# Bunkbed conjecture for complete bipartite graphs and related classes of graphs 

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#### Abstract

Let $G=(V, E)$ be a simple finite graph. The corresponding bunkbed graph $G^{ \pm}$consists of two copies $G^{+}=\left(V^{+}, E^{+}\right), G^{-}=\left(V^{-}, E^{-}\right)$of $G$ and additional edges connecting any two vertices $v_{+} \in V_{+}, v_{-} \in V_{-}$that are the copies of a vertex $v \in V$. The bunkbed conjecture states that for independent bond percolation on $G^{ \pm}$, for all $v, w \in V$, it is more likely for $v_{-}, w_{-}$to be connected than for $v_{-}, w_{+}$to be connected. While this seems very plausible, so far surprisingly little is known rigorously. Recently the conjecture has been proved for complete graphs. Here we give a proof for complete bipartite graphs, complete graphs minus the edges of a complete subgraph, and symmetric complete $k$-partite graphs.


## 1 Introduction

In this introduction we state the bunkbed conjecture and review some of the pertinent results. We start with a short recap of independent bond percolation. Let $G=(V, E)$ be a (simple, undirected) graph, and let $p_{e} \in[0,1]$ for $e \in E$ be a family of edge weights. Let $z_{e}, e \in E$, be independent, $\{0,1\}$-valued random variables (on some probability space) with $\mathbb{P}\left(z_{e}=1\right)=p_{e}$ for all $e \in E$. An edge $e \in E$ is called open if $\mathcal{Z}_{e}=1$ and closed if $Z_{e}=0$. The stochastic process $\left(\mathcal{Z}_{e}\right)_{e \in E}$ is called independent bond percolation on $G$ with edge probabilities $\left(p_{e}\right)_{e \in E}$. We may think of percolation as the random subgraph of $G$ obtained from $G$ by removing all closed edges. The connected components of this random subgraph are called clusters and for $v, w \in V$ we write $v \leftrightarrow w$, if $v, w$ are contained in the same cluster, i.e. if $v$ and $w$ are joined by a path consisting of finitely many open edges. Usually percolation is considered on infinite graphs with some regularity, such as infinite lattices, and often the edge weights are assumed to be constant, i.e. for some $p \in[0,1]$ we have $p_{e}=p$ for all $e \in E$. We refer to [G] for an introduction to percolation.

We restrict our attention to percolation on bunkbed graphs. For a given graph $G=(V, E)$, the corresponding bunkbed graph $G^{ \pm}=\left(V^{ \pm}, E^{ \pm}\right)$is defined by $V^{ \pm}:=V \times\{+,-\}$ and $E^{ \pm}:=\left\{u_{+} v_{+}, u_{-} v_{-}: u v \in E\right\} \cup\left\{u_{+} u_{-}: u \in V\right\}$, where we have written $v_{+}:=(v,+)$ and $v_{-}:=(v,-)$ and used the usual edge

[^0]notation, e.g. $u v:=\{u, v\}$. Sometimes $V^{+}:=V \times\{+\}$ and $V^{-}:=V \times\{-\}$ are called upstairs and downstairs layer respectively, and edges of the form $u_{+} v_{+}$ or $u_{-} v_{-}$are called horizontal edges, whereas edges of the form $u_{+} u_{-}$are called vertical edges. For given weights $p_{G}=\left(p_{x}\right)_{x \in E \cup V} \in[0,1]^{E \cup V}$ we consider corresponding weights $\left(p_{e}\right)_{e \in E^{ \pm}}$on $G^{ \pm}$such that $p_{u_{+} v_{+}}=p_{u_{-} v_{-}}:=p_{u v}$ and $p_{u_{+} u_{-}}:=p_{u}$ for all $u, v \in V$. Slightly abusing notation, we also write $p_{G}$ for $\left(p_{e}\right)_{e \in E^{ \pm}}$. We will use $p_{G}$ as edge probabilities in our percolation model on $G^{ \pm}$, and we write $\mathbb{P}_{p_{G}}$ instead of $\mathbb{P}$ to indicate the choice of edge weights in our notation of probabilities. We note that the reflection map $\varphi_{ \pm}: V^{ \pm} \rightarrow V^{ \pm}$, $\varphi_{ \pm}\left(u_{+}\right):=u_{-}, \varphi_{ \pm}\left(u_{-}\right):=u_{+}$for all $u \in V$, is a graph automorphism of $G^{ \pm}$, which preserves weights $p_{G}$ as defined above. Thus our independent percolation model on $G^{ \pm}$is invariant under this reflection.

