MONOCHROMATIC BOXES OF UNIT VOLUME

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Abstract. Erdős and Graham asked whether, for any coloring of the Euclidean plane $\mathbb{R}^2$ in finitely many colors, some color class contains the vertices of a rectangle of every given area. We give the negative answer to this question and its higher-dimensional generalization: there exists a finite coloring of the Euclidean space $\mathbb{R}^n$, $n \geqslant 2$, such that no color class contains the 2^n vertices of a rectangular box of volume 1. The present note is a very preliminary version of a longer treatise on similar problems.

1. Introduction

Systematic study of the Euclidean Ramsey theory was initiated by Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus [5, 6, 7] in the 1970s. Graham [9] answered positively a question of Gurevich by showing that for any finite coloring of the Euclidean plane some color class contains the vertices of a triangle of any given area. In fact, he could choose the triangles to be right-angled with axis-aligned legs. Erdős and Graham [3, p. 331] (in a paper which is also published as a chapter in the well-known problem book [4]), after mentioning Graham's result on triangles, posed a natural follow-up question:

Is this also true for rectangles?

The following precise wording of this problem is literally taken from Thomas Bloom's website *Erdős problems* [1].

Problem 1 ([1, #189]). *If $\mathbb{R}^2$ is finitely coloured then must there exist some colour class which contains the vertices of a rectangle of every area?*

To the best of author's knowledge, Problem 1 has not been addressed in the literature so far. It was explicitly mentioned as being open in *Mathematical Reviews* MR0558877 by Karsten Steffens and MR2106573 by Sheila Oates-Williams. Finally, Problem 1 (for rectangles, but also for a few other configurations) was stated as an open problem in the 2015 edition of the book *Rudiments of Ramsey theory* by Graham and Butler [8, p. 56].

Here we show that the question in Problem 1 has the negative answer.

Theorem 2. *It is possible to partition $\mathbb{R}^2$ into 25 color classes such that none of them contains the vertices of a rectangle of area 1.*

We have singled out the two-dimensional case in Theorem 2 above, because its proof is particularly elegant. However, with a bit more work one can show a natural generalization to higher dimensions.

Theorem 3. *For every integer $n \geqslant 2$ there exists a finite coloring of the Euclidean space $\mathbb{R}^n$ such that there is no rectangular box of volume 1 with all of its 2^n vertices colored the same.*

In the proofs of these theorems, i.e., for any coloring constructed in this paper, we will choose each color class to be a countable union of Jordan-measurable sets. For this reason our constructions can also disprove some density theorems that could potentially hold for subsets $A \subseteq \mathbb{R}^n$ of positive upper density; see Remark 5 for an example. Namely, in any partition of

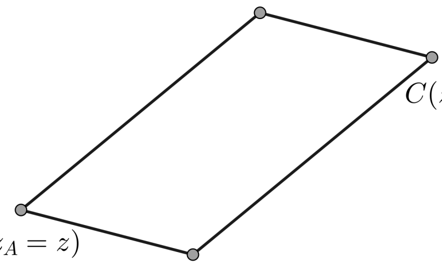

FIGURE 1. Coordinatization of a parallelogram.

$\mathbb{R}^n$ into finitely many Lebesgue-measurable color classes, at least one of the classes needs to have strictly positive upper density.

The purpose of this note is to motivate further questions related to Problem 1, which actually have positive answers and their proofs require more substantial tools. Some of these questions will be addressed by the author in the next version of this manuscript, which is still in preparation.

2. Proof of Theorem 2

We are about to give a coloring of $\mathbb{R}^2$ that uses 25 colors and has a slightly stronger property: no color class will contain the vertices of a parallelogram such that the product of lengths of its two consecutive sides equals 1. For rectangles this clearly specializes to the property of their area being equal to 1.

