Covering the edges of a graph with perfect matchings

Olha Silina*

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Abstract

An r-graph is an r-regular graph with no odd cut of size less than r. A well-celebrated result due to Lovász says that for such graphs the linear system $Ax = \mathbb{1}$ has a solution in $\mathbb{Z}/2$, where A is the $0,1$ edge to perfect matching incidence matrix. Note that we allow x to have negative entries. In this paper, we present an improved version of Lovász's result, proving that, in fact, there is a solution x with all entries being either integer or $+1/2$ and corresponding to a linearly independent set of perfect matchings. Moreover, the total number of $+1/2$'s is at most $6k$, where k is the number of Petersen bricks in the tight cut decomposition of the graph.

1 Introduction

First, we introduce some terminology.

Def 1 *A* ***cut*** *in the graph* $G = (V, E)$ *is the set of edges with exactly one endpoint in* S *for some set* $S \subseteq V$. *In this case,* S *and* $V \backslash S$ *are called the* ***shores*** *of the cut.*

Def 2 *A cut is* ***odd*** *if both shores have odd cardinality. Notice that this requires* V *to be even cardinality.*

Def 3 *A set of edges* $M \subseteq E$ *is a* ***perfect matching*** *if every vertex is incident with exactly one edge in* M.

Def 4 *A connected nontrivial graph is* ***matching-covered*** *if every edge belongs to a perfect matching. A graph is an* r***-graph*** *if every vertex has degree* r *and all odd cuts have size at least* r.

*The author is a graduate student at the Department of Mathematical Sciences at Carnegie Mellon University. Email: `osilina@andrew.cmu.edu`

Extensive work has been done around matching-covered graphs, starting with [3], [4], [5]. The notion of r-graphs was first introduced by Seymour, who also showed that an r-graph is always matching-covered [9]. One question that naturally arises here is whether an r-graph contains r disjoint perfect matchings, or alternatively, can the edges of an r-graph be covered by r perfect matchings. One of the central results in this area is due to Lovász, who got a characterization for the lattice $\mathcal{L} = \{\sum \alpha_i A_i : \alpha_i \in \mathbb{Z}\}$ generated by the perfect matchings A_i of a matching-covered graph G using dual lattice theory [6]. It follows from the characterization above that for an r-graph, the vector $2 \cdot \mathbb{1}$ belongs to $\mathcal{L}$. Hence, $\mathbb{1}$ can be obtained as a half-integral (but not necessarily non-negative) combination of the perfect matchings of G. In this paper, we strengthen this result by proving the existence of a half-integral solution in which there are only a few fractional coefficients, all equal to exactly $+1/2$.

More formally, given an r-graph $G = (V, E)$ with n vertices and m edges (notice $2m = nr$), we want to study the set of solutions to $Ax = \mathbb{1}$, where A is a $\{0, 1\}$ incidence matrix with rows corresponding to the edges of G and columns corresponding to the perfect matchings in G (we will adopt this notation throughout the paper). We consider graphs with no loops, but parallel edges are allowed.

We prove the following:

Theorem 1 (Main theorem) *Let G be an r-graph with m edges and n vertices and let A be its edge to perfect matching incidence matrix. Then there is a solution x^* to $Ax = \mathbb{1}$ satisfying the following conditions:*

1. *x^* has at most $m - n + 1$ non-zero entries, which correspond to a linearly independent set of perfect matchings of G;*
2. *all non-integral entries of x^* are equal to $+1/2$, and the total number of such entries is at most $6p$, where p is the number of Petersen bricks of G.*

The notions of a brick and the Petersen brick are defined later, but for a reference, p is bounded from above by the total number of vertices, n.

2 Tight cut decomposition

Tight cut decomposition plays an essential role in many results regarding perfect matchings.