Two special cases of the above model are of interest: Given $p \in[0,1]$ and $H \subset V$, we may consider probability weights given by $p_{u v}:=p$ for all $u v \in E$, $p_{u}:=1$ for all $u \in H$ and $p_{u}:=0$ for all $u \in V \backslash H$, i.e. we have constant weights on horizontal edges and deterministic vertical edges. Alternatively, given $p \in[0,1]$, we may consider constant probability weights given by $p_{u v}:=p$ for all $u v \in E$ and $p_{u}:=p$ for all $u \in V$. If we consider these special cases we write $\mathbb{P}_{p, H}$ or $\mathbb{P}_{p}$ respectively.

We are now ready to state the bunkbed conjecture, which seems to go back to Kasteleyn (1985) as remarked in [BK].

Conjecture 1 Three versions of the bunkbed conjecture. For independent percolation on the bunkbed graph $G^{ \pm}$for a given (simple, undirected) finite graph $G=(V, E)$ we have:
(B1) $\forall p_{G} \in[0,1]^{E \cup V}, v, w \in V: \quad \mathbb{P}_{p_{G}}\left(v_{-} \leftrightarrow w_{-}\right) \geq \mathbb{P}_{p_{G}}\left(v_{-} \leftrightarrow w_{+}\right)$.
(B2) $\forall p \in[0,1], H \subset V, v, w \in V: \quad \mathbb{P}_{p, H}\left(v_{-} \leftrightarrow w_{-}\right) \geq \mathbb{P}_{p, H}\left(v_{-} \leftrightarrow w_{+}\right)$.
(B3) $\forall p \in[0,1], v, w \in V: \quad \mathbb{P}_{p}\left(v_{-} \leftrightarrow w_{-}\right) \geq \mathbb{P}_{p}\left(v_{-} \leftrightarrow w_{+}\right)$.
In each case the two connection probabilities do not change if $v, w$ are interchanged and/or,+- are interchanged, which easily follows from the symmetry of $\leftrightarrow$ and the reflection symmetry of the probability models. The bunkbed conjecture thus compares the connection probability for two vertices in the same layer with that of the two corresponding vertices in different layers. Indeed the bunkbed conjecture is motivated by the heuristics that vertices in different layers are - in some sense - further apart from each other than the corresponding vertices in the same layer, and increasing the 'distance' between vertices in general should make it harder for the vertices to be connected w.r.t. independent percolation.

Remark 1 Versions of the bunkbed conjecture.

- All three versions (B1), (B2), (B3) of the bunkbed conjecture are believed to hold for every finite graph $G$. We note that the three versions are obviously related: For fixed $G$ (B1) implies (B2) and (B2) implies (B3). The latter implication is even true for fixed $p$, which follows via conditioning
on all vertical edges. Also, if (B2) holds for every finite graph $G$, then so does (B1). Indeed, a stronger implication is the main result of [RS].
- Horizontal edges $e$ with $p_{e}=0$ may be removed and horizontal edges with $p_{e}=1$ may be contracted, so one may want to exclude these values in the above conjecture. (B2) and (B3) trivially hold in case of $p \in\{0,1\}$.
- All versions of the bunkbed conjecture can also be formulated for infinite graphs. Indeed, if the conjecture is true for all finite graphs, then it also holds for all infinite graphs (by a simple limiting argument).

In spite of the strong intuition that the bunkbed conjecture should hold (in any of the versions presented above), surprisingly little is known rigorously, and the conjecture remains wide open almost 40 years after its first formulation. It is easy to see that a version of (B2) holds for $p$ sufficiently small (depending on $G, H)$, and similarly a version of (B2) holds for $p$ sufficiently large (depending on $G, H$ ), see [HNK]. There are also a few results for special classes of graphs. For graphs with very few connections, an inductive approach can be used. It is not too difficult to see that (B1) holds for all trees and cycle graphs. A result for outer planar graphs has recently been withdrawn, see [L1], [L2]. Furthermore, (B2) has been shown for complete graphs, see [HL and earlier partial results in [B]. The result for complete graphs mainly relies on the high degree of symmetry of the these graphs. Our aim is to deal with situations that are slightly less symmetric. We will present our results in the following section. The sections after that are devoted to the proofs of our results.

## 2 Results

Let us first review the result [HL of Hintum and Lammers for the complete graph. It is useful to note that the proof of this result does not use the full symmetry of the complete graph. Indeed, the proof shows the following:

Theorem 1 Bunkbed conjecture for neighboring vertices with a local symmetry. Let $G=(V, E)$ be a finite complete graph with weights $p_{G} \in[0,1)^{E} \times[0,1]^{V}$. Let $v, w \in V$ such that $p_{v w}>0$. In case of $p_{v}, p_{w} \neq 1$ suppose that for all $u \in V$ with $p_{u w}>0$ and $p_{u} \neq 1$ there is an automorphism $\varphi$ of the weighted graph $G$ such that $\varphi(v w)=u w$. Then we have

$$
\mathbb{P}_{p_{G}}\left(v_{-} \leftrightarrow w_{-}\right) \geq \mathbb{P}_{p_{G}}\left(v_{-} \leftrightarrow w_{+}\right)
$$

For the convenience of the reader we have included the proof of Theorem 1 in Section 5, but we note that this proof follows the argument given in [HL] (even if our presentation makes it look somewhat different).

