Spring School 2023

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Preface

Spring school on Combinatorics has been a traditional meeting organized for almost 40 years for faculty and students participating in the Combinatorial Seminar at Faculty of Mathematics and Physics of the Charles University. It is internationally known and regularly visited by students, postdocs and teachers from our cooperating institutions in the DIMATIA network. As it has been the case for several years, this Spring School is supported by Computer Science Institute (IÚUK) of Charles University, the Department of Applied Mathematics (KAM) and by some of our grants (SVV, Progres). This year we are glad we can also acknowledge generous support by the RSJ Foundation.

The Spring Schools are entirely organized and arranged by our students. The topics of talks are selected by supervisors from the Department of Applied Mathematics (KAM) and Computer Science Institute (IÚUK) of Charles University as well as from other participating institutions. In contrast, the talks themselves are almost exclusively given by students, both undergraduate and graduate. This leads to a unique atmosphere of the meeting, which helps the students in further studies and their scientific orientation.

This year the Spring School is organized in Jáchymov (in Ore Mountains in northwestern Bohemia) with a great variety of possibilities for outdoor activities.

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Series Talks

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Introduction to Cooperative Game Theory
as part of series Cooperative game theory

Introduction

In this talk, I will present you the basics of cooperative game theory. It is an introduction to a series of talks solving different problems with the model.

Definition 1 (Cooperative game) *A cooperative game is an ordered pair* (N, v)*, where* N *is a set of players and* $v\colon 2^N \to \mathbb{R}$ *is the characteristic function. Further,* $v(\emptyset) = 0$*.*

Definition 2 (Payoff vector) *Payoff vector is* $x \in \mathbb{R}^n$*, where* x_i *represents payoff of player* i*. It is efficient, if* $\sum_{i\in N} x_i = v(N)$*. It is individually rational, if* $x_i \geq v(i)$*. We denote* $x(S) = \sum_{i\in S} x_i$*.*

Definition 3 (Core) *For a cooperative game* (N, v)*, the core* $\mathcal{C}(v)$ *is*

$$\mathcal{C}(v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \wedge (S) \geq v(S), \forall S \subseteq N\}.$$

Observation 4 (Emptyness of the core) *There are cooperative games* (N, v) *with empty core.*

Theorem 5 (Weak Bondareva-Shapley) *Cooperative game* (N, v) *has non-empty core if and only if*

$$v(N) \geq \sum_{S\subseteq N} y_S v(S) \text{ for all feasible } y \in \mathbb{R}^{(2^n-1)}.$$

Definition 6 *For a cooperative game* (N, v)*, the Shapley value* $\varphi(v)$ *of player* i *is*

$$\varphi_i(v) = \sum_{S\subseteq N\setminus i} \frac{s!(n-s-1)!}{n!} \left(v(S\cup i) - v(S)\right).$$

Definition 7 (Marginal vector) *For cooperative game* (N, v) *and permutation* $\sigma \in \Sigma_n$ *is* m_v^σ *marginal vector, where* $(m_v^\sigma)_i = v\left(S_{\sigma(i)} \cup i\right) - v\left(S_{\sigma(i)}\right)$ *and* $S_{\sigma(i)} = \{j \in N \mid \sigma(j) < \sigma(i)\}$*.*

Definition 8 (Weber set) *For cooperative game* (N, v) *the Weber set is*

$$\mathcal{W}(v) = conv\{m_v^\sigma \mid \sigma \in \Sigma_n\}.$$

Lemma 9 (Shapley value and Weber set) *For a cooperative game* (N, v) *it holds:*

$$\varphi(v) \in \mathcal{W}(v).$$

Moreover, $\varphi(v)$ *is the center of gravity of* $\mathcal{W}(v)$*.*

Theorem 10 (Weber set and core) *For every cooperative game* (N, v)*, it holds* $\mathcal{C}(v) \subseteq \mathcal{W}(v)$*.*

Definition 11 (Classes) *The cooperative game* (N, v) *is said to be*

- *monotonic game* $\equiv (S \subseteq T \subseteq N)\,(v(S) \leq v(T))$*.*
- *superadditive game* $\equiv (S, T \subseteq N, S \cap T = \emptyset)\,(v(S) + v(T) \leq v(S \cup T))$*.*
- *convex game* $\equiv (S, T \subseteq N)\,(v(S) + v(T) \leq v(S \cap T) + v(S \cup T))$*.*

- *essential game* $\equiv v(N) \geq \sum_{i \in N} v(i)$.
- *balanced game* $\equiv \mathcal{C}(v) \neq \emptyset$.

Theorem 12 (Balanced and essential) *Balanced cooperative games are essential.*

Theorem 13 (Convex and superadditive) *Convex cooperative games are superadditive.*

Theorem 14 (Core of convex games) *For a convex cooperative game (N, v), it holds $\mathcal{C}(v) = \mathcal{W}(v)$.*

Corollary 15 (Shapley value and convex games) *For a convex cooperative game (N, v), it holds:*

1. $\varphi(v) \in \mathcal{C}(v)$.
2. $\varphi(v)$ *is the centre of gravity of* $\mathcal{C}(v)$.

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On Structure of Cooperative Games
as part of series Cooperative game theory

Introduction

Many problems, ranging from cost sharing to AI explainability, are modelled using cooperative game theory. The goal is to decide if cooperation is rational in a given situation and how the profit should be divided. However, the applications are limited to only selected cases with relatively low number of players. This is because to distribute the profit, values of all sub-coalitions of players is required — this being exponential in the number of players n. Incomplete cooperative game theory offers a way to describe the game using only selected coalition values. Additional structure of the game can be leveraged to obtain more information about the missing values.

However, the lack of full information can be used by each player in order to bargain with the others about his payoff. This in turn makes it difficult to agree on the profit distribution. A central *arbiter* may want to limit such exploitation by gathering more information about the game, i.e. obtaining values of coalitions which were previously unknown. The arbiter wants to choose such coalitions that, after their values are revealed, the strategic behaviour is limited as much as possible. This "coalition revealing strategy" offers insights into which coalitions are most important for a given class of games.

Background

Definition 1 (Cooperative game) *A* cooperative game *is an ordered pair* (N, v) *where* $N = \{1, \dots n\}$ *is the set of* players *and* $v\colon 2^N \to \mathbb{R}$ *is the* characteristic function *of the cooperative game. Further,* $v(\emptyset) = 0$.

Definition 2 (Coalition) *A* coalition *is any set* $S \subseteq N$. *Coalitions of size* 1 *are called* singletons *and* N *is the* grand coalition.

Definition 3 *The game* (N, v) *is said to be*

- monotone $\equiv (\forall S \subseteq T \subseteq N)(v(S) \leq v(T))$,
- super-additive $\equiv (\forall S, T \subseteq N : S \cap T = \emptyset)(v(S) + v(T) \leq v(S \cup T))$,
- convex $\equiv (\forall S, T \subseteq N)(v(S) + v(T) \leq v(S \cap T) + v(S \cup T))$.

The classes of *monotone, super-additive,* and *convex* games with n players are denoted $\mathbb{M}^n$, $\mathbb{S}^n$ and $\mathbb{C}^n$, respectively.

Definition 4 (Payoff vector) *Let* (N, v) *be a cooperative game. Then the* payoff vector *is* $x \in \mathbb{R}^n$, *where* x_i *represents the payoff of player* i. *The payoff vector is*

- efficient $\equiv x(N) := \sum_{i \in N} x_i = v(N)$,
- individually rational $\equiv (\forall i)(x_i \geq v(i))$.

Definition 5 (Solution concept) *Let* (N, v) *be a cooperative game. A* solution concept *of* (N, v) *is any function* $\mathcal{S} : \Gamma^n \to 2^{\mathbb{R}^n}$.

A solution concept essentially assigns each game a set of possible payoff vectors.