Proof of Theorem 2. Let us place a (possibly degenerate) parallelogram $\mathcal{P} = ABCD$ in the complex plane, so that its vertices A, B, C, D are respectively coordinatized by the complex numbers z_A, z_B, z_C, z_D as in Figure 1. Consider a complex quantity $\mathscr{I}(\mathcal{P})$ defined as

$$\mathscr{I}(\mathcal{P}) := z_A^2 - z_B^2 + z_C^2 - z_D^2. \tag{2.1}$$

In this definition we specify the vertex A to be the one with the smallest coordinate z_A in the lexicographic ordering of $\mathbb{C} \equiv \mathbb{R}^2$. Otherwise, $\mathscr{I}(\mathcal{P})$ would have only been determined up to multiplication by ± 1.

There exist $u, v, z \in \mathbb{C}$ such that the vertices of $\mathcal{P}$ have complex coordinates

$$z_A = z, \quad z_B = z + u, \quad z_C = z + u + v, \quad z_D = z + v;$$

see Figure 1 again. The quantity $\mathscr{I}(\mathcal{P})$ now simplifies as

$$\mathscr{I}(\mathcal{P}) = z^2 - (z+u)^2 + (z+u+v)^2 - (z+v)^2 = 2uv.$$

Consecutive side lenghts of $\mathcal{P}$ are $|u|$ and $|v|$, so we have

$$|\mathscr{I}(\mathcal{P})| = 2$$

whenever their product equals 1. As we have already mentioned, this holds in particular if $\mathcal{P}$ is a rectangle of area 1. Therefore, it remains to find a coloring of $\mathbb{C}$ such that, if all vertices of $\mathcal{P}$ are assigned the same color, then the complex number $\mathscr{I}(\mathcal{P})$ does not lie on the circle

$$\{w \in \mathbb{C} : |w| = 2\}. \tag{2.2}$$

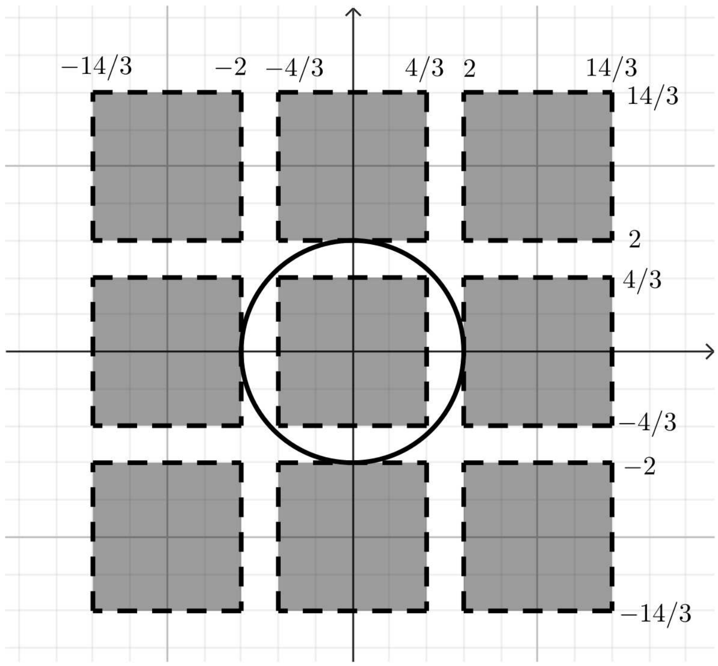

FIGURE 2. The circle misses the squares.

For each pair $(j,k) \in \{0,1,2,3,4\}^2$ define a color class $\mathscr{C}_{j,k}$ as

$$\mathscr{C}_{j,k} := \Big\{ z \in \mathbb{C} \,:\, z^2 \in \frac{10}{3}\Big(\mathbb{Z} + \mathbb{i}\mathbb{Z} + \frac{j+\mathbb{i}k}{5} + \Big[0,\frac{1}{5}\Big) + \mathbb{i}\Big[0,\frac{1}{5}\Big)\Big)\Big\}.$$

If the four vertices of $\mathcal{P} = ABCD$ belonged to the same color class, then, by the definition (2.1), we would clearly have

$$\mathscr{I}(\mathcal{P}) \in \frac{10}{3}\Big(\mathbb{Z} + \mathbb{i}\mathbb{Z} + \Big(-\frac{2}{5},\frac{2}{5}\Big) + \mathbb{i}\Big(-\frac{2}{5},\frac{2}{5}\Big)\Big).$$

The above set does not intersect the circle (2.2); see Figure 2. Indeed, the central square lies fully inside (2.2) because of $4\sqrt{2}/3 < 2$, while all remaining open squares clearly belong to its exterior. □

We can say that the above solution uses the help of the invariant quantity $|\mathscr{I}(\mathcal{P})|$ assigned to rectangles of area 1. It does not generalize to all higher dimensions, since it uses multiplication of complex numbers, so we will resort to an "almost invariant" quantity in the next section.