Remark 2 Bunkbed conjecture for neighboring vertices with a local symmetry.

- In our formulation of Theorem 1 we have excluded $p_{e}=1$ in order to simplify the local symmetry condition. We note that for any edge with $p_{e}=$ 1 its two incident vertices can be combined into a single vertex without changing connectivity probabilities, so this is not really a restriction.
- The local symmetry is formulated in terms of the existence of automorphisms. We note that an automorphism of a graph $G=(V, E)$ with probability weights $p_{G}$ is simply a graph automorphism $\varphi: V \rightarrow V$ that preserves the probability weights in that $p_{\varphi(v)}=p_{v}$ for all $v \in V$ and $p_{\varphi(u) \varphi(v)}=p_{u v}$ for all $u v \in E$. What is actually needed in the proof is the weaker (but less explicit) condition that the probabilities $\mathbb{P}_{p_{G}}\left(u_{-} \leftrightarrow w_{-}\right)$ and $\mathbb{P}_{p_{G}}\left(u_{-} \leftrightarrow w_{+}\right)$are constant for $u \in V$ with $p_{u w}>0$ and $p_{u} \neq 1$.
- The theorem is formulated for the model $\mathbb{P}_{p_{G}}$, and it immediately implies corresponding results for $\mathbb{P}_{p, H}$ and $\mathbb{P}_{p}$. In order to see how the assumptions simplify, let us formulate the result for $\mathbb{P}_{p}:$ Let $G=(V, E)$ be a finite graph and $p \in[0,1]$. Let $v, w \in V$ such that $v w \in E$. Suppose that for all $u \in V$ such that $u w \in E$ there is an graph automorphism $\varphi$ of $G$ such that $\varphi(v w)=u w$. Then we have $\mathbb{P}_{p}\left(v_{-} \leftrightarrow w_{-}\right) \geq \mathbb{P}_{p}\left(v_{-} \leftrightarrow w_{+}\right)$.

If we aim at proving the bunkbed conjecture, say in form of (B3), for certain graphs $G$ using only the above theorem, we are somewhat limited, since $v, w$ are required to be neighbors. The theorem immediately implies that (B3) (and in fact also (B2)) holds for complete graphs, but we can't go any further. However, if we are only interested in neighboring vertices of graphs with appropriate symmetries, the above theorem may be helpful:

Corollary 1 Let $G=(V, E)$ be a finite, edge-transitive graph. Then we have

$$
\forall p \in[0,1], v w \in E: \mathbb{P}_{p}\left(v_{-} \leftrightarrow w_{-}\right) \geq \mathbb{P}_{p}\left(v_{-} \leftrightarrow w_{+}\right) .
$$

This is an immediate consequence of Theorem 1 (in the special case of constant probability weights). Most interesting examples of edge-transitive graphs are symmetric (i.e. vertex- and edge-transitive) or indeed arc-transitive. For relations between the different notions of graph transitivity and examples, we refer to GR]. In order to give some concrete examples, we note that in particular the above corollary applies to the following interesting classes of graphs: complete graphs, cycle graphs, the graphs corresponding to regular polyhedra (such as the icosahedral graph), hypercube graphs (such as the cubic graph), the Petersen graph. It also applies to complete bipartite graphs. Here in fact all pairs of vertices that are non-neighboring have the same set of neighbors. This is our motivation for investigating the following situation, which is the main result of our paper:

Theorem 2 Bunkbed conjecture for vertices with the same neighbors. Let $G=$ $(V, E)$ be a finite complete graph with probability weights $p_{G} \in[0,1]^{E \cup V}$. Let $v, w \in V$ such that for all $u \in V \backslash\{v, w\}$ we have $p_{v u}=p_{w u}$. Then we have

$$
\mathbb{P}_{p_{G}}\left(v_{-} \leftrightarrow w_{-}\right) \geq \mathbb{P}_{p_{G}}\left(v_{-} \leftrightarrow w_{+}\right)
$$

The proof of Theorem 2 is presented in the next section. It relies on a suitable decomposition of the events, conditioning and symmetrization that allows us to use the symmetry assumption. While these ideas are rather elementary, the way they have to be combined to give the result is somewhat tricky.

Remark 3 Bunkbed conjecture for vertices with the same neighbors.