Definition 6 (Shapley value) *Let (N, v) be a cooperative game. Then the* Shapley value *for player i is*

$$\varphi_i(v) := \sum_{S \subseteq N \setminus \{i\}} \frac{s!\,(n-s-1)!}{n!} \left(v(S \cup i) - v(S)\right).$$

Incomplete Cooperative Games

Definition 7 (Incomplete game) *An* incomplete game *is a cooperative game, in which we do not know the values of v for all coalitions. Formally, it is a tuple $(N, \mathcal{K}, v)$, where*

- $N = \{1, \dots, n\}$ *is the set of* players*,*
- $\mathcal{K} \subseteq 2^N$ *is the set of* coalitions with a known value*,* $\emptyset \in \mathcal{K}$*,*
- $v : \mathcal{K} \to \mathbb{R}$ *is the* characteristic function*,* $v(\emptyset) = 0$*.*

Further, an incomplete game is said to be minimal $\equiv \mathcal{K} = \{\emptyset, N\} \cup \bigcup_i \{i\}$*.*

Definition 8 (C-extension) *Let $C \subseteq \Gamma^n$ be a class of cooperative games. Then the game $(N, w) \in C$ is a C-*extension *of the incomplete game $(N, \mathcal{K}, v) \equiv \forall S \in \mathcal{K} : v(S) = w(S)$.*

*The class of all C-extensions of $(N, \mathcal{K}, v)$ is denoted by $C(\mathcal{K}, v)$. If $C(\mathcal{K}, v)$ is non-empty, then we say that $(N, \mathcal{K}, v)$ is C-*extensible*.*

Coalition Revealing Game

Each player in the game can leverage the lack of knowledge of v to increase his profit as much as possible.

Definition 9 (Gain) *Let $C \subseteq \Gamma^n$, $(N, \mathcal{K}, v) \in C$, $i \in N$ and (N, v) be a C-extension of $(N, \mathcal{K}, v)$. Then the C-*gain *is*

$$g_i^C(N, \mathcal{K}, v) = \max_{w \in C(\mathcal{K}, v)} \varphi_i(w) - \varphi_i(v) \geq 0.$$

The C-exploitability measures how much all can exploit the system.

Definition 10 (Exploitability) *Let $C \subseteq \Gamma^n$, $(N, \mathcal{K}, v) \in C$ and (N, v) be a C-extension of $(N, \mathcal{K}, v)$. Then the C-*exploitability *is*

$$E^C(N, \mathcal{K}, v) = \sum_{i \in N} g_i^C(N, \mathcal{K}, v) = \sum_{i \in N} \left(\max_{w \in C(\mathcal{K}, v)} \varphi_i(w) \right) - v(N).$$

PPO

Definition 11 (Return) *Let $r(a, s)$ be the reward gained by taking action a in state s. Then for a sequence $s_0, a_0, s_1, \dots s_T$ the return is*

$$R_T = \sum_{t=1}^{T} r(a_t, s_t).$$

Definition 12 ((q)-value) *Let $\pi : \mathcal{S} \to \Delta^{|\mathcal{A}|}$ be the policy and $\tau : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \Delta^{|\mathcal{S}|}$ be the transition functon. Then*

$$v(s) = \mathbb{E}_{a \sim \pi} \left[\sum_{t=1}^{T} r(a_t, s_t) \right],$$

$$q(a, s) = \mathbb{E}_{a \sim \pi} \left[r(a, s) + \sum_{s'} \tau(s, a, s') v(s') \right].$$

Definition 13 (PPO) *The PPO algorithm minimizes the following two losses*

$$L^{\pi}(\theta|\varphi) = -\mathbb{E}_{t\sim\mathcal{T}(\pi)}\left[\left[(R_t - v(s_t|\varphi))\frac{\pi(a_t|\theta)}{\pi(a_t|\theta_0)}\right]_{1-\varepsilon}^{1+\varepsilon}\right],$$
$$L^{v}(\varphi|\theta) = \mathbb{E}_{t\sim\mathcal{T}(\pi)}\left[(R_t - v(s_t|\varphi))^2\right],$$

along a trajectory $t \sim \mathcal{T}$ sampled under policy π.

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Interpreting model predictions via the Shapley value *as part of series* Cooperative game theory

Introduction

Understanding why a model makes a certain prediction can be as crucial as the prediction's accuracy in many applications. It creates appropriate user trust, provides insight into how a model may be improved, and supports understanding of the process being modeled. However, the highest accuracy for modern datasets is often achieved by complex models, such as deep learning models, that are difficult to interpret. This is bringing to the forefront the trade-off between accuracy and interpretability of a model's output.

In response to this problem, various methods have been proposed, but it is often unclear in which way they are related and when one method is preferable over another. This paper presents a unified approach for interpreting model predictions called SHAP (SHapley Additive exPlanations), based on results from cooperative game theory.

Properties

Definition 1 (Additive feature attribution methods) *Additive feature attribution methods have an explanation model that is a linear function of binary variables:*

$$g(z') = \varphi_0 + \sum_{i=1}^{M} \varphi_i z'_i,$$

where $z' \in \{0,1\}^M$, M is the number of simplified input features, and $\varphi_i \in \mathbb{R}$.

Property 1 (Local accuracy)

$$f(x) = g(x') = \varphi_0 + \sum_{i=1}^{M} \varphi_i x'_i$$

The explanation model $g(x')$ matches the original model $f(x)$ when $x = h_x(x')$.

Property 2 (Missingness)

$$x'_i = 0 \Longrightarrow \varphi_i = 0$$

Missingness constrains features where $x'_i = 0$ to have no attributed impact.

Property 3 (Consistency) *Let $f_x(z') = f(h_x(z'))$ and $z' \backslash i$ denote setting $z'_i = 0$. For any two models f and f', if*

$$f'_x(z') - f'_x(z' \backslash i) \geq f_x(z') - f_x(z' \backslash i)$$

for all inputs $z' \in \{0,1\}^M$, then $\varphi_i(f', x) \geq \varphi_i(f, x)$.

Theorem 2 *Only one possible explanation model g follows Definition 1 and satisfies Properties 1, 2, and 3:*

$$\varphi_i(f, x) = \sum_{z' \subseteq x'} \frac{|z'|!(M - |z'| - 1)!}{M!} [f_x(z') - f_x(z' \backslash i)]$$

where $|z'|$ is the number of non-zero entries in z', and $z' \subseteq z'$ represents all z' vectors where the non-zero entities are a subset of the non-zero entities in x'.

Theorem 3 (Shapley kernel) *Under Definition 1, the specific forms of $\pi_{x'}, L,$ and Ω that make solutions of Equation 2 consistent with Properties 1 through 3 are:*

$$\Omega(g) = 0,$$

$$\pi_{x'}(z') = \frac{(M-1)}{(M \ choose \ |z'|)|z'|(M-|z'|)},$$

$$L(f, g, \pi_{x'}) = \sum_{z' \in Z} [f(h_x^{-1}(z')) - g(z')]^2 \pi_{x'}(z'),$$

where $|z'|$ is the number of non-zero elements in z'.

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Cooperative games in stochastic form
as part of series Cooperative game theory

Introduction

In cooperative game theory we usually need to know values of all the possible coalitions of players. It is not necessarily possible to obtain all these values exactly. We could therefore define stochastic characteristic function to incorporate the randomness. Then we are able to model situation in which we need to make a decision before a random event is observed or in a situation where we just do not know exact values of coalitions. In this talk we look at a few models incorporating randomness and solution concepts which can follow from them. We focus on both the generalization of the concept of core to the stochastic games and on the approach using optimization of various types of objective functions. We try to not only to look at the solution concepts but also to compare them and discuss their advantages and drawbacks. Development of such a model is one of the possible ways how to work with uncertainty in the setting of cooperative (coalitional) games. Another approaches to the problem of missing information are topics of other talks like Incomplete or Interval games or not presented fuzzy games.