Construction of the above coloring could be thought of as a complex modification of the approach of Erdős at al. [5, §3] who used $|z|^2$ in place of z^2.

Let us illustrate the coloring constructed in the previous proof. Boundaries of the color classes are given in the (x,y)-coordinate system by the equations

$$x^2 - y^2 = \frac{2a}{3} \quad \text{and} \quad xy = \frac{b}{3}$$

for arbitrary $a,b \in \mathbb{Z}$. These are two mutually orthogonal families of hyperbolas (including degenerate ones for $a=0$ or $b=0$), depicted in Figure 3.

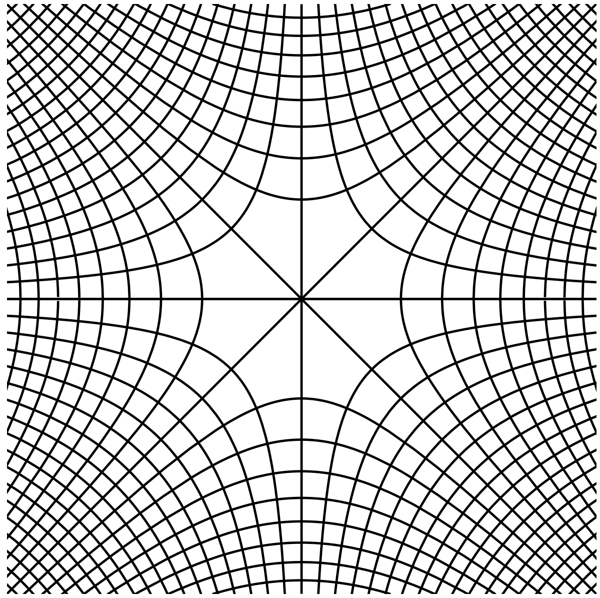

FIGURE 3. Boundaries of color classes $\mathscr{C}_{j,k}$.

3. Proof of Theorem 3

The idea is based on the following simple observation. Let us first take an axes-aligned box in $\mathbb{R}^n$ with edge lengths $a_1, a_2, \ldots, a_n \in (0,\infty)$. Its vertices can be enumerated by subsets T of $\{1,2,\ldots,n\}$ as

$$\Big(\mathbf{q} + \sum_{j\in T} a_j \mathbf{e}_j : T \subseteq \{1,2,\ldots,n\}\Big), \tag{3.1}$$

where $\mathbf{q} = (q_1,\ldots,q_n) \in \mathbb{R}^n$ is some point. Let us compute the alternating sum of the product of coordinates of the box vertices,

$$\sum_{T\subseteq\{1,2,\ldots,n\}} (-1)^{n-|T|} \Big(\prod_{j\in T^c} q_j\Big)\Big(\prod_{j\in T}(q_j + a_j)\Big) = \prod_{j=1}^{n}(-q_j + q_j + a_j) = a_1 a_2 \cdots a_n,$$

and notice that we have obtained precisely the box volume.

The crucial part of the proof will be to show that the same quantity is almost invariant for slightly rotated rectangular boxes, where the "slightness" can be prescribed uniformly over all box eccentricities. Note that this property is not quite obvious and it is sensitive to the pattern shape, as the uniformity will fail already for parallelograms in $\mathbb{R}^2$. After we construct a coloring that prohibits those slightly tilted boxes, in the last step we will rotate it by finitely many matrices $U_1,\ldots,U_m \in \mathrm{SO}(n)$, thanks to compactness of the rotation group.

Before the proof we need a simple identity.