- Here the symmetry assumption can also be formulated in terms of graph automorphisms: We assume that the transposition map $\varphi_{v, w}: V \rightarrow V$, $\varphi_{v, w}(v):=w, \varphi_{v, w}(w):=v, \varphi_{v, w}(u):=u$ for all $u \in V \backslash\{v, w\}$ is an automorphism of the weighted graph $G$.
- While Theorem 1 only applies to vertices with graph distance 1, the above result only applies to vertices with graph distance $\leq 2$. Unfortunately the symmetry assumption is again rather strong.
- Again the theorem immediately implies corresponding results for $\mathbb{P}_{p, H}$ and $\mathbb{P}_{p}$. E.g. let us formulate the result for $\mathbb{P}_{p}$ : Let $G=(V, E)$ be a finite graph and $p \in[0,1]$. Let $v, w \in V$ such that for all $u \in V \backslash\{v, w\}$ we have $v u \in E$ iff $w u \in E$. Then we have $\mathbb{P}_{p}\left(v_{-} \leftrightarrow w_{-}\right) \geq \mathbb{P}_{p}\left(v_{-} \leftrightarrow w_{+}\right)$.

The combination of Theorems 1 and 2 proves the bunkbed conjecture for new classes of graphs.

Corollary 2 Let $G=(V, E)$ be a finite complete graph, let $V_{1}, V_{2}$ be a disjoint composition of $V$, let $H \subset V$, and let $p, p^{\prime} \in[0,1)$. Define probability weights $p_{G} \in[0,1]^{E \cup V}$ by $p_{u v}=0$ for all $u \neq v \in V_{1}, p_{u v}=p$ for all $u \in V_{1}, v \in V_{2}$. $p_{u v}=p^{\prime}$ for all $u \neq v \in V_{2}, p_{u}=1$ for all $u \in H$ and $p_{u}=0$ for all $u \notin H$. Then we have

$$
\forall v, w \in V: \mathbb{P}_{p_{G}}\left(v_{-} \leftrightarrow w_{-}\right) \geq \mathbb{P}_{p_{G}}\left(v_{-} \leftrightarrow w_{+}\right) .
$$

In particular the bunkbed conjecture in its strong form (B2) holds for all complete bipartite graphs, and for all complete graphs minus the edges of an arbitrary complete subgraph.

In fact the corollary still holds for arbitrary weights $p_{v} \in[0,1], v \in V$ instead of the deterministic weights above, which easily follows by conditioning on the states of all vertical edges. We have not found other interesting classes of graphs for which a combination of the two theorems proves the strong form of the bunkbed conjecture (B2). For the weaker version (B3) of the bunkbed conjecture we get a little bit more:

Corollary 3 Let $k \geq 1$ and let $G=(V, E)$ a symmetric complete $k$-partite graph, i.e. suppose that $V$ can be decomposed into disjoint subsets $V_{1}, \ldots, V_{k}$ such that $\left|V_{1}\right|=\ldots=\left|V_{k}\right|$ and $E=\left\{u v: \exists i \neq j: u \in V_{i}, v \in V_{j}\right\}$. Then for $G$ the weak form (B3) of the bunkbed conjecture holds.

The straightforward proofs for the two preceding corollaries are relegated to Section 4.

Finally we note that the above theorems prove inequalities as in (B2) or (B3) for many graphs and suitable fixed choices of $H \subset V, v, w \in V$ or $v, w \in V$ respectively. However, in order to treat further classes of graphs, something is missing. E.g. for general complete $k$-partite graphs with vertex set decomposition $V_{1}, \ldots, V_{k}$ the case of $v, w \in V_{i}$ for some $i$ can be treated as above, but in
case of $v \in V_{i}, w \in V_{j}$ for $i \neq j$ the symmetry assumption in Theorem 1 is too strong and does not hold. We also stress that results that rely on symmetry assumptions obviously will not be strong enough to prove the bunkbed conjecture in full generality. However one might hope that the ideas of the proof of Theorem 2 could lead to further progress towards a proof of the conjecture.

## 3 Vertices with the same neighbors: Theorem 2

For the proof of Theorem 2 let $v, w \in V$ satisfy the given symmetry assumption. We will write $\mathbb{P}:=\mathbb{P}_{p_{G}}$. By reflection invariance we have $\mathbb{P}\left(v_{-} \leftrightarrow w_{+}\right)=$ $\mathbb{P}\left(v_{+} \leftrightarrow w_{-}\right)$and $\mathbb{P}\left(v_{-} \leftrightarrow w_{-}\right)=\mathbb{P}\left(v_{+} \leftrightarrow w_{+}\right)$, so we need to prove

$$
d:=\mathbb{P}\left(v_{+} \leftrightarrow w_{+}\right)+\mathbb{P}\left(v_{-} \leftrightarrow w_{-}\right)-\mathbb{P}\left(v_{+} \leftrightarrow w_{-}\right)-\mathbb{P}\left(v_{-} \leftrightarrow w_{+}\right) \geq 0
$$

