2}} + ((\omega(G) - 1)^{\frac{1}{2}} - 1) + 1 \\ &\leq (\omega(G) - 1)^{\frac{1}{2}} \cdot \omega(G) \\ &\leq \omega^{\frac{3}{2}}(G).\end{aligned}$$

Thus, $\chi(G) \leq \omega^{\frac{3}{2}}(G)$ holds and so does Theorem 1.3. ■

Data availability

No data was used for the research described in the article.

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