Definitions and Theorems

Definition 1 (Cooperative game in stochastic form (SCG)) *Cooperative game in stochastic form is a pair (N, v), where N is a set of players and $v : (2^N, \Omega) \longrightarrow E$ for which holds:*

- $v(S) : \Omega \longrightarrow E,\ S \subseteq N$
- Ω *possible outcomes of a random variable*
- E *measurable space (for us mostly $\mathbb{R}$)*

Definition 2 (Allocation in SCG with transfer payments) *Distributing the value $v(S)$ of a coalition S the allocation of the player i is equal to:*

$$x_i = d_i + r_i(v(S) - \mathbb{E}(v(S))),$$

where

- $d_i \in \mathbb{R}$ *and* $\sum_{i \in S} d_i = \mathbb{E}v(S)$
- $r_i \geq 0$ *and* $\sum_{i \in S} r_i = 1$

Definition 3 (Solution concept for SCG with transfer payments) *Solution concept for Cooperative game in stochastic form is given by $d_i, r_i \subseteq \mathbb{R}\ \forall i \in N$.*

Definition 4 (Model with preferences) *It is a triple $(N, v, (\succeq_i)_{i \in N})$, i.e., a model of SCG with defined preferences $\succeq_i$ over the random variables is defined for each player. Preferences over random variables:*

- *α-quantiles (inverse of a distribution function) of a random variable X denoted by u_α^X: $X \succeq Y \iff u_\alpha^X \geq u_\alpha^Y$*

- *Let* $b \in \mathbb{R}$ *then* $X \succeq Y \iff \mathbb{E}(X) + b \cdot Var(X) \geq \mathbb{E}(Y) + b \cdot Var(Y)$
- *(First order stochastic dominance) Let* F_X *be a distribution function of random variable* X *then* $X \succeq Y \iff F_X(z) \geq F_Y(z) \forall z \in \mathbb{R}$

Definition 5 (Objective function model) *It is a quadruple* $(N, v, (Cov(v(S), v(N)))_{S \subseteq N}, f)$*, where* f *is a given objective function following from the* v*. The* r_i *part of the allocation is not restricted, i.e.,* $r_i \in \mathbb{R}\ \forall i \in N$*.*

Definition 6 (Core) *If no coalition has incentive to split off for a given allocation then the allocation is in the core of the game.*

Theorem 7 (Core in the model with preferences) *Let* $\Gamma_\alpha = (N, v, (\succeq_i)_{i \in N})$ *be a game in the form of model with preferences with allocation given by Definition 2. Then coalition has no incentive to split off if and only if*

$$\sum_{i \in S} \Big(d_i + r_i(u_{\alpha_i}^{v(N)} - \mathbb{E}(v(N)) \Big) \geq max_{i \in S} u_{\alpha_i}^{v(S)}$$

Exercise 8 *Minimize* $\sum_{S \subseteq N} \mathbb{E}[e(S,x) - \overline{e}(S,x)]^2$*, where* x *is an allocation,* $e(S,x) = v(S) - x(S)$ *is an excess of the coalition* S *and* $\overline{e}(S,x) = \frac{1}{2^n - 1} \sum_{S \subseteq N} e(S,x)$ *is an average excess.*

Theorem 9 (Solution in the objective function model) *Solution for the exercise 8 is given by:*

$$r_i = \frac{1}{n} + \frac{nc_i(v) - \sum_{j \in N} c_j(v)}{n2^{n-2} Var(v(N))}, \forall i \in N$$

$$d_i = \frac{1}{n} \mathbb{E}(v(N)) + \frac{ne_i(v) - \sum j \in Ne_j(v)}{n2^{n-2}}, \forall i \in N,$$

where $c_i(v) = \sum_{S \ni i} cov(v(S), v(N))$ *and* $e_i(v) = \sum_{S \ni i} \mathbb{E}(v(S))$*.*

Additional definitions

Definition 10 (State (Scenario) model) *Let* $S = (S_1, \ldots, S_k)$ *be a set of possible states and* p_i *the probability of being in the state* S_i *with* $\sum_{i \in N} p_i = 1$*. State model is the set* S *with probabilities* $p_i\ \forall i \in N$*, where* $S_i = (N, v^i)$ *is a deterministic cooperative game.*

Definition 11 (Model with preferences II) *Model is defined as the model with preferences with allocation being:* $x_i = r_i(v(S)),\ r_i \geq 0,\ \sum_{i \in N}$*.*

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Interval cooperative games *as part of series* Cooperative game theory

Introduction

I will introduce cooperative interval games, in which worth of every coalition corresponds to closed interval. I will show 2 possible approaches to interval games, and some relations between them. First approach is based on weak ordering. Other approach based on selections. Selections are all possible outcomes of the interval game with no additional uncertainty.

I will show some results about core coincidence.

Background

Definition 1 (Interval) *An interval X is a set*

$$X := [\underline{X}, \bar{X}] = \{x \in \mathbb{R} : \underline{X} \leq x \leq \bar{X}\}$$

with $\underline{X}$ being the lower bound and $\bar{X}$ being the upper bound of the interval. By interval we mean closed interval.We denote set of real intervals by $\mathbb{IR}$

Definition 2 (Interval arithmetics) *For every $X, Y, Z \in \mathbb{IR}$ and $0 \notin Z$ define*

- $X + Y := [\underline{X} + \underline{Y}, \bar{X} + \bar{Y}]$
- $X - Y := [\underline{X} - \underline{Y}, \bar{X} - \bar{Y}]$
- $X * Y := [min(S), max(S)], S = \{\underline{X}/\bar{Y}, \bar{X}\underline{Y}, \underline{XY}, \bar{X}\bar{Y}\}$
- $X/Z := [min(S), max(S)], S = \{\underline{X}/\bar{Z}, \bar{X}/\underline{Z}, \underline{X}/\underline{Z}, \bar{X}/\bar{Z}\}$

Definition 3 (Cooperative interval game) *A cooperative interval game is an ordered pair (N, w), where $N = \{1, 2, ..., n\}$ is a set of players and $w : 2^N \to \mathbb{IR}$ is a characteristic function of the cooperative game. We further assume that $w(\emptyset) = [0, 0]$.*

The set of all interval cooperative games on a player set N is denoted by IG^N

Definition 4 (border games) *For every $(N, w) \in \mathbb{N}$, border games $(N, \underline{w}) \in G^N$ (lower border game) and $(N, \bar{w}) \in G^N$ (upper border game) are given by $\underline{w}(S) = \underline{w(S)}$ and $\underline{w}(S) = \overline{w(S)}$ for every $S \in 2^N$*

Definition 5 (Weakly better operator) *Interval I is weakly better than interval J ($J \succeq I$) if and only if $\underline{I} \geq \underline{J}$ and $\bar{I} \geq \bar{J}$.*

Definition 6 *Set of all interval imputations of $(N, w) \in G^N$:*

$$\mathcal{I}(w) := \{(I_1, I_2, ..., I_N) \in \mathbb{IR}^{\mathbb{N}} | \sum_{i \in N} I_i = w(N), I_i \succeq w(i), \forall i \in N\}$$

Definition 7 *Set of interval selection core of $(N, w) \in G^N$:*

$$\mathcal{C}(w) := \{(I_1, I_2, ..., I_N) \in \mathcal{I}(w) | \sum_{i \in S} I_i \succeq w(S), \forall S \in 2^N \setminus \emptyset\}$$

Definition 8 (Selection) *A game $(N, v) \in G^N$ is a selection of $(N, w) \in IG^N$ if for every $S \subseteq N$ we have $v(S) \in w(S)$. Set of all selections of (N, w) is denoted by $Sel(w)$*