First we decompose

$$
\begin{aligned}
\mathbb{P}\left(v_{+} \leftrightarrow w_{+}\right) & =\mathbb{P}\left(v_{-} \leftrightarrow v_{+} \leftrightarrow w_{+} \leftrightarrow w_{-}\right)+\mathbb{P}\left(v_{-} \not \leftrightarrow v_{+} \leftrightarrow w_{+} \leftrightarrow w_{-}\right) \\
& +\mathbb{P}\left(v_{-} \leftrightarrow v_{+} \leftrightarrow w_{+} \not \leftrightarrow w_{-}\right)+\mathbb{P}\left(v_{-} \not \leftrightarrow v_{+} \leftrightarrow w_{+} \not \leftrightarrow w_{-}\right)
\end{aligned}
$$

and similarly for the other three events. In the corresponding decomposition of $d$, the probability with all four vertices connected appears twice with a positive and twice with a negative sign, and each probability with three of the vertices connected appears once with a positive and once with a negative sign. Thus

$$
\begin{aligned}
d & =\mathbb{P}\left(v_{-} \nrightarrow v_{+} \leftrightarrow w_{+} \nleftarrow w_{-}\right)+\mathbb{P}\left(v_{+} \nleftarrow v_{-} \leftrightarrow w_{-} \nrightarrow w_{+}\right) \\
& -\mathbb{P}\left(v_{-} \not \leftrightarrow v_{+} \leftrightarrow w_{-} \nleftarrow w_{+}\right)-\mathbb{P}\left(v_{+} \nleftarrow v_{-} \leftrightarrow w_{+} \nleftarrow w_{-}\right) .
\end{aligned}
$$

Next we condition on $z_{v}$. We note that on $z_{v}=1$ we have $v_{+} \leftrightarrow v_{-}$and thus $\mathbb{P}\left(v_{-} \not \leftrightarrow v_{+} \leftrightarrow w_{+} \nleftarrow w_{-} \mid z_{v}=1\right)=0$ and similarly for the other three probabilities. Thus it suffices to consider the conditional probabilities w.r.t. $\mathbb{P}\left(. \mid \mathcal{Z}_{v}=0\right)$, i.e. w.l.o.g. we may assume $p_{v}=0$. Similarly we may assume $p_{w}=0$. Next we consider the edge $v w$. Let $A:=\left\{z_{v_{+} w_{+}}=z_{v_{-} w_{-}}=0\right\}$. On $A^{c}$ we either have $v_{+} \leftrightarrow w_{+}$or $v_{-} \leftrightarrow w_{-}$, so

$$
\mathbb{P}\left(v_{-} \not \leftrightarrow v_{+} \leftrightarrow w_{-} \nleftarrow w_{+} \mid A^{c}\right)=0=\mathbb{P}\left(v_{+} \nleftarrow v_{-} \leftrightarrow w_{+} \not \leftrightarrow w_{-} \mid A^{c}\right) .
$$

Thus it suffices to consider the conditional probabilities w.r.t. $\mathbb{P}(. \mid A)$, i.e. w.l.o.g. we may assume $p_{v w}=0$. Next let $W=\left\{v_{-}, v_{+}, w_{-}, w_{+}\right\}$and let $G_{v w}^{ \pm}=\left(V_{v w}^{ \pm}, E_{v w}^{ \pm}\right)$denote the subgraph of $G^{ \pm}$induced by $V_{v w}^{ \pm}:=V_{+} \cup V_{-} \backslash W$. For any decomposition $C=\left\{C_{i}: i \in I\right\}$ of $V_{v w}^{ \pm}$into disjoint connected sets, we let $A_{C}$ denote the event that $z_{e}, e \in E_{v w}^{ \pm}$, produce clusters given by $C$. Let $\mathbb{P}_{C}:=\mathbb{P}\left(. \mid A_{C}\right)$ denote the conditional distribution. Let

$$
\begin{aligned}
d_{C} & =\mathbb{P}_{C}\left(v_{-} \nleftarrow v_{+} \leftrightarrow w_{+} \nleftarrow w_{-}\right)+\mathbb{P}_{C}\left(v_{+} \nleftarrow v_{-} \leftrightarrow w_{-} \not \leftrightarrow w_{+}\right) \\
& -\mathbb{P}_{C}\left(v_{-} \not \leftrightarrow v_{+} \leftrightarrow w_{-} \nleftarrow w_{+}\right)-\mathbb{P}_{C}\left(v_{+} \nleftarrow v_{-} \leftrightarrow w_{+} \nleftarrow w_{-}\right) .
\end{aligned}
$$

It suffices to prove $d_{C} \geq 0$ for all $C$, and we fix $C$ for the remainder of the proof. We now look at these four events from the perspective of the clusters $C_{i}$. For $u \in W$ and $i \in I$ we write $u \sim C_{i}$ if $z_{u u^{\prime}}=1$ for some $u^{\prime} \in C_{i}$.

