C_{10} has positive Turán density in the hypercube

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Abstract. The n-dimensional hypercube Q_n is a graph with vertex set $\{0,1\}^n$ and there is an edge between two vertices if they differ in exactly one coordinate. For any graph H, define $\mathrm{ex}(Q_n, H)$ to be the maximum number of edges of a subgraph of Q_n without a copy of H. In this short note, we prove that for any $n \in \mathbb{N}$

$$\mathrm{ex}(Q_n, C_{10}) > 0.024 \cdot e(Q_n).$$

Our construction is strongly inspired by the recent breakthrough work of Ellis, Ivan, and Leader [12], who showed that "daisy" hypergraphs have positive Turán density with an extremely clever and simple linear-algebraic argument.

1. Introduction

For each $n \in \mathbb{N}$, define the n-dimensional hypercube Q_n to be the graph with vertex set $\{0,1\}^n$, and with an edge between two vertices if they differ in exactly one coordinate. Erdős [13] initiated the study of $\mathrm{ex}(Q_n, H)$, the maximum number of edges in an H-free subgraph of Q_n, in the special case where H is an even cycle. We say that H has positive Turán density in the hypercube if there is some constant $\alpha > 0$ such that for every $n \in \mathbb{N}$

$$\mathrm{ex}(Q_n, H) \geqslant \alpha \cdot e(Q_n),$$

where $e(Q_n)$ denotes the number of edges of Q_n.

It will be useful to identify in the usual way the vertices of Q_n with the subsets of $[n] := \{1, \ldots, n\}$. For any $r \leqslant n$, we define $L_r(n)$, the r-th (edge) layer of Q_n, to be the subgraph of Q_n formed by the edges between $\binom{[n]}{r-1}$ and $\binom{[n]}{r}$, where

$$\binom{[n]}{r} := \big\{ S \subset [n] : |S| = r \big\}.$$

It is easy to see that $C_4 \not\subset L_r(n)$, thus taking every second layer gives that

$$\mathrm{ex}(Q_n, C_4) \geqslant \frac{1}{2} \cdot e(Q_n).$$

Erdős [14] offered \$100 for an answer to whether this bound is optimal, and it remains open until the present date. The best known upper bound, due to Baber [4], is approximately $0.60318 \cdot e(Q_n)$ (see [9, 17, 5] for previous upper bounds).

Erdős [13] suggested that longer even cycles might not have positive Turan density, but this was shown to be false for C_6 by Chung [9] and Brouwer, Dejter, and Thomassen [7]. On

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the other hand, it was proved by Chung [9] for every even $t \geqslant 4$, and by Füredi and Özkahya [15, 16] for every odd $t \geqslant 7$, that

$$\mathrm{ex}(Q_n, C_{2t}) = o(e(Q_n)),$$

and a unified proof for all of the above cases was given by Conlon [11]. Thus, the only case for which the problem remained open was C_{10}. The main result of this paper completes the picture, by showing that C_{10} has positive Turán density in the hypercube.

Theorem 1.1. *For all $n \in \mathbb{N}$*

$$\mathrm{ex}(Q_n, C_{10}) > 0.024 \cdot e(Q_n).$$

In fact, our proof gives the constant $c/12$, where

$$c = \prod_{k=1}^{\infty} \left(1 - \frac{1}{2^k}\right) > 0.288.$$

We also remark that the graph we construct to prove Theorem 1.1 is free of C_4, C_6 and C_{10} simultaneously. The previous best known lower bound in this case was

$$\mathrm{ex}(Q_n, C_{10}) \geqslant \frac{e(Q_n)}{(\log n)^{\alpha}}$$

for some constant $\alpha > 0$, recently shown by Axenovich, Martin, and Winter [3, Theorem 5].

The Ramsey problem for even cycles in the hypercube has also attracted a great deal of attention over the years. In particular, Chung [9] and Brouwer, Dejter, and Thomassen [7] found 4-colourings of the hypercube without a monochromatic C_6, and Conder [10] found a 3-colouring with the same property, implying the best known lower bound for this case of $\mathrm{ex}(n, C_6) \geqslant \frac{1}{3} e(Q_n)$. However, it was shown by Alon, Radoičić, Sudakov, and Vondrák [1] that for any k-colouring of the edges of Q_n there is a monochromatic copy of C_{10}. Our result therefore gives, to the best of our knowledge, the first example of a Ramsey graph which has positive Turán density in the hypercube. The best upper bound we are aware of is $\mathrm{ex}(Q_n, C_{10}) \leqslant \frac{1}{\sqrt{2}} \cdot e(Q_n)$ by Axenovich and Martin [2, Theorem 3.3].