Lemma 4. *The identity*

$$\sum_{T\subseteq\{1,2,\ldots,n\}} (-1)^{n-|T|} \prod_{k=1}^{n}\Big(p_k + \sum_{j\in T} v_{j,k}\Big) = \sum_{\sigma\in S_n} v_{1,\sigma(1)} v_{2,\sigma(2)} \cdots v_{n,\sigma(n)} \tag{3.2}$$

holds for real numbers $(p_k)_{1\leqslant k\leqslant n}$ and $(v_{j,k})_{1\leqslant j,k\leqslant n}$.

It is peculiar to notice that the right hand side RHS of (3.2) is the permanent of the matrix $(v_{j,k})_{j,k}$.

Proof of Lemma 4. We will prove the identity by induction on n. The basis case $n = 1$ is trivial as then the identity reads

$$-p_1 + (p_1 + v_{1,1}) = v_{1,1}.$$

Take a positive integer $n \geqslant 2$. Observe that the left hand side LHS of (3.2) is a homogeneous polynomial of degree n in $n^2 + n$ variables p_k and $v_{j,k}$, but the degree of each of the variables in it is at most 1. Differentiating it with respect to the variable $v_{n,l}$ for some $l \in \{1, \dots, n\}$ we obtain

$$\begin{aligned}\frac{\partial}{\partial v_{n,l}}\mathrm{LHS} &= \sum_{T\subseteq\{1,\dots,n-1\}} (-1)^{n-1-|T|} \prod_{\substack{1\leqslant k\leqslant n\\ k\neq l}} \Big(p_k + v_{n,k} + \sum_{j\in T} v_{j,k}\Big)\\ &= \sum_{\substack{\sigma\in S_n\\ \sigma(n)=l}} v_{1,\sigma(1)} v_{2,\sigma(2)} \cdots v_{n-1,\sigma(n-1)},\end{aligned}$$

where in the last equality we applied the inductions hypothesis to a smaller collection of numbers

$$(p_k + v_{n,k})_{1\leqslant k\leqslant n, k\neq l}, \quad (v_{j,k})_{1\leqslant j\leqslant n-1, 1\leqslant k\leqslant n, k\neq l}$$

and renamed the indices. Therefore

$$\mathrm{LHS} - \sum_{l=1}^{n} v_{n,l} \frac{\partial}{\partial v_{n,l}}\mathrm{LHS} = \mathrm{LHS} - \mathrm{RHS}$$

and this quantity can be evaluated by plugging $v_{n,1} = \cdots = v_{n,n} = 0$ into the left hand side:

$$\sum_{T\subseteq\{1,\dots,n\}} (-1)^{n-|T|} \prod_{k=1}^{n} \Big(p_k + \sum_{\substack{1\leqslant j\leqslant n-1\\ j\in T}} v_{j,k}\Big) = 0.$$

The last sum equals 0 because its terms can be paired to cancel each other: adding/subtracting the element n to/from the set T gives exactly the same term, only with the opposite sign $(-1)^{n-|T|}$. □

Now we turn to the proof of the announced result.

Proof of Theorem 3. To each rectangular box $\mathcal{R}$ in $\mathbb{R}^n$ we assign a real quantity $\mathscr{J}(\mathcal{R})$ defined as

$$\mathscr{J}(\mathcal{R}) := \sum_{\mathbf{x}=(x_1,\dots,x_n) \text{ is a vertex of } \mathcal{R}} (-1)^{n-\mathrm{par}(\mathbf{x})}\, x_1 \cdots x_n. \tag{3.3}$$

Here $\mathrm{par}(\mathbf{x})$ denotes the *parity* of the vertex $\mathbf{x}$, which is computed as follows. Choose the base vertex of $\mathcal{R}$ to be the one with the smallest coordinate representation in the lexicographic ordering of $\mathbb{R}^n$. The parity of any vertex of $\mathcal{R}$ is defined to be the parity of its distance from the base vertex in the 1-skeleton graph of $\mathcal{R}$. This number is either 0 or 1 and it changes as we move from a vertex to its neighbor along the 1-edge of $\mathcal{R}$. Clearly, the expression (3.3) has 2^n terms, half of them get the $+$ sign and half of them come with the $-$ sign; see the illustration of these signs in Figure 4.