Definition 9 (Selection interval imputations) *Set of all selection interval imputations of* $(N, w) \in IG^N$*:*

$$\mathcal{SL}(w) = \bigcup\{\mathcal{I}(v) | v \in Sel(w)\}$$

Definition 10 (Interval selection core) *Set of interval selection core of* $(N, w) \in IG^N$*:*

$$\mathcal{CL}(w) = \bigcup\{\mathcal{C}(v) | v \in Sel(w)\}$$

Definition 11 (Selection monotonic interval game) *An interval game* (N, w) *is selection monotonic if all its selections are monotonic games. The class of such games on set of* N *players is denoted by* $SeMIG^N$

Theorem 12 (Theorem 1) *An interval game* (N, w) *is selection monotonic if and only if for every* $S, T \in 2^N, S \subset T$

$$\bar{w}(S) \leq \underline{w}(T).$$

Definition 13 *The function* $gen : 2^{\mathbb{IR}^{\mathbb{N}}} \to 2^{\mathbb{R}^N}$ *maps to every set of interval vectors a set of real vectors. It is defined as:*

$$gen(S) = \bigcup_{s \in S} \{(x_1, x_2, ..., x_n) | x_i \in s_i\}$$

Theorem 14 (Theorem 4) *For every interval game* (M, w) *we have* $gen(\mathcal{C}(w)) \subseteq \mathcal{SC}(w)$

Theorem 15 (Theorem 5) *For every interval game* (N, w) *we have* $gen(\mathcal{C}(w)) = \mathcal{SC}(w)$ *if and only if for every* $x \in \mathcal{SC}(w)$ *there exist non-negative vectors* $l^{(x)}$ *and* $u^{(x)}$ *such that*

4.1

$$\sum_{i \in N} (x_i - l_i^{(x)}) = \underline{w}(N),$$

4.2

$$\sum_{i \in N} (x_i + u_i^{(x)}) = \overline{w}(N),$$

4.3

$$\sum_{i \in S} (x_i - l_i^{(x)}) \geq \underline{w}(S), \forall S \in 2^N \setminus \{\emptyset\},$$

4.4

$$\sum_{i \in S} (x_i + u_i^{(x)}) \geq \overline{w}(S), \forall S \in 2^N \setminus \{\emptyset\}.$$

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Cooperative games with skills in Open Anonymous Environments

Introduction

Traditional models of cooperative game theory consider agents as entities without more subtle differentiation. Agent as a whole brings some value to already existing coalition and this value is represented in the characteristic function. Instead, we can see agents not as “magical” units but as entities with some sets of skills and these skills they bring to the coalitions. This model allows us to represent more fine-grained picture of reality.

In open anonymous environments (such as internet), it is very easy for agents to collude (and create a bigger, non-existing agent) which has both their capabilities, separate themselves (and create smaller agents) or hide some of their skills in order to increase their profits. Traditional solution concepts are vulnerable to these 3 ways of manipulation and new concepts based on traditional ones are proposed.

Basic definitions

Definition 1 (Skills and agents) *Let T be the set of all possible skills. Each agent t has a subset of skills $S_t \subseteq T$. We assume that the skills are unique: $\forall t \neq u, S_t \cap S_u = \emptyset$.*

Definition 2 (Characteristic function over skills) *A characteristic function $v : 2^T \to \mathbb{R}$ assigns a value to each set of skills.*

Definition 3 (Hiding skills) *If agent i has a set of skills S_i, for any $S'_i \subseteq S_i$, it can declare that it has only S'_i.*

Definition 4 (False names or separation) *Agent i can use multiple identifiers and declare that each identifier has a subset of skills S_i. Since we assume each skill is unique, two different identifiers cannot declare they have the same skill. Thus, a false-name manipulation by agent i corresponds to a partition of S_i into multiple identifiers. (If the manipulation is combined with a skill-hiding manipulation, only a subset of S_i is partitioned.)*

Definition 5 (Collusion) *Multiple agents can collude and pretend to be a single agent. They can declare the skills of this agent to be the union of their skills (or a subset of this union, in case we combine the manipulation with a skillhiding manipulation).*

Definition 6 (Solution concept - Shapley value) *Give an ordering o of the set of agents W in the coalition, let $X(o, i)$ be the set of agents in W that appear before i in ordering o. Then the Shapley value for agent i is defined as*

$$Sh(W, i) = \frac{1}{|W|!} \sum_{o} \left(w(X(o, i) \cup \{i\}) - w(X(o, i)) \right)$$

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Where are graphs in cooperative games?
as part of series Cooperative game theory

Introduction

In this talk, you will see graph theory being applied in the analysis of cooperative games. The problem with the standard model of a cooperative game arises when not all of the coalitions $S \subseteq N$ are feasible. We restrict to coalitions where each player is known by at least one other player from the coalition. Under this restriction, the goal is to study an equivalent of the Shapley value. You will see a derivation of a so called *graph value*, which, in a sense, generalises the Shapley value.

Games and graphs

Let $\mathcal{A}(G) \subseteq 2^N$ be the set of all subsets of vertices representing connected subgraphs of a graph $G = (N, E)$.

Definition 1 (Game on graph) *A characteristic function* $v\colon \mathcal{A}(G) \to \mathbb{R}$ *is a* coalitional game *on* G*. Further,* $v(\emptyset) = 0$*.*

We denote $\mathcal{G}$ the set of all coalitional games on graphs with vertices N.

Definition 2 (Automorphism) *A permutation* $\pi \in \Pi_N$ *is an* automorphism *of* G *if* $\pi(S) \in \mathcal{A}(G)$ *for all* $S \in \mathcal{A}(G)$*.*

Lemma 3 *A permutation* π *is an automorphism of* G *if and only if for every pair* $i, j \in N$*,* $(i, j) \in E$ *implies that* $(\pi(i), \pi(j)) \in E$*.*

The Shapley value satisfies several axioms for which we define their weaker forms for games on graphs.

Lemma 4 *Let* φ_i *be a value for* i *on* $\mathcal{G}$ *satisfying linearity. Then there is collection of constants* $\{a_S\}_{S\in\mathcal{A}(G)}$ *such that for all* $v \in \mathcal{G}$*,*

$$\varphi_i(v) = \sum_{S\in\mathcal{A}(G)} a_S v(S).$$

Lemma 5 *Let* φ_i *be a value for* i *on* $\mathcal{G}$ *satisfying linearity, dummy and monotonity axioms. Then* φ_i *is a probabilistic value, i.e.*

$$\varphi_i(v) = \sum_{S\in\mathcal{A}(G)^{-i}} p^i_S \left[v(S \cup i) - v(S)\right],$$

where $\{p^i_S\}$ *is a probability distribution over* $S \in \mathcal{A}(G)^{-i}$*.*

Definition 6 (Graph value) *A value* $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ *over graph* $G = (N, E)$ *is a* graph value *if it satisfies linearity, dummy, monotonicity, efficiency and symmetry axioms.*

Theorem 7 *A value* φ *over* G *is a graph value if and only if it is a random order value (a special probabilistic value), whose weights are* $Aut(G)$ *invariant.*

Corollary 8 *For a complete graph* G*, there is a unique graph value equivalent to the Shapley value.*

Corollary 9 *There is graph* G *such that the graph value is not unique.*

Corollary 10 *There is graph* G *such that the graph value is unique but not equal to the Shapley value.*

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Feasible reasoning *as part of series* Introduction to bounded arithmetic

Introduction

Bounded arithmetic is a branch of mathematical logic that studies the strength and limitations of the restricted forms of arithmetic. It serves as a unifying framework for a number of different areas of mathematics and computer science, including model theory, proof complexity, circuit complexity, TFNP search problems and classical complexity theory. Historically, it has also played a role in the formalization of *feasible reasoning*. In this talk I will introduce the basic concepts of bounded arithmetic together with the context needed to understand the theory. I will discuss some classical results and mention open problems in the area. This talk is part of the series *Introduction to bounded arithmetic*.