$$
\mathcal{A}_{i}=\left\{u \in W: u \sim C_{i}\right\}
$$

denotes the (random) set of vertices of $W$ directly connected to $C_{i}$. We note that $v_{-} \not \leftrightarrow v_{+} \leftrightarrow w_{+} \not \leftrightarrow w_{-}$iff for all $i$ we have $\left|\mathcal{A}_{i}\right| \leq 1$ or $\mathcal{A}_{i}=\left\{v_{-}, w_{-}\right\}$ or $\mathcal{A}_{i}=\left\{v_{+}, w_{+}\right\}$, and the latter occurs at least once. (Here we use that $p_{v}=p_{w}=p_{v w}=0$, so all connections between the points of $W$ have to go via one of the clusters $C_{i}$.) Thus we have

$$
\begin{aligned}
& \mathbb{P}_{C}\left(v_{-} \not \leftrightarrow v_{+} \leftrightarrow w_{+} \not \leftrightarrow w_{-}\right) \\
& =\sum_{J, K, L: K \neq \emptyset} \mathbb{P}_{C}\left(\forall i \in J:\left|\mathcal{A}_{i}\right| \leq 1, \forall i \in K: \mathcal{A}_{i}=\left\{v_{+}, w_{+}\right\}, \forall i \in L: \mathcal{A}_{i}=\left\{v_{-}, w_{-}\right\}\right) \\
& =\sum_{J, K, L: K \neq \emptyset} \prod_{i \in J} \mathbb{P}_{C}\left(\left|\mathcal{A}_{i}\right| \leq 1\right) \prod_{i \in K} \mathbb{P}_{C}\left(\mathcal{A}_{i}=\left\{v_{+}, w_{+}\right\}\right) \prod_{i \in L} \mathbb{P}_{C}\left(\mathcal{A}_{i}=\left\{v_{-}, w_{-}\right\}\right),
\end{aligned}
$$

where the sum is over all disjoint decompositions of the index set $I$. We note that in the last step we may use independence, since the edge sets connecting $W$ to $C_{i}$ and the edge set $E_{v w}^{ \pm}$are disjoint. We have similar decompositions for the other three terms in $d_{C}$. With

$$
\left.p_{M}(\leq 1):=\prod_{i \in M} \mathbb{P}_{C}\left(\left|\mathcal{A}_{i}\right| \leq 1\right) \quad \text { and } \quad p_{M}\left(W^{\prime}\right):=\prod_{i \in M} \mathbb{P}_{C}\left(\mathcal{A}_{i}=W^{\prime}\right)\right)
$$

for $M \subset I, W^{\prime} \subset W$, we thus obtain

$$
\begin{aligned}
& d_{C}=\sum_{J, K, L: K \neq \emptyset} p_{J}(\leq 1) d_{K, L}, \quad \text { where } \\
& \begin{aligned}
d_{K, L}: & =p_{K}\left(v_{+},\right. \\
\left.w_{+}\right) & p_{L}\left(v_{-}, w_{-}\right)+p_{K}\left(v_{-}, w_{-}\right) p_{L}\left(v_{+}, w_{+}\right) \\
& \quad-p_{K}\left(v_{+}, w_{-}\right) p_{L}\left(v_{-}, w_{+}\right)-p_{K}\left(v_{-}, w_{+}\right) p_{L}\left(v_{+}, w_{-}\right) .
\end{aligned}
\end{aligned}
$$

Thus it suffices to show that $d_{K, L} \geq 0$ for all $K, L$. Writing

$$
p_{i}(u):=\mathbb{P}_{C}\left(u \sim C_{i}\right) \text { for } u \in W
$$

we note that the given symmetry assumption implies that

$$
p_{i}\left(v_{+}\right)=1-\prod_{u \in C_{i} \cap V_{+}}\left(1-p_{v_{+} u}\right)=1-\prod_{u \in C_{i} \cap V_{+}}\left(1-p_{w_{+} u}\right)=p_{i}\left(w_{+}\right)=: p_{i+}
$$

and similarly $p_{i}\left(v_{-}\right)=p_{i}\left(w_{-}\right)=: p_{i-}$. Thus we can write

$$
\begin{aligned}
& d_{K, L}=\prod_{i \in K} p_{i+}^{2}\left(1-p_{i-}\right)^{2} \prod_{i \in L} p_{i-}^{2}\left(1-p_{i+}\right)^{2}+\prod_{i \in K} p_{i-}^{2}\left(1-p_{i+}\right)^{2} \prod_{i \in L} p_{i+}^{2}\left(1-p_{i-}\right)^{2} \\
& \quad-2 \prod_{i \in K} p_{i-} p_{i+}\left(1-p_{i-}\right)\left(1-p_{i+}\right) \prod_{i \in L} p_{i-} p_{i+}\left(1-p_{i-}\right)\left(1-p_{i+}\right) \\
& =\left(\prod_{i \in K} p_{i+}\left(1-p_{i-}\right) \prod_{i \in L} p_{i-}\left(1-p_{i+}\right)-\prod_{i \in K} p_{i-}\left(1-p_{i+}\right) \prod_{i \in L} p_{i+}\left(1-p_{i-}\right)\right)^{2},
\end{aligned}
$$

which implies that $d_{K, L} \geq 0$ and thus we have completed the proof.