Every rectangular box $\mathcal{R}$ can be obtained by rotating an axes-aligned box $\mathcal{R}_0$ about the origin, where the rotation is given by a special orthogonal transformation, $U \in \mathrm{SO}(n)$:

$$\mathcal{R} = U\mathcal{R}_0. \tag{3.4}$$

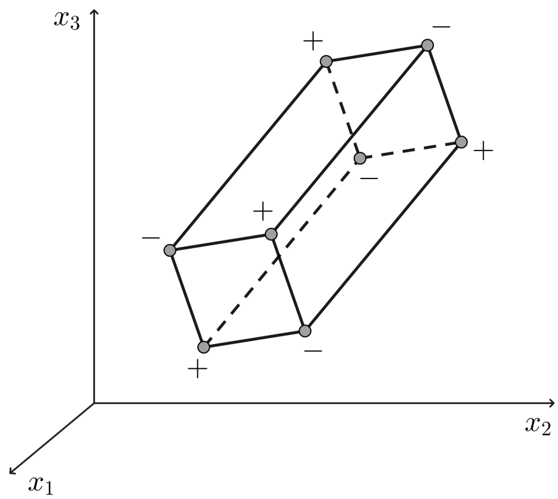

FIGURE 4. Signs attached to box vertices.

Suppose that the vertices of $\mathcal{R}_0$ are (3.1), while the vertices of $\mathcal{R}$ are given by

$$\Big(\mathbf{p} + \sum_{j\in T} \mathbf{v}_j : T \subseteq \{1, 2, \ldots, n\}\Big),$$

where $\mathbf{p} = U\mathbf{q}$ and $\mathbf{v}_j = a_j U\mathbf{e}_j$ for each index $1 \leqslant j \leqslant n$. Also, write coordinate-wise:

$$\mathbf{p} = (p_k)_{1\leqslant k\leqslant n}, \quad \mathbf{v}_j = (v_{j,k})_{1\leqslant k\leqslant n}$$

and note that Lemma 4 gives

$$\mathscr{J}(\mathcal{R}) = \pm \sum_{\sigma\in S_n} v_{1,\sigma(1)} v_{2,\sigma(2)} \cdots v_{n,\sigma(n)}. \tag{3.5}$$

First, suppose that the rotation U satisfies $\|U - I\|_{\mathrm{op}} < \varepsilon$, where

$$\varepsilon := \frac{1}{2^{n+2} n!}.$$

In particular,

$$\Big|\frac{1}{a_j}\mathbf{v}_j - \mathbf{e}_j\Big| = |(U - I)\mathbf{e}_j| < \varepsilon,$$

so that

$$|v_{j,k} - a_j \delta_{j,k}| < \varepsilon a_j.$$

Consequently,

$$\begin{aligned}
&\Big| \sum_{\sigma\in S_n} v_{1,\sigma(1)} v_{2,\sigma(2)} \cdots v_{n,\sigma(n)} - a_1 a_2 \cdots a_n \Big| \\
&\leqslant |v_{1,1} v_{2,2} \cdots v_{n,n} - a_1 a_2 \cdots a_n| + \sum_{\substack{\sigma\in S_n \\ \sigma\neq \mathrm{id}}} |v_{1,\sigma(1)} v_{2,\sigma(2)} \cdots v_{n,\sigma(n)}| \\
&< n\varepsilon(1+\varepsilon)^{n-1} a_1 a_2 \cdots a_n + (n! - 1)\varepsilon(1+\varepsilon)^{n-1} a_1 a_2 \cdots a_n \\
&\leqslant 2^n n! \varepsilon a_1 a_2 \cdots a_n \leqslant \frac{1}{4} a_1 a_2 \cdots a_n.
\end{aligned} \tag{3.6}$$

Let us now additionally assume that $\mathcal{R}$ has volume 1. Then (3.5) combined with (3.6) and $a_1 a_2 \cdots a_n = 1$ gives

$$\mathscr{J}(\mathcal{R}) \in \Big(-\frac{5}{4}, -\frac{3}{4}\Big) \cup \Big(\frac{3}{4}, \frac{5}{4}\Big). \tag{3.7}$$