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Complexity theory through the eyes of bounded arithmetic
as part of series Introduction to bounded arithmetic

Introduction

What if we show that P=NP cannot be proven? In the second talk of this series, we will more thoroughly explore the relationship between the hierarchy of fragments of bounded arithmetic and the polynomial hierarchy. We will then introduce the theories based on Cook's PV then talk about the strength of PV_1 and recent results which show that it cannot prove P to be computable with $\mathrm{SIZE}[n^k]$ circuits for a fixed k.

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Introduction to Proof Complexity
as part of series Topics in Proof Complexity

Introduction

Proof complexity is a field on the border of logic and computational complexity. I will talk about the basic goals and results of this field. I will show a connection with games and, if time permits, with graph pebbling.

Proof systems

Definition 1 *A* literal *is a variable or its negation. A* clause *is a disjunction of literals. A* term *is a conjunction of literals. A* CNF *is a formula which is a conjunction of clauses. A* DNF *is a formula which is a disjunction of terms. A formula is* satisfiable *if for at least one assignment to variables the formula evaluates to 1. A formula is a* tautology *if for every assignment to variables the formula evaluates to 1. The set of all DNF tautologies is denoted by* TAUT.

Definition 2 *For $m > n$, let PHP^m_n denote the unsatisfiable CNF consisting of the following clauses:*
Pigeon axioms: *Each of the m pigeons sits in at least one of n holes*

$$x_{i,1} \lor x_{i,2} \lor \cdots \lor x_{i,n} \quad \text{for every } i = 1, \ldots, m.$$

Hole axioms: *No two pigeons sit in one hole*

$$\neg x_{i_1,j} \lor \neg x_{i_2,j} \quad \text{for every } i_1 \neq i_2 \text{ and } j = 1, \ldots, n.$$

Definition 3 *A* propositional proof system *is a binary relation $Q \subseteq \{0,1\}^* \times \{0,1\}^*$ such that*

- *Q is decidable in polynomial time,*
- *for any α, w, if $Q(\alpha, w)$ holds then $\alpha \in \mathrm{TAUT}$, and*
- *for any $\alpha \in \mathrm{TAUT}$ there is $w \in \{0,1\}^*$ such that $Q(\alpha, w)$ holds.*

The system is p-bounded *if there exists $c \geq 1$ such that every $\alpha \in \mathrm{TAUT}$ has a proof of length at most $|\alpha|^c + c$.*

Definition 4 (Resolution) *A resolution refutation of a CNF F is a sequence of clauses $C_1, \ldots, C_\ell$ such that each C_j is either one of the clauses of F, or it is derived from some C_{j_1} and C_{j_2} with $j_1, j_2 < j$ using the resolution rule*

$$\frac{C \lor p \quad D \lor \neg p}{C \lor D},$$

where $C_{j_1} = C \lor p$, $C_{j_2} = D \lor \neg p$ and $C_j = C \lor D$. The last clause of the refutation C_ℓ is the empty clause.

Definition 5 (Nullstellensatz proof system) *Let $\mathbf{F}$ be a field. Consider a CNF φ with clauses $C_1, \ldots, C_m$ over variables $x_1, \ldots, x_n$. We translate each clause C_j into a polynomial equation $P_j = 0$. For example $x_1 \lor \neg x_2 \lor x_3$ becomes $(1 - x_1)x_2(1 - x_3) = 0$.*

Then the NS/$\mathbf{F}$-refutation of φ is a set of polynomials $Q_1, \ldots, Q_m$ and $R_1, \ldots, R_n$ satisfying

$$\sum_{j=1}^{m} Q_j P_j + \sum_{i=1}^{n} R_i(x_i^2 - x_i) = 1.$$

Theorem 6 (Cook-Reckhow) *A p-bounded proof system exists iff* $\mathsf{NP} = \mathsf{coNP}$.

Lower bound for tree-like resolution

Theorem 7 *For any* $m > n$*, any tree-like resolution refutation proof of* PHP_n^m *has size* $2^{\Omega(n \log n)}$.

Definition 8 ((a, b)-game) *We have* a *and* b *such that* $1/a + 1/b = 1$*. The game has two players, Prover and Delayer. The game starts with an empty partial assignment. The game proceeds in rounds, in each round:*

- *Prover suggests a variable* x_i *to be set in this round, and*
- *Delayer either chooses a value 0 or 1 for* x_i*, or leaves the choice to Prover.*
- *The number of points earned by Delayer is*
  - *0 if Delayer chooses the value for* x_i*,*
  - $\log_2 a$ *if Prover sets* x_i *to 0, and*
  - $\log_2 b$ *if Prover sets* x_i *to 1.*

The game ends when some of the clauses of F *is falsified.*

Lemma 9 *Let* F *be an unsatisfiable CNF. If Delayer can earn* r *points in some* (a, b)*-game, then any tree-like resolution refutation proof of* F *has size at least* 2^r*.*

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Presented paper by Mika Göös, Pritish Kamath, Robert Robere and Dmitry Sokolov
Adventures in Monotone Complexity and TFNP
as part of series Topics in Proof Complexity
(https://drops.dagstuhl.de/opus/volltexte/2018/10131/)

TFNP: the class of total efficiently verifiable search problems

The field of computational complexity has traditionally been focused on *decision problems*: Given a language $\mathcal{L} \subseteq \{0, 1\}^*$ and some input $x \in \{0, 1\}^*$, how hard is it to decide whether $x \in \mathcal{L}$? Even if this paradigm seems to be as general as possible, there are computational tasks that don't fit in. One such example are *search problems* that are *total*:

1. Given an integer $n \in \mathbb{N}$, *find* its prime factors.
2. Given a two player game, *find* a Nash equilibrium.
3. Given a cubic graph G with an Hamiltonian cycle, *find* a different Hamiltonian cycle of G.

Each of those problem is guaranteed to have a solution by virtue of mathematical proof; thus, their decision-counterpart are moot. Indeed, "does game A have a Nash equilibrium?" is trivial to answer (it's yes!). Still, it seems those tasks *do* carry some computational hardness: People have been trying to solve Nash equilibrium problems to no avail for more than half a century.

Definition 1 *Let $S \subseteq \{0, 1\}^* \times \{0, 1\}^*$, be a poly-time computable relation. $S \in$ TFNP if for any $x \in \{0, 1\}^*$, there exists some $y \in \{0, 1\}^{\text{poly}(|x|)}$ with $S(x, y)$.*

Definition 2 *Let $S, R \in$ TFNP, $S \leq R$ if there exists a pair of poly-time computable function f, g such that for any $x \in \{0, 1\}^*$, if $(f(x), z) \in R$, then $(x, g(x, z)) \in S$.*

Since TFNP is a semantic class, it is unlikely to have any complete problem. To circumvent this limitation, subclasses of TFNP have been invented based on the type of mathematical proof guaranteeing the totality of the problem. For instance, the class PPA is based on the *parity principle*: Any graph has an even number of odd-degree vertices. More formally, we define the problem Leaf as taking for input a small circuit $C : \{0, 1\}^n \to \{0, 1\}^{2n}$ that defines implicitly the graph $G = (V, E)$ with $V = \{0, 1\}^n$ and $\{u, v\} \in E$ if $u \in C(v)$ and $v \in C(u)$. A solution to a Leaf circuit is the vertex 0^n if it has even degree or any other vertex $v \neq 0^n$ with odd degree. Note that Leaf $\in$ TFNP because the existence of a solution is guaranteed by the parity principle. PPA is then simply defined as PPA $:= \{S \in \text{TFNP} : S \leq \text{Leaf}\}$. The hardness of solving Leaf stems from the fact that short instances represent exponential-sized graphs. Thus, any algorithm trying to solve Leaf which only uses the circuit as a *black-box* is bound to make an exponential number of queries to the circuit. On the other hand, we do not (yet) know how to take advantage of knowing the internals of a circuit – this might explain that there is no known efficient algorithm for PPA.