## 4 Special classes of graphs: Corollaries 2 and 3

For the proof of Corollary 2 let $v, w \in V$. In case of $w \in V_{1}, v \in V_{2}$ we note that the assertion is trivial for $p=0$, and for $p>0$ we use Theorem $\mathbb{1}$ noting that $p_{v w}>0$. If $p_{v}, p_{w} \neq 1$ and $u \in V$ is an arbitrary vertex with $p_{u w}>0$ and $p_{u} \neq 1$, then necessarily $u \in V_{2}$ and the transposition $\varphi_{v, u}$ of $v$ and $u$ is an automorphism of the weighted graph $G$. We have thus verified the required symmetry assumption. (The case $w \in V_{2}, v \in V_{1}$ is the same.)

In case of $w, v \in V_{1}$ we use Theorem 2 noting $p_{v u}=p_{w u}=0$ for all $u \in$ $V_{1} \backslash\{v, w\}$ and $p_{v u}=p_{w u}=p$ for all $u \in V_{2}$. Similarly, in case of $w, v \in V_{2}$ we use Theorem 2 noting $p_{v u}=p_{w u}=p^{\prime}$ for all $u \in V_{2} \backslash\{v, w\}$ and $p_{v u}=p_{w u}=p$ for all $u \in V_{1}$.

For the last part of the corollary we may assume w.l.o.g. that $p<1$. We note that any complete bipartite graph can be obtained from the graph under consideration by setting $p^{\prime}:=0$, and any complete graph minus the edges of an arbitrary complete subgraph can similarly be obtained by setting $p^{\prime}:=p$.

For the proof of Corollary 3 let $p \in[0,1]$ and $v, w \in V$. W.l.o.g. $p \in(0,1)$. In case of $w \in V_{i}, v \in V_{j}$ for $i \neq j$ we use Theorem 1. Noting that the given graph is edge-transitive, the symmetry assumption is satisfied. In case of $v, w \in V_{i}$ for some $i$ we use Theorem 2 noting that for $u \in V$ we have $v u \in E \Leftrightarrow u \notin V_{i} \Leftrightarrow w u \in E$.

## 5 Neighboring vertices: Theorem 1

For the proof of Theorem $\rceil$ let $v, w \in V$ satisfy the given symmetry assumption. We write $\mathbb{P}=\mathbb{P}_{p_{G}}$. As in the proof of Theorem 2 we may assume that $p_{w}=0$ and $p_{v} \neq 1$. We define

$$
d:=2 \sum_{u} c_{u w}\left(\mathbb{P}\left(u_{-} \leftrightarrow w_{-}\right)-\mathbb{P}\left(u_{-} \leftrightarrow w_{+}\right)\right), \text {where } c_{u w}:=-\ln \left(1-p_{u w}\right) .
$$

We note that $c_{u w} \in[0, \infty)$ since $p_{u w} \in[0,1)$. For all $u$ such that $p_{u w}=0$ we have $c_{u w}=0$. For all $u$ such that $p_{u w}>0$ and $p_{u}=1$ we have $\mathbb{P}\left(u_{-} \leftrightarrow w_{-}\right)=$ $\mathbb{P}\left(u_{-} \leftrightarrow w_{+}\right)$. For all $u$ such that $p_{u w}>0$ and $p_{u} \neq 1$ we have $\mathbb{P}\left(v_{-} \leftrightarrow w_{-}\right)$ $=\mathbb{P}\left(u_{-} \leftrightarrow w_{-}\right)$and $\mathbb{P}\left(v_{-} \leftrightarrow w_{+}\right)=\mathbb{P}\left(u_{-} \leftrightarrow w_{+}\right)$, since the distribution of percolation on the bunkbed graph is invariant under all automorphisms of the underlying weighted graph. Thus

$$
d=2 c\left(\mathbb{P}\left(v_{-} \leftrightarrow w_{-}\right)-\mathbb{P}\left(v_{-} \leftrightarrow w_{+}\right)\right), \text {where } c:=\sum_{u: p_{u w}>0, p_{u}<1} c_{u w} \text {. }
$$

Since $c \geq c_{v w}>0$ it suffices to show that $d \geq 0$. As in the proof of Theorem 2 reflection invariance gives

$$
d=\sum_{u} c_{u w}\left(\mathbb{P}\left(u_{-} \leftrightarrow w_{-}\right)-\mathbb{P}\left(u_{-} \leftrightarrow w_{+}\right)+\mathbb{P}\left(u_{+} \leftrightarrow w_{+}\right)-\mathbb{P}\left(u_{+} \leftrightarrow w_{-}\right)\right) .
$$