Partition $\mathbb{R}^n$ into the sets

$$\mathscr{S}_l := \Big\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 x_2 \cdots x_n \in \frac{3}{2}\Big(\mathbb{Z} + \Big[\frac{l}{3\cdot 2^n}, \frac{l+1}{3\cdot 2^n}\Big)\Big)\Big\}$$

for $0 \leqslant l \leqslant 3\cdot 2^n - 1$. We claim that the vertices of $\mathcal{R}$ cannot all belong to the same set $\mathscr{S}_l$. Namely, if they did, then the definition of $\mathscr{J}(\mathcal{R})$ would give

$$\mathscr{J}(\mathcal{R}) \in \frac{3}{2}\mathbb{Z} + \Big(-\frac{1}{4}, \frac{1}{4}\Big) = \cdots \cup \Big(-\frac{7}{4}, -\frac{5}{4}\Big) \cup \Big(-\frac{1}{4}, \frac{1}{4}\Big) \cup \Big(\frac{5}{4}, \frac{7}{4}\Big) \cup \cdots . \tag{3.8}$$

However, (3.7) and (3.8) together lead to a contradiction as the sets on their right hand sides are disjoint.

Finally, we handle completely arbitrary rectagular boxes $\mathcal{R}$ of unit volume. Consider an open neighborhood $\mathcal{O}$ of the identity I in the rotation group $\mathrm{SO}(n)$ defined as

$$\mathcal{O} := \{ V \in \mathrm{SO}(n) \,:\, \|V - I\|_{\mathrm{op}} < \varepsilon \}.$$

Then the family $\{U\mathcal{O} : U \in \mathrm{SO}(n)\}$ constitutes an open cover of the compact space $\mathrm{SO}(n)$, so it can be reduced to a finite subcover $\{U_1\mathcal{O}, U_2\mathcal{O}, \ldots, U_m\mathcal{O}\}$. The color classes of the desired coloring of $\mathbb{R}^n$ can now be defined as

$$\mathscr{C}_{l_1,l_2,\ldots,l_m} := (U_1\mathscr{S}_{l_1}) \cap (U_2\mathscr{S}_{l_2}) \cap \cdots \cap (U_m\mathscr{S}_{l_m}),$$

where $(l_1, l_2, \ldots, l_m)$ run over all m-tuples of elements from $\{0, 1, 2, \ldots, 3\cdot 2^n - 1\}$. Suppose that the vertices of $\mathcal{R}$ belong to the same color class $\mathscr{C}_{l_1,l_2,\ldots,l_m}$. Let $U \in \mathrm{SO}(n)$ be as in (3.4), but without any assumption on the norm of $U - I$. Take an index $i \in \{1, \ldots, m\}$ such that $U \in U_i\mathcal{O}$. Then the box

$$\mathcal{R}' := U_i^{-1}\mathcal{R}$$

satisfies

$$\mathcal{R}' = U_i^{-1}U\mathcal{R}_0, \quad \|U_i^{-1}U - I\|_{\mathrm{op}} < \varepsilon$$

and all of its vertices are in the set

$$U_i^{-1}\mathscr{C}_{l_1,l_2,\ldots,l_m} \subseteq U_i^{-1}U_i\mathscr{S}_{l_i} = \mathscr{S}_{l_i}.$$

This contradicts the previous part of the proof. □

The number of colors needed in the above proof grows superexponentially in n.

The trick of using compactness of the underlying transformation group (which is $\mathrm{SO}(n)$ in our case) is essentially due to Straus [11]. However, note that its applicability is far from automatic; see Remark 6.

4. Other configurations

Here we give a couple of simple observations on further applicability of constructions from Sections 2 and 3.

