A BOOK PROOF OF THE MIDDLE LEVELS THEOREM

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Abstract. We give a short constructive proof for the existence of a Hamilton cycle in the subgraph of the $(2n+1)$-dimensional hypercube induced by all vertices with exactly n or $n+1$ many 1s.

The *n-dimensional hypercube* Q_n is the graph that has as vertices all bitstrings of length n, and an edge between any two bitstrings that differ in a single bit. The *weight* of a vertex x of Q_n is the number of 1s in x. The *kth level* of Q_n is the set of vertices with weight k.

Theorem 1. *For all $n \geq 1$, the subgraph of Q_{2n+1} induced by levels n and $n+1$ has a Hamilton cycle.*

Theorem 1 solves the well-known *middle levels conjecture*, and it was first proved in [Müt16] (see this paper for a history of the problem). A shorter proof was presented in [GMN18] (12 pages). Here, we present a proof from 'the book'.

Proof. We write D_n for all Dyck words of length $2n$, i.e., bitstrings of length $2n$ with weight n in which every prefix contains at least as many 1s as 0s. We also define $D := \bigcup_{n\geq 0} D_n$. Any $x \in D_n$ can be decomposed uniquely as $x = 1u0v$ with $u, v \in D$, or alternatively as $x = u1v0$ with $u, v \in D$. Dyck words of length $2n$ can be identified by ordered rooted trees with n edges as follows; see Figure 1: Given $x = 1u0v \in D_n$, the corresponding rooted tree has a root whose leftmost child is the root of the subtree corresponding to u, and whose remaining children together with the root form the subtree corresponding to v. For any bitstring x, we write $\sigma^s(x)$ for the cyclic right rotation of x by s steps. We write A_n and B_n for the vertices of Q_{2n+1} in level n or $n+1$, respectively, and we define $M_n := Q_{2n+1}[A_n \cup B_n]$. For any $x \in D_n$, $b \in \{0,1\}$ and $s \in \{0,\ldots,2n\}$ we define $\langle x,b,s\rangle := \sigma^s(xb)$. Note that we have $A_n = \{\langle x,0,s\rangle \mid x \in D_n \wedge 0 \leq s \leq 2n\}$ and $B_n = \{\langle x,1,s\rangle \mid x \in D_n \wedge 0 \leq s \leq 2n\}$. Thus, we think of every vertex of M_n as a triple $\langle x,b,s\rangle$, i.e., an ordered rooted tree x with n edges referred to as the *nut*, a bit $b \in \{0,1\}$, and an integer $s \in \{0,\ldots,2n\}$ referred to as the *shift*.

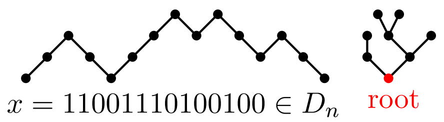

Figure 1. A Dyck word (left) and the corresponding ordered rooted tree (right).

The first step is to construct a cycle factor in the graph M_n. For this we define a mapping $f : A_n \cup B_n \to A_n \cup B_n$ as follows. Given an ordered rooted tree $x = 1u0v \in D_n$, a *tree rotation* yields the tree $\rho(x) := u1v0 \in D_n$; see Figure 2. We define $f(\langle x,0,s\rangle) := \langle \rho(x),1,s+1\rangle$ and $f(\langle x,1,s\rangle) := \langle x,0,s\rangle$. Note that f changes only a single bit, and that it is a bijection. Indeed, the inverse mapping is $f^{-1}(\langle x,0,s\rangle) = \langle x,1,s\rangle$ and $f^{-1}(\langle x,1,s\rangle) = \langle \rho^{-1}(x),0,s-1\rangle$. We also note that $f^2(\langle x,0,s\rangle) = \langle \rho(x),0,s+1\rangle \neq \langle x,0,s\rangle$. Consequently, for any vertex y of M_n, the sequence $C(y) := \{f^i(y) \mid i \geq 0\}$ is a cycle, and $F_n := \{C(y) \mid y \in A_n \cup B_n\}$ is a cycle factor in M_n.

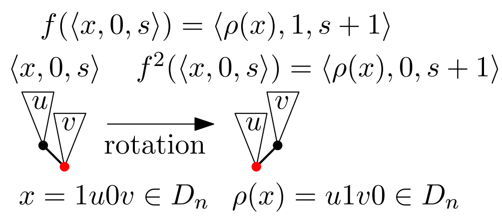

Figure 2. Cycles of F_n and tree rotation.