Yet another example of encoding a totality principle in an exponentially-sized graph is the class PLS which encodes "any non-empty acyclic DAG must have a sink"; Refer to Figure 1 for an overview of the TFNP hierarchy. The theory of TFNP has proven to be successful at pinning down the complexity of many tasks, perhaps culminating in the proof of PPAD-completeness for

Nash equilibriums [5, 6]. Beyond completeness results, a theoretician's dream would be to *separate classes*. For instance, we know that PPAD $\subseteq$ PPA but could it be that PPAD = PPA? Proving PPAD $\subsetneq$ PPA unconditionally would show P $\neq$ NP [7]. On the other hand, most known reductions within TFNP are *black-box*. A reduction (f, g) from S to R (where S and R are problems defined by circuits) is said to be **black-box** if f and g treat the circuit of S exclusively as an oracle and restrain from looking at its internals. On a high level, such reductions are simply *decision trees*, where queries are made adaptively to S. If it is out of reach to separate TFNP subclasses, we might still hope for the next best thing: can we rule out *black-box* reductions? The answer is yes, and the techniques involved are worth to look at.

Decision-tree analogues of TFNP

Query analogues of TFNP subclasses will be denoted by appending a *dt* superscript, e.g. PLS^{dt} is the query analogue of PLS. Since decisions tree are non-uniform models of computation, problems in TFNP^{dt} are actually sequences of relation $S = (S_n)_{n\in\mathbb{N}}$ where each $S_n \subseteq \{0, 1\}^n \times \{0, 1\}^n$ is *total* and *efficiently verifiable*. In our model, efficiently verifiable means that for each y, there exists a decision tree o_y that can decide $S(x, y)$ by querying at most polylog(n) bits of x. Note that the logarithm is natural here: In the Leaf problem, the input circuit had size poly(n) but could be probed on an exponential number $N = 2^n$ of points. Still, Leaf $\in$ TFNP because only poly(n) = polylog(N) numbers of circuit-queries are necessary to check that a vertex has odd-degree. PPA^{dt} could be defined by carefully crafting a decision-tree analogue of Leaf and then define a notion of decision-tree reductions within TFNP^{dt}. Instead, we will use the more convenient (but equivalent) notion of PPA-formulation.

Definition 3 *Fix some $S_n \subseteq \{0, 1\}^n \times \{0, 1\}^n \in \text{TFNP}^{dt}$. A PPA-**formulation** of S is a tuple $\mathcal{F} = (V, \{t_v\}_{v\in V}, \{o_v\}_{v\in V})$ where each t_v is a decision tree over $\{0, 1\}^n$ with labels in V^2 and each o_v is a decision tree over $\{0, 1\}^n$ with labels in $\{0, 1\}^n$. On input $x \in \{0, 1\}^n$, $\{t_v\}$ implicitly defines a graph $G_x = (V, E_x)$ where $\{u, v\} \in E_x$ if $u \in t_v(x)$ and $v \in t_u(x)$. $\mathcal{F}$ is **correct** if for any x and any Leaf solution u of G_x it holds that $S(x, o_u(x)) = 1$. The cost of the formulation is defined as:*

$$\text{cost}(\mathcal{F}) = \max_{v\in V}\{\text{depth}(t_v)\} + \max_{v}\{\text{depth}(o_v)\} + \log(|V|)$$

$\text{PPA}^{dt}(S)$ *is defined as the least cost of a* PPA*-formulation of S. Finally,* $\text{PPA}^{dt} = \{S \in \text{TFNP}^{dt} : \text{PPA}^{dt}(S) = \text{polylog}(n)\}$.

Note that this definition carries the essence of black-box reductions and upon further inspection, the family $\{t_v\}$ essentially implements the function f and $\{o_v\}$ the function h in the original definition of reduction within TFNP. One difference is that the reduction is allowed to be non-uniform (i.e. be different for each n). It turns out the above definition is strong enough for our purpose. For instance, showing $\text{PPAD}^{dt} \subsetneq \text{PPA}^{dt}$ rules out the possibility of a black-box reduction from PPA to PPAD. Actually, something slightly stronger holds: this separation in the query-world implies the existence of a generic oracle $\mathcal{O}$ such that $\text{PPA}^{\mathcal{O}} \subsetneq \text{PPAD}^{\mathcal{O}}$ (see [1]). We are now left with proving lower-bounds for PPA^{dt}.

Characterizations and separations of query analogues

There is a rich connection between query analogues of TFNP subclasses and various proof systems. As a first hint, observe that any CNF contradiction $\varphi = C_1 \wedge \cdots \wedge C_m$ can be re-casted as a total search problem. If the variables of φ are $x_1, \ldots, x_n$, define $S(\varphi) \subseteq \{0, 1\}^n \times [m]$ as the following search problem: Given $x \in \{0, 1\}^n$, *find* some C_i which is not satisfied by x. If each clause is of size polylog(n), then $S(\varphi) \in \text{TFNP}^{dt}$ because verifying a clause is dissatisfied amounts to querying its literals. In the converse direction, a TFNP^{dt} problem can be encoded as a low-width CNF contradiction that roughly says "this instance has no solution". Those observations go even further: it turns out proof systems *characterize* query-analogues of TFNP!

Theorem 4 *For any low-width CNF contradiction φ:*

1. $\mathrm{PPA}^{dt}(S(\varphi)) = \mathbb{F}_2 - \mathrm{NSdegree}(\varphi)$ *[3]*
2. $\mathrm{PLS}^{dt}(S(\varphi)) = \mathrm{ResolutionWidth}(\varphi)$ *[2]*

Most other classes in TFNP also have a corresponding proof system, see Figure 1 for an overview. Those characterizations come in very handy, because results from proof complexity (a very rich field) can be directly imported into search problems theory, as the following showcases.

Theorem 5 $\mathrm{PLS}^{dt} \not\subseteq \mathrm{PPA}^{dt}$ *and* $\mathrm{PPA}^{dt} \not\subseteq \mathrm{PLS}^{dt}$*.*

Proof The first result is originally due to [2], but a direct proof can be obtained from the above characterizations. Indeed, [4] shows that there exists a family of constant-degree acyclic DAGs $\{G_n\}_{n\in\mathbb{N}}$ with reversible pebbling number $\mathrm{RP}(G_n) \geq \Omega(n^{1/3})$. Let φ_G be the corresponding pebbling contradiction. A theorem from this set of notes shows $\mathbb{F}_2 - \mathrm{NS}(\varphi_G) = \Theta(\mathrm{RP})(G)$ so that using the above characterization, $\mathrm{PPA}^{dt}(S(\varphi_G)) \geq \Omega(n^{1/3})$, or in other words $S(\varphi_G) \notin \mathrm{PPA}^{dt}$. On the other hand, φ_G is a Horn formula (at most one positive literal per clause) so that φ_G has a constant-degree resolution proof and $S(\varphi_G) \in \mathrm{PLS}^{dt}$. For the second result, one can use Tseitin's formula (which encode the parity principle). □

It also happens that results within the structure of TFNP imply new discoveries in proof complexity. One such example is the recent collapse $\mathrm{SoPL} = \mathrm{PLS} \cap \mathrm{PPADS}$ (which holds in the query world) [9]. Using the corresponding characterization theorem, we get that a low-width refutation CNF has a low-width resolution proof *and* a low-degree unary Sheralli-Adams proof if and only if it has a low-width reversible resolution proof [8].