Our proof of Theorem 1.1 is heavily inspired by a recent breakthrough work of Ellis, Ivan, and Leader [12], who gave an extremely clever and simple linear-algebraic construction which shows that "daisies" have positive Turán density in hypergraphs, disproving a conjecture of Bollobás, Leader, and Malvenuto [6] and Bukh [8]. The argument we use is essentially a slight modification of their approach.

We will need another important idea, first observed by Alon, Radoičić, Sudakov, and Vondrák [1] and Axenovich and Martin [2]: it is possible to find a 4-colouring of $E(Q_n)$ without monochromatic *chordless* C_{10}. More precisely, define $\mathrm{ex}^*(Q_n, C_6^-)$ to be the maximum number of edges of a subgraph of Q_n such that there is no embedding of C_6 minus one edge that can be extended to a C_6 in Q_n. We will make use of the following result of Axenovich, Martin, and Winter [3], which is based on the 3-colouring given by Conder [10].

Theorem 1.2 (Lemma 19 in [3])**.**

$$\mathrm{ex}(Q_n, C_{10}) \geqslant \frac{1}{3} \cdot \mathrm{ex}^*(Q_n, C_6^-).$$

Thus, the core of the proof will be to show the following theorem.

Theorem 1.3. *For any $r, n \in \mathbb{N}$ with $r \leqslant n$, there exists a C_6-free induced subgraph G_r of $L_r(n)$ with*

$$e(G_r) > \frac{c}{2} \cdot e(L_r(n)).$$

Proof of Theorem 1.1 from Theorem 1.3. Take G to be the union of the graphs G_r given by Theorem 1.3 for every odd $r \leqslant n$, so that G_r is disconnected from $G_{r'}$ for every $r \neq r'$.

Observe that every induced subgraph of a layer containing a C_6^- also contains a C_6; thus, since G_r is a C_6-free induced subgraph of $L_r(n)$, it is also free of C_6^-. This implies

$$\mathrm{ex}^*(Q_n, C_6^-) \geqslant e(G) > \frac{c}{2} \sum_{r=1}^{\lceil n/2 \rceil} e\big(L_{2r-1}(n)\big) = \frac{c}{4} \cdot e(Q_n),$$

which, by Theorem 1.2, concludes the proof. □

2. Proof of Theorem 1.3

Similarly to the argument used in [12], we consider the vector space $\mathbb{F}_2^r$. Also we fix a nonzero vector $v_0 \in \mathbb{F}_2^r$. For each $i \in [n]$, pick a uniform random vector $v_i \in \mathbb{F}_2^r \setminus \{0\}$. Now each vertex of our hypercube layer $L_r(n)$ corresponds to a certain collection of vectors in $\mathbb{F}_2^r$ (of size r or $r-1$). Define the sets

$$B_r = \left\{ S \in \binom{[n]}{r} \;\middle|\; \big\{v_i : i \in S\big\} \text{ form a basis of } \mathbb{F}_2^r \right\},$$

$$B_{r-1} = \left\{ S \in \binom{[n]}{r-1} \;\middle|\; \big\{v_i : i \in S\big\} \cup \{v_0\} \text{ form a basis of } \mathbb{F}_2^r \right\},$$

and consider the induced subgraph G_r on $B_{r-1} \cup B_r$.

We will show that this (random) graph G_r is C_6-free (deterministically) and has a large expected number of edges, and therefore has the desired properties with positive probability.

Claim 2.1. *G_r is C_6-free.*

Proof. Suppose there is a copy of C_6 in G_r. There are exactly three different coordinates flipped by its edges, so it must form the middle layer of some 3-dimensional subcube of Q_n. More precisely, by permuting the coordinates, we may assume that this copy of C_6 is exactly of the form $L_r(n)\big[A_1 \cup A_2\big]$, where

$$A_1 = \big\{\{i\} \cup I \;\mid\; 1 \leqslant i \leqslant 3\big\} \qquad \text{and} \qquad A_2 = \big\{\{i, j\} \cup I \;\mid\; 1 \leqslant i < j \leqslant 3\big\}$$

for some $I \subset [n] \setminus \{1, 2, 3\}$ with $|I| = r - 2$. Also let $U = \{v_i \mid i \in I\}$ be the collection of vectors corresponding to the coordinates in I.