Remark 5. Let us conveniently work in the complex plane again. Take a positive integer $n \geqslant 2$, positive numbers $a_1, \ldots, a_n$, and complex numbers $z, u_1, \ldots, u_n$. If 2^n points

$$z + r_1u_1 + r_2u_2 + \cdots + r_nu_n \quad \text{for } (r_1, r_2, \ldots, r_n) \in \{0,1\}^n \tag{4.1}$$

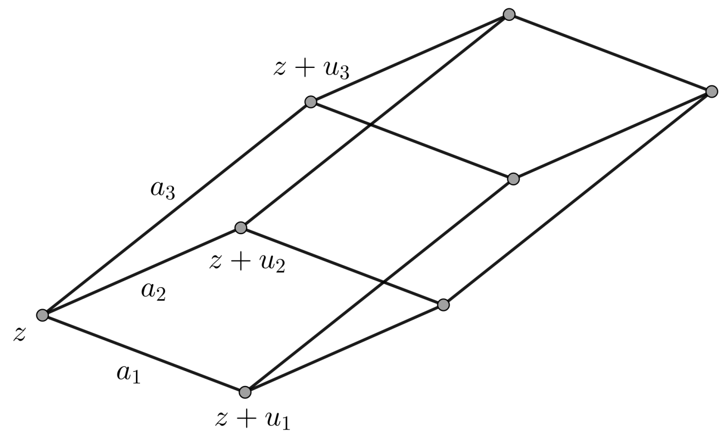

FIGURE 5. Embedding of a 1-skeleton of an n-box.

are mutually distinct and $|u_j| = a_j$ for $1 \leqslant j \leqslant n$, then we can say that (4.1) is an *embedding* in the plane of a 1-skeleton of an n-dimensional box

$$[0,a_1] \times [0,a_2] \times \cdots \times [0,a_n];$$

see Figure 5.

Predojević and the author [10] showed that a measurable subset $A \subseteq \mathbb{R}^2$ of positive upper density contains all sufficiently large dilates of a fixed 1-skeleton of an n-box; for instance we can take $a_1 = \cdots = a_n = \lambda$ for all sufficiently large numbers $\lambda \in (0,\infty)$. On the other extreme, a fixed 1-skeleton, such as the one with $a_1 = \cdots = a_n = 1$ need not be embeddable in A, simply because there is no reason why A should contain two point at distance 1 apart. Moreover, by a simple modification of the construction from Section 2 we can even find a measurable finite coloring of $\mathbb{R}^2$ such that no color class contains a 1-skeleton of an n-box satisfying $a_1 \cdots a_n = 1$. Namely, let us first observe that the identity (3.2) remains to hold for complex numbers p_k, $v_{j,k}$, simply by acknowledging the same proof given in Section 2. By taking $p_k = z$ and $v_{j,k} = u_j$ for each index k, we obtain a simpler identity

$$\sum_{T\subseteq\{1,2,\ldots,n\}} (-1)^{n-|T|} \Big(z + \sum_{j\in T} u_j \Big)^n = n!\, u_1 u_2 \cdots u_n$$

for $z, u_1, u_2, \ldots, u_n \in \mathbb{C}$. (Actually, its analogous inductive proof is even notationally slightly simpler.) It is now easy to color $\mathbb{C}$ appropriately, according to where z^n lies with respect to the Gaussian integers $\mathbb{Z} + \mathbb{i}\mathbb{Z}$.

Remark 6. In this paper we do not study parallelograms, which were also mentioned by Erdős and Graham [3, p. 331], but we comment on them rather briefly. The same quantity $\mathscr{J}$ from Section 3 is actually an invariant for parallelograms with one side parallel to the horizontal axis. Thus, a finite coloring of $\mathbb{R}^2$ can certainly avoid axis-parallel parallelograms. Using the same trick of compactness of the group of rotations SO(2) one can easily also construct a finite coloring of $\mathbb{R}^2$ that avoids monochromatic parallelograms with a fixed angle. For these reasons the author believes that the variant of Problem 1 for parallelograms is quite difficult. There is another reason adding to this belief. Erdős remarked [2, p. 324] (without a reference) that he and Mauldin constructed a set $S \subseteq \mathbb{R}^2$ of infinite measure that does not contains vertices of a parallelogram of area 1. Even if this has no implications to finite colorings of $\mathbb{R}^2$ or to positive density subsets of $\mathbb{R}^2$, it still hints that sets that avoid the vertices of parallelograms of unit area can be quite large.

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