Def 5 *A **tight cut** is an odd cut $C (\subseteq E)$ such that every perfect matching intersects C in exactly one edge. A cut is **trivial** if one of its shores is a single vertex. Otherwise, it is **non-trivial**.*

One can notice that all trivial cuts are tight since every perfect matching covers any vertex once. Let us prove the following statement:

Lemma 1 *In a matching-covered graph G with $Ax = \mathbb{1}$ feasible, all tight cuts have the same size. In particular, G is regular.*

Proof: Consider any tight cut C and its characteristic vector $\mathbb{1}_C$. Then $A_i^T\mathbb{1}_C = 1$ for all columns A_i of the matrix A because every perfect matching intersects the tight cut at exactly one edge. Moreover, $\mathbb{1}^T\mathbb{1}_C = |C|$. Thus, multiplying both sides of $Ax = \mathbb{1}$ by $\mathbb{1}_C$ we get $\sum_i (A_i^T\mathbb{1}_C)x_i = \sum_i x_i = |C|$. Notice that the left-hand side of the equation $\sum_i x_i = |C|$ does not depend on the choice of C. □

Further, we define bricks and braces.

Def 6 *A graph G is a **brick** if it is 3-connected and for every pair x, y of vertices, $G - x - y$ has a perfect matching. A bipartite graph G with bipartition U, W with $|U| = |W|$ is a **brace** if each subset $X \subset U$ with $0 < |X| < |U| - 1$ has at least $|X| + 2$ neighbors in W.*

The following facts are due to Lovász [6]:

Theorem 2
1. *The result of contracting one shore of a tight cut of a matching-covered graph is a matching-covered graph.*
2. *A matching-covered graph has no non-trivial tight cuts if and only if it is either a brick or a brace.*

This suggests the following decomposition process on an r-graph: find a tight cut C of G, consider two graphs obtained from G by contracting one of the shores of C, call them G_1 and G_2, and continue the decomposition for them. We stop when a graph has no non-trivial tight cuts i.e. when it is either a brick or a brace. Notice that the property of being an r-graph is also preserved by the contraction: any odd cut of G_1 corresponds to the same odd cut of G and has the same size and all degrees are preserved because any tight cut must have size exactly r. Let us remark that all graphs are allowed to contain parallel edges.

In the next sections we discuss how to use this decomposition to inductively construct solutions to $Ax = \mathbb{1}$ with the desired properties.

3 Solution construction

A classical approach is described in Murty [7] and it can be summarized as follows.

Let G be an r-graph and C be its tight cut. Consider the graphs G_1 and G_2 obtained from G by contracting one shore of C and let E_1, E_2 be the edges of G_1, G_2, respectively. Furthermore, let A_1 and A_2 be the edge to perfect matching incidence matrices for G_1 and G_2, respectively. Given y, t satisfying $A_1 y = \mathbb{1}$ and $A_2 t = \mathbb{1}$, we will construct a vector x satisfying $Ax = \mathbb{1}$.

Fix an edge $e \in C$ and let $\{M_i^e\}_{i \in I^e}$ be the set of perfect matchings of G_1 using e, and let $\{N_j^e\}_{j \in J^e}$ be the set of perfect matchings of G_2 using e. Clearly, all of these matchings do not use any other edges of C. It is easy to see that $M_i^e \cup N_j^e =: K_{ij}^e$ is a perfect matching in G for all indices i, j. Define $x_{ij}^e := y_i^e t_j^e$, and let us prove that $Ax = \mathbb{1}$, which is equivalent to

$$\sum_{e \in C} \sum_{i \in I^e, j \in J^e} x_{ij}^e K_{ij} = \mathbb{1}.$$

It suffices to prove this for projections on E_1, E_2, and on C. Every edge $e \in C$ appears with coefficients $y_i t_j$, so the total sum of the entries corresponding to e in Ax is

$$\sum_{i \in I^e, j \in J^e} y_i t_j = (\sum_{i \in I^e} y_i)(\sum_{j \in J^e} t_j) = 1,$$

where the last equality follows from $A_1 y = \mathbb{1}$ and $A_2 t = \mathbb{1}$.