Next let $G_{w}^{ \pm}=\left(V_{w}^{ \pm}, E_{w}^{ \pm}\right)$denote the subgraph of $G^{ \pm}$induced by $V_{w}^{ \pm}:=V_{+} \cup$ $V_{-} \backslash\left\{w_{+}, w_{-}\right\}$. For any decomposition $C=\left\{C_{i}: i \in I\right\}$ of $V_{w}^{ \pm}$into disjoint connected sets, let $A_{C}$ denote the event that $\mathcal{Z}_{e}, e \in E_{w}^{ \pm}$, produce clusters given by $C$, let $\mathbb{P}_{C}:=\mathbb{P}\left(. \mid A_{C}\right)$ denote the conditional distribution, and let

$$
\begin{aligned}
& d_{C}:=\sum_{u} c_{u w}\left(\mathbb{P}_{C}\left(u_{-} \leftrightarrow w_{-}\right)-\mathbb{P}_{C}\left(u_{-} \leftrightarrow w_{+}\right)+\mathbb{P}_{C}\left(u_{+} \leftrightarrow w_{+}\right)-\mathbb{P}_{C}\left(u_{+} \leftrightarrow w_{-}\right)\right) \\
& =\sum_{u} c_{u w}\left(\mathbb{P}_{C}\left(u_{-} \leftrightarrow w_{-} \nleftarrow w_{+}\right)-\mathbb{P}_{C}\left(u_{-} \leftrightarrow w_{+} \nleftarrow w_{-}\right)\right. \\
& \left.\quad+\mathbb{P}_{C}\left(u_{+} \leftrightarrow w_{+} \nleftarrow w_{-}\right)-\mathbb{P}_{C}\left(u_{+} \leftrightarrow w_{-} \nleftarrow w_{+}\right)\right)
\end{aligned}
$$

where we have canceled $\mathbb{P}_{C}\left(v_{+} \leftrightarrow w_{+} \leftrightarrow w_{-}\right)$and $\mathbb{P}_{C}\left(v_{-} \leftrightarrow w_{+} \leftrightarrow w_{-}\right)$. It suffices to show that $d_{C} \geq 0$ for all $C$, and we fix $C$ for the remainder of the proof. For $i \in I$ and $w^{\prime} \in\left\{w_{+}, w_{-}\right\}$we write $w^{\prime} \sim C_{i}$ iff $z_{w^{\prime} v^{\prime}}=1$ for some $v^{\prime} \in C_{i}$, and we let

$$
p_{i+}:=\mathbb{P}_{C}\left(w_{+} \sim C_{i}\right) \quad \text { and } \quad p_{i-}:=\mathbb{P}_{C}\left(w_{-} \sim C_{i}\right)
$$

We note that in case of $u_{-} \in C_{i}$ we have

$$
\begin{aligned}
& \mathbb{P}_{C}\left(u_{-} \leftrightarrow w_{-} \not \leftrightarrow w_{+}\right)=\mathbb{P}_{C}\left(w_{-} \sim C_{i}, w_{+} \nsim C_{i}, \forall j \neq i: w_{-} \nsim C_{j} \text { or } w_{+} \nsim C_{j}\right) \\
& =p_{i-}\left(1-p_{i+}\right) r_{i}, \text { where } r_{i}:=\prod_{j \neq i}\left(1-p_{j-} p_{j+}\right)
\end{aligned}
$$

Here the first step is due to equality of the two events, and in the second step we have used independence due to the events depending on disjoint edge sets. With similar calculations for the other probabilities we obtain

$$
\begin{aligned}
d_{C}= & \sum_{i} \sum_{u_{-} \in C_{i}} c_{u w}\left(p_{i-}\left(1-p_{i+}\right) r_{i}-p_{i+}\left(1-p_{i-}\right) r_{i}\right) \\
& \left.+\sum_{i} \sum_{u_{+} \in C_{i}} c_{u w}\left(p_{i+}\left(1-p_{i-}\right) r_{i}-p_{i-}\left(1-p_{i+}\right) r_{i}\right)\right) \\
= & \sum_{i}\left(\left(p_{i-}-p_{i+}\right) r_{i} \sum_{u_{-} \in C_{i}} c_{u w}+\left(p_{i+}-p_{i-}\right) r_{i} \sum_{u_{+} \in C_{i}} c_{u w}\right)
\end{aligned}
$$

By definition of the weights $c_{u w}$ we have

$$
\sum_{u_{-} \in C_{i}} c_{u w}=-\ln \prod_{u_{-} \in C_{i}}\left(1-p_{u w}\right)=-\ln \left(1-p_{i-}\right)
$$

and similarly for the other sum, so

$$
d_{C}=\sum_{i} r_{i}\left(p_{i-}-p_{i+}\right)\left(-\ln \left(1-p_{i-}\right)+\ln \left(1-p_{i+}\right)\right)
$$

which indeed is nonnegative, since $f(x)=-\ln (1-x)$ is increasing. This finishes the proof of the theorem.

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