Another proof of Seymour's 6-flow theorem

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Abstract

In 1981 Seymour proved his famous 6-flow theorem asserting that every 2-edge-connected graph has a nowhere-zero flow in the group $\mathbb{Z}_2 \times \mathbb{Z}_3$ (in fact, he offers two proofs of this result). In this note we give a new short proof of a generalization of this theorem where $\mathbb{Z}_2 \times \mathbb{Z}_3$-valued functions are found subject to certain boundary constraints.

Throughout we permit loops and parallel edges. Let $G = (V, E)$ be a digraph and let $v \in V$. We define $\delta^+(v)$ ($\delta^-(v)$) to be the set of edges with tail (head) v. Let Γ be an abelian group written additively and let $\phi : E \to \Gamma$. The *boundary* of ϕ is the function $\partial\phi : V \to \Gamma$ given by the rule:

$$\partial\phi(v) = \sum_{e\in\delta^+(v)} \phi(e) - \sum_{e\in\delta^-(v)} \phi(e).$$

Note that the condition $\sum_{v\in V} \partial\phi(v) = 0$ is always satisfied since every edge contributes 0 to this quantity. We say that ϕ is *nowhere-zero* if $0 \notin \phi(E)$ and we say that ϕ is a Γ-*flow* if $\partial\phi$ is the constant 0 function. Let us comment that reversing an edge and replacing the value assigned to this edge by its additive inverse preserves the boundary and maintains the condition nowhere-zero. So, in particular, the question of when a graph has a nowhere-zero function $\phi : E \to \Gamma$ with a given boundary is independent of the orientation. Setting $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ we may state Seymour's theorem as follows.

Theorem 1 (Seymour [2]). *Every 2-edge-connected digraph has a nowhere-zero $\mathbb{Z}_6$-flow.*

This result combines with a theorem of Tutte [3] to show that every 2-edge-connected digraph has a nowhere-zero 6-flow (i.e. a $\mathbb{Z}$-flow with range a subset of $\{\pm1, \pm2, \ldots, \pm5\}$). In this article we prove the following generalization of Seymour's theorem (set $T = U = \emptyset$ to derive Theorem 1). Here supp(f) denotes the support of a function f and for a graph G and a set $X \subseteq V(G)$ we use $d(X)$ to denote the number of edges with exactly one end in X.

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Theorem 2. *Let $G = (V, E)$ be a connected digraph and let $T \subseteq U \subseteq V$ have $|T|$ even and $|U| \neq 1$. Assume further that every $\emptyset \neq V' \subset V$ with $V' \cap U = \emptyset$ satisfies $d(V') \geq 2$. Then for $k = 2, 3$ there exist functions $\phi_k : E(G) \to \mathbb{Z}_k$ satisfying the following properties:*

- $(\phi_2(e), \phi_3(e)) \neq (0, 0)$ *for every* $e \in E$,
- $\mathrm{supp}\,(\partial\phi_2) = T$, *and*
- $\mathrm{supp}\,(\partial\phi_3) = U$.

Under the stronger hypothesis that G is 3-edge-connected, a theorem of Jaeger et. al. [1] shows that one may find a nowhere-zero $\mathbb{Z}_6$-valued function with any desired zero-sum boundary function. The main novelty of our result is that the hypotheses are relatively weak and the result has a quick proof by induction. In particular, we do not require the standard reductions to 3-connected cubic graphs.

Our notation is fairly standard. For sets X, Y we use $X \oplus Y = (X \setminus Y) \cup (Y \setminus X)$ to denote symmetric difference. If G is a graph and (G_1, G_2) is a pair of subgraphs satisfying $E(G_1) \cup E(G_2) = E(G)$, $E(G_1) \cap E(G_2) = \emptyset$, and $|V(G_1) \cap V(G_2)| = k$, then we call (G_1, G_2) a k-separation. This separation is *proper* if $V(G_1) \setminus V(G_2) \neq \emptyset \neq V(G_2) \setminus V(G_1)$. Note that a graph with at least $k+1$ vertices is k-connected if and only if it has no proper $(k-1)$-separation.