Now, projecting Ax on $E_1 \backslash C$:

$$\sum_{e \in C} x_{ij}^e K_{ij}^e = \sum_{e \in C} (\sum_{i \in I^e} y_i^e M_i^e \sum_{j \in J^e} t_j^e) = \sum_{e \in C} \sum_{i \in I^e} y_i^e M_i^e = \mathbb{1},$$

and similarly for $E_2 \backslash C$.

There are other ways of assigning coefficients x_{ij}^e to K_{ij}^e, which we will see in the next section. As long as for each matching M_i^e we ensure $\sum_{j \in J} x_{ij}^e = y_i$ and similarly for N_j^e, the proof of feasibility stays the same.

4 Main fact

In this section, we give a different way of combining two solutions for tight cut contractions, which is a crucial ingredient of 1. A similar approach appears in the proof of the matching polytope theorem in [8].

4.1 A better way to combine solutions

Lemma 2 *There is a solution x^* to $Ax = \mathbb{1}$ with all entries being either integral, or equal to $+1/2$.*

Proof: We prove this by induction on the number of bricks and braces in a tight cut decomposition of the graph. We will prove the base case for bricks and braces in the next section, so for now assume it is true.

For the inductive step, consider any graph G with a tight cut C and consider any edge $e \in C$. Keeping the notations of A_1, A_2, y, t from the previous section, let further S_1 and S_2 be the indices at which y and t have negative integers as entries, P_1 and P_2 be the indices with positive integers, and H_1 and H_2 be the indices with $+1/2$. From $A_1 y = \mathbb{1}$ and $A_2 t = \mathbb{1}$ it follows that

$$\sum_{i \in P_1} y_i + \sum_{i \in H_1} 1/2 - \sum_{i \in S_1} |y_i| = 1 = \sum_{i \in P_2} t_i + \sum_{i \in H_2} 1/2 - \sum_{i \in S_2} |t_i|.$$

Say $-L_1 := \sum_{i \in S_1} y_i$ and $-L_2 := \sum_{i \in S_2} t_i$ (so that both L_1, L_2 are nonnegative). If $L_1 \neq L_2$, then without loss of generality $L_1 < L_2$ and let us duplicate one matching of G_1 from P_1 with the largest coefficient s and split it as $s = (s - L_1 + L_2) + (L_1 - L_2)$, where $s - L_1 + L_2$ will now correspond to the nonnegative entries P_1 and $L_1 - L_2$ will be negative. This way, the sums of negative coefficients are equal and $\sum_{i \in P_1} y_i + \sum_{i \in H_1} 1/2 = \sum_{i \in P_2} t_i + \sum_{i \in H_2} 1/2$.

Now, we will show how to pair positive and negative coefficients separately. Suppose we have two sets $a_1, \dots, a_n$ and $b_1, \dots, b_m$ (the positive (negative) entries of y, t) satisfying $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j =: d$. We may assume both sequences are non-decreasing. Mark the points of the form $a_1 + a_2 + \ldots + a_i$ and $b_1 + b_2 + \ldots + b_j$ (a total of $n + m$ points including 0 and d) on the line segment $[0, d]$. For every line segment (w, w') of this partition not containing any other partition points, let i be the smallest index with $a_1 + \ldots + a_i \geq w'$ and similarly let j be the smallest index for which $b_1 + \ldots + b_j \geq w'$. Then let $w' - w$ be the $i + j - 1$-th entry of the new vector c (notice that this is well-defined since each subsequent point will have either i or j index increased). This c has all entries integral, and its $i + j - 1$'st entry will correspond to the union of the perfect matchings associated with a_i and b_j.