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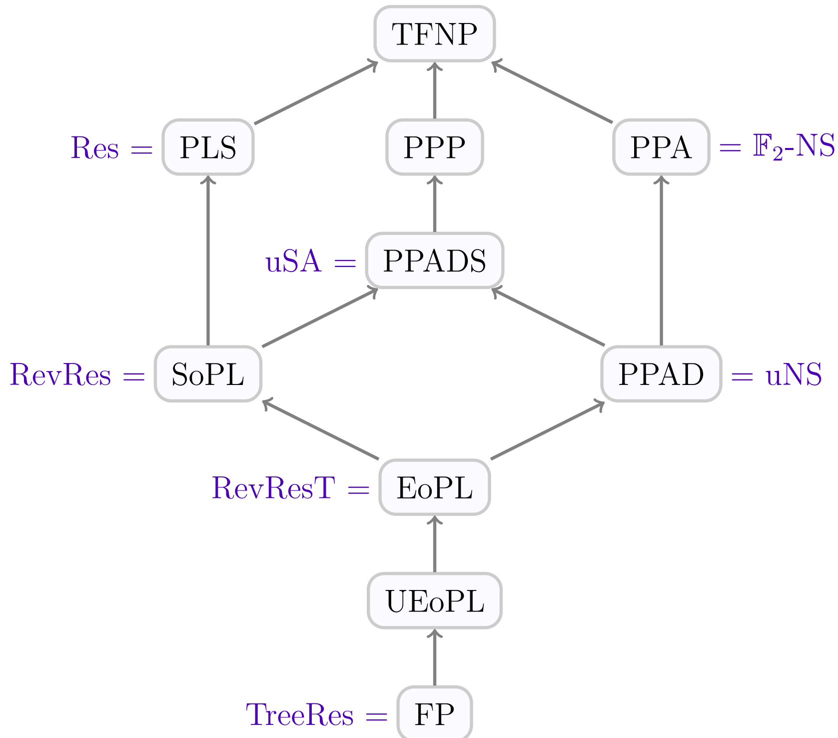

Figure 1: A subset of the TFNP hierarchy together with the corresponding proof systems (if available). All the above classes are separated with respect to black box reductions. Figure taken from [8]

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A Combinatorial Characterization of Resolution Width *as part of series* Topics in Proof Complexity

(https://www.sciencedirect.com/science/article/pii/S0022000007000876)

Introduction

Suppose we are given a proof system and a statement α under a reasonable formalization of mathematics, say, sequent calculus of first-order logic. Then, in order to prove $\nvdash \alpha$ (nonexistence of a proof of α), the only thing we need to do is to find a model which satisfies $\neg\alpha$. This is remarkable since nonexistence of a combinatorial object "proof" is characterized by existence of an algebraic object "model."

Now, what if we consider "$\nvdash_n \alpha$" (α does not have a proof of *complexity* $\leq n$)? Of course this question is informal and its rigorous meaning depends heavily on our formalization of *mathematical proofs* and the *complexity measure* of a proof. At least, if we focus on propositional logic and *resolution* proof system, and think *width* as the complexity measure, then we obtain a satisfactory characterization of nonexistence of a short proof, namely, existence of a winning strategy for *Liar* in a kind of pebbling games related to α.

In this talk, we first introduce *resolution* and *width* and see why *width* is important in the study of proof complexity, following the exhibition of [3]. Then we prove the characterization above and go through its applications according to [1]. In the course of it, we will also touch briefly other estimation techniques for proof complexity of resolution.

Preliminaries

Resolution is a propositional proof system which has been studied very vigorously due to its importance in SAT solving and automated theorem proving. It is interesting also because it can be regarded as *sequent calculus* whose formulae in each sequent are all restricted to be literals.

We fix a countable set $\mathcal{V} = \{p_0, p_1, \ldots\}$ of *(propositional) variables* and $\bar{\mathcal{V}} = \{\bar{p_0}, \bar{p_1}, \ldots\}$ of *the negations (or, the complements) of variables* throughout this talk.

Definition 1 (clause) *A clause is a finite subset of* $\mathcal{V} \cup \bar{\mathcal{V}}$.

Definition 2 (partial assignments) *A partial (truth) assignment is a map from a subset of* $\mathcal{V}$ *to* $\{0,1\}$. *If the domain is the whole* $\mathcal{V}$, *it is simply called an assignment. For a partial assignment* f, *we denote its domain by* $\mathrm{Dom}(f)$.

Definition 3 (natural extension) *Let* $f\colon \mathrm{Dom}(f) \to \{0,1\}$ *be a partial assignment. Then we denote its natural extension to a map*

$$\mathcal{V} \cup \bar{\mathcal{V}} \to \{0,1\} \cup \mathcal{V} \cup \bar{\mathcal{V}}$$

by $\tilde{f}$.

Definition 4 (restriction, and disjunctions as clauses) *Let* C *be a clause and* $f\colon \mathrm{Dom}(f) \to$

$\{0,1\}$ *be a partial assignment. The* restriction $C \restriction_f$ *is defined as follows:*

$$C \restriction_f := \begin{cases} 1 & \text{(if there exists } l \in C \text{ such that } \tilde{f}(l) = 1\text{)} \\ 0 & \text{(if each } l \in C \text{ satisfies } \tilde{f}(l) = 0\text{)} \\ \tilde{f}(C) & \text{(otherwise)} \end{cases}$$

When $C \restriction_f$ *is* 0 *(resp.*1*), we say* C *is* falsified *(resp.* satisfied*)* by f*.*

Definition 5 (CNFs as sets of clauses) *Let* $\mathcal{C}$ *be a set of clauses.* $\mathcal{C}$ *is* satisfiable *if and only if there exists an assignment which satisfies the all* $C \in \mathcal{C}$*.*

Definition 6 (resolution) *Let* $\mathcal{C}$ *be a set of clauses and* D *be a clause. A* (dag-like) resolution derivation of D from $\mathcal{C}$ *is a finite sequence*

$$\pi = \langle C_1, r_1 \rangle, \ldots, \langle C_s, r_s \rangle$$

of pairs of a clause C_i *and a* rule indicator r_i*, which is a two-element set, such that every pair* $\langle C_i, r_i \rangle$ *satisfies the following:*

Axiom: *If* $r_i = \emptyset$*, then* $C_i \in \mathcal{C}$*.*

Weakening: *If* $r_i = \{j\}$*, then* $j < i$ *and* $C_j \subset C_i$*.*

Resolution: *If* $r_i = \{j, k\}$ *such that* $j \neq k$*, then* $j, k < i$ *and* C_j, C_k, C_i *have the following forms:*

$$C_j = X \sqcup \{p_t\}, C_k = Y \sqcup \{\bar{p_t}\}, C_i = X \cup Y.$$

Conclusion: $C_s = D$*.*

We denote the situation by $\pi \colon \mathcal{C} \vdash D$*. In particular, when* $D = \emptyset$*, we say* π *is a* resolution refutation of $\mathcal{C}$*.*

Note 7 (underlying graph) *It is a simple observation that a resolution derivation can be regarded as a directed acyclic graph with each vertex labeled by a clause. Each* r_i *corresponds to the arrows from a vertex labeled with* C_i*.*

Definition 8 (tree-like resolution) *Let* π *be a dag-like resolution derivation. When the underlying graph of* π *is a directed tree, we say* π *is* tree-like*.*

Proposition 9 *Resolution gives a propositional proof system (in the sense of Cook and Reckhow).*

Note 10 *Strictly speaking, we have to "extend" resolution to enable it to deal with propositional formulae of arbitrary depth (not just CNFs or DNFs). However, we do not get into it this time although we see the trick we need here later in Example 18.*

Definition 11 (complexity measures) *Let* $\pi = \langle C_1, r_1 \rangle, \ldots, \langle C_s, r_s \rangle$ *be a resolution derivation. Then set:*

$$\mathbf{size}(\pi) := \sum_{i=1}^{s} |C_i|,$$

$$\mathbf{width}(\pi) := \max_{i=1,\ldots,s} |C_i|.$$

Furthermore, for an unsatisfiable set $\mathcal{C}$ *of clauses, define*

$$\mathbf{minsize}(\mathcal{C} \vdash \emptyset) := \min_{\pi \colon \mathcal{C} \vdash \emptyset} \mathbf{size}(\pi),$$

$$\mathbf{minwidth}(\mathcal{C} \vdash \emptyset) := \min_{\pi \colon \mathcal{C} \vdash \emptyset} \mathbf{width}(\pi).$$

Moreover, we abuse the notation **width** *and set:*

$$\mathbf{width}(\mathcal{C}) := \max_{C \in \mathcal{C}} |C|.$$

Let $\mathcal{C}$ be a set of clauses and D be a clause and $k \in \mathbb{N}$. We write $\mathcal{C} \vdash_k D$ if and only if there exists $\pi\colon \mathcal{C} \vdash D$ such that $\mathbf{width}(\pi) \leq k$.