Proof of Theorem 2. We proceed by induction on $|E|$. The case when $|V| \leq 2$ holds by inspection, so we may assume $|V| \geq 3$. If there exists $v \in V \setminus U$ with $\deg(v) = 2$, then the result follows by contracting an edge incident with v (to eliminate this vertex) and applying induction. So we may assume no such vertex exists. Next suppose that G has a proper 1-separation (G_1, G_2) with $\{v\} = V(G_1) \cap V(G_2)$. First suppose that $U \subseteq V(G_1)$. In this case the result follows by applying the theorem inductively to G_1 with the given sets T, U and to G_2 with $T = U = \emptyset$. Next suppose that U contains a vertex in both $V(G_1) \setminus \{v\}$ and $V(G_2) \setminus \{v\}$. For $i = 1, 2$ let $U_i = (V(G_i) \cap U) \cup \{v\}$ and choose $T_i = T \cap V(G_i)$ or $T_i = (T \cap V(G_i)) \oplus \{v\}$ so that $|T_i|$ is even. For $i = 1, 2$ apply the theorem inductively to G_i with T_i, U_i to obtain the functions ϕ_2^i and ϕ_3^i. Now taking $\phi_2 = \phi_2^1 \cup \phi_2^2$ and a suitable choice of $\phi_3 = \phi_3^1 \cup \pm\phi_3^2$ gives the desired functions for G. (To see this note that by choosing $\pm\phi_3^2$ we may arrange for $\partial\phi_3(v)$ to be zero or nonzero as desired).

By the above arguments, we may now assume that G is 2-connected. If $U = \emptyset$, choose an edge $e = uw$ and apply the theorem inductively to $G' = G - e$ with $T' = \emptyset$ and $U' = \{u, w\}$ to obtain ϕ_2' and ϕ_3'. Extend ϕ_2' to a function $\phi_2 : E \to \mathbb{Z}_2$ by setting $\phi_2(e) = 0$. Since $\sum_{v \in V} \partial\phi_3' = 0$ we have $\partial\phi_3'(u) = -\partial_3'(w) = \pm 1$. Therefore, we may extend ϕ_3' to a function $\phi_3 : E \to \mathbb{Z}_3$ by setting $\phi_3(e) = \pm 1$ so that $\partial\phi_3 = 0$, thus completing the proof in this case.

Now we may assume $|U| \geq 2$. By Menger's theorem we may choose a nontrivial path P so that both ends of P are in U and furthermore, some component, say H, of $G - E(P)$

contains both ends of P. Over all such paths, choose one P so that H is maximal. Suppose (for a contradiction) that some component $H' \neq H$ of $G - E(P)$ satisfies $V(H') \cap U \neq \emptyset$. Choose $u \in V(H') \cap U$ and choose two internally vertex-disjoint paths $Q_1, Q_2 \subseteq H'$ starting at u and ending in $V(P)$. Now we may choose a nontrival path $P' \subseteq P \cup Q_1$ with one end u and the other an end of P so that $H \cup Q_2$ is contained in some connected component of $G - E(P')$, thus contradicting our choice of P. Therefore, every component of $G - E(P)$ apart from H contains no vertices in U. Note that by our choice, every interior vertex of P is in $V \setminus U$ (and thus has degree ≥ 3).

Let $\{u_1, u_2\}$ be the ends of P and let $G' = G - E(P)$. Define $T' = T \oplus \{u_1, u_2\}$ and $U' = U \cup V(P)$. Now apply the theorem inductively to each component of G' with the corresponding restrictions of T' and U' to obtain $\phi_2' : E(G') \to \mathbb{Z}_2$ and $\phi_3' : E(G') \to \mathbb{Z}_3$. Extend ϕ_2' to a function $\phi_2 : E \to \mathbb{Z}_2$ by defining $\phi_2(e) = 1$ for every $e \in E(P)$ and note that $\mathrm{supp}(\partial\phi_2) = T$ as desired. By possibly reorienting we may assume that P is a directed path with edges in order $e_1, \ldots, e_k$. By greedily assigning values to these edges in order, we may extend ϕ_3' to a function $\phi_3 : E \to \mathbb{Z}_3$ with the property that $\partial\phi_3(v) = 0$ for every internal vertex v of P. Now the function ϕ_3 satisfies the desired boundary condition at every vertex except possibly the ends of P. Let $t \in \mathbb{Z}_3$ and modify ϕ_3 by adding t to $\phi_3(e)$ for every $e \in E(P)$. This has no effect on the boundaries of the internal vertices of P, and for some $t \in \mathbb{Z}_3$ the resulting function will have nonzero boundary at both ends of P. This gives us our desired functions ϕ_2 and ϕ_3. □

References

[1] F. Jaeger, N. Linial, C. Payan, M. Tarsi, *Group connectivity of graphs–A nonhomogeneous analogue of nowhere-zero flow properties*, J. Combin. Theory Ser. B 56 (1992), no.2, 165–182.

[2] P.D. Seymour, *Nowhere-zero 6-flows*, J. Combin. Theory Ser. B 30 (1981), 130–135.

[3] W. T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. 6 (1954), 80–91.