We make a small adjustment if there are entries equal to 1/2 (by the hypothesis, the only non-integral entries are 1/2). By construction, the fractional coefficients of a and b are listed first. Assuming without loss of generality that a has fewer fractional terms, each 1/2 in a's is paired with a 1/2 in b's. Until the sequence of b's runs out of 1/2's, all entries of c are 1/2 and the remaining entries of c are determined the same way as the all-integral case. Hence, c will have all entries either integral, or equal to 1/2. □

4.2 Analysis

In fact, the above algorithm of combining two solutions preserves several other properties. Let us prove the following statement.

Proposition 1 *Suppose the method described in Lemma 2 results in a solution x^* after combining two solutions y, t of the two contractions. Then, the following properties hold:*

i. the support size of x^* satisfies $supp(x^*) \leq supp(y) + supp(t)$;

ii. the largest entry of x^* satisfies $\|x^*\|_\infty \leq \max(\|y\|_\infty, \|t\|_\infty)$;

iii. the total number of $+1/2$'s in x^* is at most number of $+1/2$'s in y and t combined;

iv. if both y and t only used linearly independent perfect mathcings, then so does x^*.

For convenience, let us say a perfect matching is *used* in x^* if the corresponding coefficient of x^* is non-zero.

Proof: Using the same notation as in the proof of Lemma 2, for every $e \in C$ the total number of positive terms in y (t) is $H_{1(2)} + P_{1(2)}$ and negative is $S_{1(2)}$. In the proposed algorithm to match a_i and b_j, the total number of entries of c is $\leq m+n-1$. Since we might have added an extra term to make $L_1 = L_2$, we create a total of at most $\sum_{i=1,2}(H_i + P_i + S_i)$ nonzero entries of x^* corresponding to e. Summing this over all $e \in C$ gives (i). Similarly, each entry c_{i+j-1} is less than both a_i and b_j, thus all entries of c are at most $\min(\max(a_i), \max(b_j))$. Finally, notice that at most one of a_i, b_j was artificially augmented by making $L_1 = L_2$, so at least one of $\max(a_i), \max(b_j)$ is $\leq \max(\|y\|_\infty, \|t\|_\infty)$, implying (ii). (iii) holds since for each $e \in C$ the construction yields at most $\max(|H_1|, |H_2|) \leq |H_1| + |H_2|$ terms equal to $+1/2$. Summing this up over all e we get the desired result.

Suppose (iv) does not hold, i.e. there is a linear combination $Aw = 0$ where w has the same support as x^*. Notice that any column of A is of the form $(M \mid \mathbb{1}_e \mid N)^T$, where $e = uv \in C$ and M, N are some perfect matchings in $G_1 - \{u, v\}$ and $G_2 - \{u, v\}$. Comparing projections on G_1 and G_2, we get that for every matching M of G_1 or G_2, the sum of entries of w that correspond to a column of A that uses M, is zero by assumption on y, t. Hence, it means that for any $e \in C$, the set of perfect matchings containing e has linear combination equal to 0. Now it suffices to prove that the perfect matchings used in x^* that correspond to the same $e \in C$ are linearly independent. Indeed, notice that each new matching (going over the matchings in order of the coefficients c) either uses a new matching of G_1 or of G_2, thus does not belong to the linear hull of the previous matchings.

□

To complete the proof of 2, we must consider the base case. We will separately treat braces, Petersen bricks, and non-Petersen bricks. The braces resulting in the tight cut decomposition of an r-graphs must necessarily be bipartite and r-regular. One can check that all regular bipartite graphs satisfy Hall's condition, and hence always contain a perfect matching. Deleting the corresponding edges from the graph again gives a regular bipartite graph. Thus, repeating this process we obtain a union of disjoint perfect matchings that uses all edges (for a more detailed explanation, one can refer to [1], Section 5.2). We remark that these perfect matchings are disjoint, and hence linearly independent. This means that in fact we obtain a $\{0, 1\}$ solution with non-zero entries corresponding to linearly independent matchings, so it satisfies the conditions of the Theorem 1.

We will now proceed to the case of bricks, which requires some deeper analysis.