Observe that, since $\{1,2\} \cup I \in A_2 \subset B_r$, the collection of vectors $\{v_1, v_2\} \cup U$ is a basis of $\mathbb{F}_2^r$, so, in particular, the vectors in U are linearly independent. Therefore, after taking the quotient of $\mathbb{F}_2^r$ by $\langle U \rangle$ we obtain a vector space V isomorphic to $\mathbb{F}_2^2$.

Define $x_0, x_1, x_2, x_3 \in V$ to be the images of v_0, v_1, v_2, v_3, respectively, under the quotient map. Since $A_2 \subset B_r$ we have that

$$\{v_i, v_j\} \cup U \text{ form a basis of } \mathbb{F}_2^r \text{ for any } 1 \leqslant i < j \leqslant 3,$$

i.e.,

$$\{x_i, x_j\} \text{ form a basis of } V \text{ for any } 1 \leqslant i < j \leqslant 3.$$

Similarly, from $A_1 \subset B_{r-1}$, by the same argument as above, we obtain

$$\{x_0, x_i\} \text{ form a basis of } V \text{ for any } 1 \leqslant i \leqslant 3.$$

In particular, this implies that $x_i \neq 0$ for each $i \in \{0,1,2,3\}$ and that $x_i \neq x_j$ for any $i \neq j$. But $|V \setminus \{0\}| = 3$, which yields a contradiction. □

Claim 2.2. $\mathbb{E}\big[e(G_r)\big] > \dfrac{c}{2} \cdot e\big(L_r(n)\big)$.

Proof. Consider an edge of $L_r(n)$ connecting two sets $x = \{j_1, \ldots, j_{r-1}\}$ and $y = \{j_1, \ldots, j_r\}$. Define the vector spaces

$$V_k = \langle v_0, v_{j_1}, \ldots, v_{j_k} \rangle \quad \text{for } 0 \leqslant k \leqslant r-1, \text{ and} \quad V_r = \langle v_{j_1}, \ldots, v_{j_r} \rangle.$$

Note that

$$\mathbb{P}\big(xy \in G_r\big) = \mathbb{P}\big(x \in B_{r-1} \text{ and } y \in B_r\big) = \mathbb{P}\big(\dim V_{r-1} = \dim V_r = r\big). \tag{1}$$

Observe that, if the vectors $v_0, v_{j_1}, \ldots, v_{j_{k-1}}$ (for $1 \leqslant k \leqslant r-1$) are fixed and linearly independent, then there are exactly $2^r - 2^k$ admissible choices for the next vector v_{j_k}. Similarly, if the vectors $v_{j_1}, \ldots, v_{j_{r-1}}$ are fixed and linearly independent, then there are $2^r - 2^{r-1}$ admissible choices for the last vector v_{j_r}. Therefore, by (1), we have

$$\begin{aligned}\mathbb{P}\big(xy \in G_r\big) &= \left(\prod_{k=1}^{r-1} \mathbb{P}\big(v_{j_k} \notin V_{k-1} \mid \dim V_{k-1} = k\big)\right) \cdot \mathbb{P}\big(v_{j_r} \notin \langle v_{j_1}, \ldots, v_{j_{r-1}} \rangle \mid \dim V_{r-1} = r\big) \\ &= \left(\prod_{k=1}^{r-1} \frac{2^r - 2^k}{2^r - 1}\right) \cdot \frac{2^r - 2^{r-1}}{2^r - 1} > \frac{1}{2} \prod_{k=1}^{\infty} \left(1 - \frac{1}{2^k}\right) = \frac{c}{2}.\end{aligned}$$

The claim now follows from linearity of expectation. □

As observed above, it follows from Claims 2.1 and 2.2 that there exists a choice of the vectors $v_1, \ldots, v_n$ such that

$$e(G_r) > \frac{c}{2} \cdot e\big(L_r(n)\big)$$

and G_r is C_6-free, as required.

□

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