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As $f^2(\langle x,0,s\rangle) = \langle \rho(x),0,s+1\rangle$, moving two steps forward along a cycle of F_n applies a tree rotation to the nut, and increases the shift by $+1$. For an ordered rooted tree $x \in D_n$, let $r > 0$ be the minimum integer such that $x = \rho^r(x)$. Note that r divides $2n$ and therefore $\gcd(r, 2n+1) = 1$, implying that all shifts of the nut x are contained in the cycle $C(\langle x,0,0\rangle)$, i.e., $\langle x,0,s\rangle \in C(\langle x,0,0\rangle)$ for all $s \in \{0,\ldots,2n\}$. As a consequence, the cycles of F_n are in bijection with equivalence classes of ordered rooted trees with n edges under tree rotation, which are known as *plane trees*. In particular, the number of cycles of F_n is the number of plane trees with n edges (OEIS A002995).

The second step is to glue the cycles of the factor F_n to a single Hamilton cycle. We call an ordered rooted tree $x \in D_n$ *pullable* if $x = 110u0v$ for $u,v \in D$, and we define $p(x) := 101u0v \in D_n$. We refer to $p(x)$ as the tree obtained from x by a *pull* operation. In words, the leftmost child of the leftmost child of the root of x is a leaf, and the edge leading to this leaf is removed and reattached as the new leftmost child of the root in $p(x)$; see Figure 3. For any pullable tree $x = 110u0v \in D_n$ with $u,v \in D$, we define $y := \langle x,0,0\rangle = x0$ and $z := \langle p(x),0,0\rangle = p(x)0$, and we consider the 6-cycle $G(x) := (y, f(y), f^6(y), f^5(y), z, f(z)) = (110u0v0, 110u1v0, 100u1v0, 101u1v0, 101u0v0, 111u0v0)$, which has the edges $(y, f(y))$ and $(f^6(y), f^5(y))$ in common with the cycle $C(y)$, and the edge $(z, f(z))$ in common with the cycle $C(z)$. Consequently, if $C(y)$ and $C(z)$ are two distinct cycles, then the symmetric difference between the edge sets of $C(y)$, $C(z)$ and $G(x)$ is a single cycle on the same set of vertices, i.e., $G(x)$ can be used to glue the cycles $C(y)$ and $C(z)$ together. We claim that for any two distinct pullable trees x and x', the gluing cycles $G(x)$ and $G(x')$ are edge-disjoint. To see this, consider the shifts of the six vertices of $G(x)$ and $G(x')$, which are $(0,1,3,3,0,1)$ in this order. It follows that if $G(x)$ and $G(x')$ share an edge, then we must have $x = x'$, $p(x) = x'$, or $x = p(x')$. These cases are ruled out by the assumption that x and x' are distinct, the fact that $p(x) = 10\cdots$ and $x' = 11\cdots$ differ in the second bit, and that $x = 11\cdots$ and $p(x') = 10\cdots$ differ in the second bit, respectively.

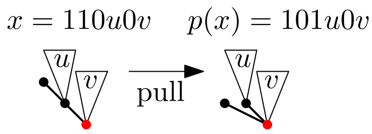

FIGURE 3. Pullable tree and pull operation.

To complete the proof, it remains to show that the cycles of the factor F_n can be glued to a single cycle via gluing cycles $G(x)$ for a suitable set of pullable trees $x \in D_n$. As all gluing cycles are edge-disjoint, none of the gluing operations interfere with each other. Using the interpretation of the cycles of F_n as equivalence classes of ordered rooted trees under tree rotation, it suffices to prove that every cycle can be glued to the cycle that corresponds to the star with n edges. As each gluing cycle corresponds to a pull operation, this amounts to proving that any ordered rooted tree $x \in D_n$ can be transformed to the star $(10)^n$ via a sequence of tree rotations and pulls. Indeed, this can be achieved by fixing any vertex c of x to become the center of the star, and by repeatedly performing pulls that bring one of the leaves in distance > 1 from c one step closer to c (with a root in distance 2 from the leaf being pulled). This completes the proof of the theorem. □

Acknowledgements

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References

[GMN18] P. Gregor, T. Mütze, and J. Nummenpalo. A short proof of the middle levels theorem. *Discrete Anal.*, Paper No. 8:12 pp., 2018.

[Müt16] T. Mütze. Proof of the middle levels conjecture. *Proc. Lond. Math. Soc.*, 112(4):677–713, 2016.