5 Petersen brick solutions

Following the proof of Lemma 2, we begin with an r-graph G and apply tight cut decomposition to it. Let B be any brick in the tight cut decomposition of G whose underlying graph is the Petersen graph. Notice that B might have parallel edges and it must be r-regular. For convenience, consider a simple graph B', isomorphic to the Petersen graph, whose edges are assigned weights equal to the corresponding number of parallel arcs in B.

First, let us see which weight assignments for the edges of B' are feasible. Let c_i be the weight of the edge i, then we are looking for the 16-tuples $(c_1, c_2, \ldots, c_{15}, r)$ of positive integers satisfying $deg(v) = r$ for every vertex v. This gives 10 constraints, which can be checked to be linearly independent. Thus, the set of solutions has dimension 6 (here, we drop the positive integer requirement). On the other hand, we can construct a dimension 6 set of solutions by taking all possible linear combinations of the 6 perfect matchings of the

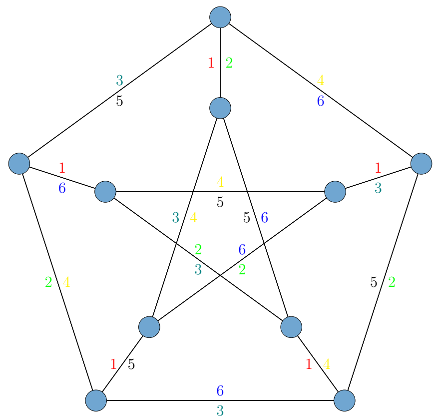

Figure 1: Edges of the Petersen graph labeled by the perfect matchings they belong to

Petersen graph. Therefore, any edge weight assignment c can be represented as $c = \sum_{i=1}^{6} \alpha_i M_i$ where M_i are the perfect matchings of the Petersen graph.

Lemma 3 *If $c \in \mathbb{Z}_{>0}$, then $\alpha_i \geq 0$ and either $\alpha_i \in \mathbb{Z}$ for all i, or $\alpha_i \in 1/2 + \mathbb{Z}$ for all i.*

Proof: Indeed, notice that the total weight of any edge is $\alpha_i + \alpha_j$ for some matchings M_i and M_j because every edge belongs to exactly two perfect matchings. Hence these indices satisfy $\alpha_i \equiv -\alpha_j \mod 1$. Moreover, any two perfect matchings intersect in one edge, meaning that for any two i, j there is an edge whose weight is exactly $\alpha_i + \alpha_j$, so the previous congruence holds for any pair $i \neq j$ of indices. This implies that for any distinct i, j, k we have $\alpha_i \equiv -\alpha_k \equiv \alpha_j$, so $\alpha_i \equiv \alpha_j$. This together with $\alpha_i \equiv -\alpha_j$ implies $2\alpha_i \equiv 0$, so either α_i is integer and hence all other α_j are integer, or $\alpha_i \equiv 1/2 \mod 1$.

Removing the edges corresponding to M_1 from B leaves us with a union of two vertex-disjoint 5-cycles, meaning that M_1 is an odd cut. Because B is an r-graph, every odd cut must have size at least r. The size of this cut is the sum of weights of the edges used in M_1, which is $5\alpha_1 + \alpha_2 + \ldots + \alpha_6$ since every edge in M_1 belongs to one other matching. Hence $r \leq 5\alpha_1 + \alpha_2 + \ldots + \alpha_6$. On the other hand, the degree of each vertex is exactly $\sum_i \alpha_i = r$, so

$$\alpha_1 + \alpha_2 + \ldots + \alpha_6 = r \leq 5\alpha_1 + \alpha_2 + \ldots + \alpha_6,$$

implying $\alpha_1 \geq 0$. Similarly, all other α_i are non-negative. □

This proves that the graph B' has its edge weights $c = \sum_{i=1}^{6} \alpha_i M_i$ for $\alpha_i \geq 0$ either all integer, or all half-integer. Now, we show how to return to the original brick B with parallel edges.

1. If all α_i are integer, write $\sum_{i=1}^{6} \alpha_i M_i$ as a sum of $\sum_{i=1}^{6} \alpha_i$ single term. Then, for each edge e of B', its weight $c(e)$ equals the total number of terms

using e. This means one can replace the instances of e with $e_1, \ldots, e_{c(e)}$, which are the parallel arc in B corresponding to e. Repeating this for every $e \in B'$, we obtain a set of perfect matchings that use each edge of B exactly once, as wanted.

2. If all α_i are half-integer. Let $\beta_i = \alpha_i - 1/2 \in \mathbb{Z}_{\geq 0}$. Cover an arbitrary subset of edges forming a Petersen graph by $\sum_{i=1}^{6} M_i$, and the remaining is reduced to the previous case.

Hence, we have a solution with at most 6 half-integer coefficients for any Petersen brick. In fact, there are either no fractional coefficients or there are exactly 6 of them. We will again remark that the perfect matchings used in the solution are linearly independent. To see this, notice that the six mathcings of the Petersen graph M_i are independent and by adding parallel arcs the set remains linearly independent.

6 Non-Petersen brick solutions

In the case of non-Petersen bricks, Carvalho, Lucchesi, and Murty [2] show the following:

Theorem 3 *For every non-Petersen brick G with m edges and n vertices, the dimension of its matching lattice is $m-n+1$ and it has an integral basis consisting of perfect matching vectors.*

Due to Lovász [6], $\mathbb{1}$ belongs to the matching lattice of G, it can be expressed as an integral combination of at most $m-n+1$ perfect matching vectors (which are also linearly independent). Moreover, the construction in [2] results in the basis of perfect matchings $M_1, M_2, \ldots, M_d$ such that each M_{i+1} contains an edge e_{i+1} not used by $M_1, \ldots, M_i$ for $1 \leq i \leq m+n-2$. Let us prove the following bound:

Lemma 4 *If x is a solution to $\sum_{i=1}^{d} M_i x_i = \mathbb{1}$, then $|x_{d-i}| \leq 2^i$.*

Proof: By induction on i. In the base case $i = 0$, project $\sum_1^d x_i M_i = \mathbb{1}$ onto e_d to get $x_d = 1$ since e_d is only used in the matching M_d. Now, assuming the statement for all previous i, projecting $\sum_1^d x_i M_i = \mathbb{1}$ onto the e_{d-i-1} coordinate, we get:

$$x_{d-i-1} = 1 - (x_{d-i} M_{d-i} + x_{d-i+1} M_{d-i+1} + \ldots + x_d M_d) e_{d-i-1}.$$

Recall that each of x_{d-j} for $j \leq i$ is bounded in absolute value by 2^j, and so are $x_{d-j} M_{d-j} e_{d-i-1}$. Also, $1 - x_d M_d e_{d-i-1} = 1 - M_d e_{d-i-1}$ is either 0 or 1, so it is bounded in absolute value by 1. Hence,

$$|x_{d-i-1}| \leq \sum_{j=1}^{i} 2^j + 1 = 2^{i+1},$$

as wanted. □

Combining the results from before, we get the following statement, which is a strengthening of Theorem 1:

Theorem 4 *Let G be an r-graph and let A be its edge to perfect matching incidence matrix. Then there is a solution x^* to $Ax = \mathbb{1}$ satisfying the following conditions:*

1. *x^* has at most $m - n + 1$ non-zero entries, which correspond to a linearly independent set of perfect matchings of G;*
2. *all non-integral entries of x^* are equal to $+1/2$, and the total number of such entries is at most $6p$, where p is the number of Petersen bricks of G;*
3. *the coefficients of x^* is are at most 2^d in absolute value where d is the largest dimension of the matching lattice (i.e. $m - n + 1$) over the non-Petersen bricks of G. If G has no non-Petersen bricks, then the largest entry of x^* in absolute value is at most 1.*

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