Example 12 *The pigeonhole principle for m pigeons and n holes ($m > n$) can be expressed by unsatisfiability of a set of clauses (let us denote it by PHP^m_n). We will exhibit a resolution refutation of it for small parameters in the talk.*

Why resolution width?

In the context of proof complexity, the first thing we want to consider is to estimate $\mathbf{minsize}(\mathcal{C} \vdash \emptyset)$. However, it turns out that the analysis of $\mathbf{minwidth}(\mathcal{C} \vdash \emptyset)$ is actually helpful for that purpose:

Theorem 13 ([2]) *Let $\mathcal{C}$ be an unsatisfiable set of clauses which contains n variables. Then the following holds:*

$$\mathbf{minsize}(\mathcal{C} \vdash \emptyset) \geq \exp\left(\Omega\left(\frac{(\mathbf{minwidth}(\mathcal{C} \vdash \emptyset) - \mathbf{width}(\mathcal{C}))^2}{n}\right)\right).$$

Furthermore, for a minimum size s of tree-like refutations of $\mathcal{C}$, the following holds:

$$s \geq 2^{(\mathbf{minwidth}(\mathcal{C} \vdash \emptyset) - \mathbf{width}(\mathcal{C}))}.$$

We also stress that there are other reasons why resolution width is important:

- It is tightly related to a complexity measure of Polynomial Local Search (PLS).
- The quantitative provability $\vdash_k$ has a good semantics, which will be presented in the next section.

Model theory of $\vdash_k$

Given an unsatisfiable set $\mathcal{C}$ of clauses, consider the following two-player game:

- There are two players; named *Liar* and *Prover*.
- *Liar* "lies" as if they had a satisfying assignment of $\mathcal{C}$, and *Prover* wants to "prove," by asking the truth value of each variable, that *Liar* is actually lying.
- If *Prover* had an unlimited ability to make queries, the "game" would become trivial. So, we parametrize the game by $k \in \mathbb{N}$.

  At the beginning of the game, *Prover* has k-pebbles in their pocket, and now the play proceeds in turn, starting from *Prover*:

  1. *Prover* chooses one pebble from their pocket and put it on a variable. Before doing it, it is allowed for *Prover* to remove several pebbles already played and put them back in their pocket (therefore, the pebbles can be reused during the game).
  2. Then *Liar* answers the truth value of the variable and passes the turn back to *Prover*.
  3. The play continues until the partial truth assignment with domain induced by the pebbles and values defined by the last answers of *Liar* falsifies some $C \in \mathcal{C}$. In that case, *Prover* wins the game. Otherwise, that is, if *Liar* can survive endlessly, then *Liar* wins.

This game serves as the semantics for $\vdash_k$. To be precise, the game is rigorously formalized as follows:

Definition 14 *Let $\mathcal{C}$ be an unsatisfiable set of clauses, and $k \in \mathbb{N}$.*

We say Liar wins Boolean existential k-pebble game on $\mathcal{C}$ if and only if there exists a family $\mathcal{H}$ of partial truth assignments which satisfies the following:

1. $\mathcal{H} \neq \emptyset$.
2. *If $f \in \mathcal{H}$ and $C \in \mathcal{C}$, then $C \restriction_f \neq 0$.*
3. *If $f \in \mathcal{H}$, $|\mathrm{Dom}(f)| \leq k$.*
4. *If $g \subset f$ and $f \in \mathcal{H}$, then $g \in \mathcal{H}$.*
5. *If $f \in \mathcal{H}$, $|\mathrm{Dom}(f)| < k$ and x is a variable, then there exists $g \in \mathcal{H}$ such that $f \subset g$ and $x \in \mathrm{Dom}(g)$.*

$\mathcal{H}$ is called a winning strategy for Liar.

The following characterization shows up:

Theorem 15 ([1]) *Let $r, k \in \mathbb{N}$, $\mathcal{C}$ be an unsatisfiable set of clauses, and suppose each $C \in \mathcal{C}$ satisfies $|C| \leq r < k$. Then the following are equivalent:*

- $\mathcal{C} \nvdash_k \emptyset$, *that is,* $\mathbf{minwidth}(\mathcal{C} \vdash \emptyset) \geq k + 1$.
- *Liar wins the Boolean $(k+1)$-pebble game on $\mathcal{C}$.*

Note 16 *In the original text [1], the condition $r < k$ is accidentally missing. However, the condition is not necessary for the backward implication, which is enough for lower-bound proofs.*

Applications

Using the criterion of Theorem 15, we can estimate the necessary width of various kinds of $\mathcal{C}$.

Example 17 *For $m > n$, $PHP^m_n \nvdash_n \emptyset$, that is,* $\mathbf{minwidth}(PHP^m_n \vdash \emptyset) \geq n + 1$.

Example 18 *Densely Linear Ordering Principle for $[n]$, that is, any linear order on $[n]$ cannot be dense, can be expressed by an unsatisfiable set of clauses (let us denote it by DLO_n). Then $DLO_n \nvdash_{n/3} \emptyset$, that is,* $\mathbf{minwidth}(DLO_n \vdash \emptyset) \geq n/3 + 1$.

Example 19 *We can formalize "necessary memory" for carrying out a resolution refutation of $\mathcal{C}$, denoted by $\mathbf{Csp}(\mathcal{C} \vdash \emptyset)$. It holds that $\mathbf{Csp}(\mathcal{C} \vdash \emptyset) \geq \mathbf{minwidth}(\mathcal{C} \vdash \emptyset) - \mathbf{width}(\mathcal{C})$. See section 4 of [1] and section 5.5 of [3] for details.*

Further reading on estimation techniques for proof complexity of resolution

We refer interested audience to chapters 5, 10, 11, 13 and the related chapters of [3].

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Standalone Talks

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Presented paper by Sepehr Assadi, Yu Chen and Sanjeev Khanna

Sublinear Algorithms for $(\Delta + 1)$ Vertex Coloring

(https://arxiv.org/pdf/1807.08886/)

Introduction

Given a graph $G(V, E)$, a proper $k-$coloring of the vertices of G is an assignment of colors from $\{1, 2, \ldots, k\}$ (called the palette) to the vertices of G such that no two adjacent vertices have the same color.

Given that the graph has maximum degree Δ, it can always be colored using $\Delta + 1$ colors and there exists a simple greedy algorithm for $(\Delta + 1)-$coloring problem. It runs in linear time and takes linear amount of space (reading the entire graph). Now the question is can $(\Delta + 1)-$coloring be solved in sublinear time and/or sublinear space?

This paper by Assadi, Chen and Khanna [1] answers the above question.

Results

The main technical result of this paper is this key theorem:

Theorem 1 ([1]Palette-Sparsification Theorem) *Let $G(V, E)$ be an n vertex graph with maximum degree Δ. Suppose for any vertex $v \in V$, we sample $\mathcal{O}(\log n)$ colors $L(v)$ from $\{1, ..., \Delta + 1\}$ independently and uniformly at random. Then with high probability there exists a proper $(\Delta + 1)$ coloring of G in which the color for every vertex v is chosen from $L(v)$.*

Using the **Palette-Sparsification Theorem**, they have designed the following two (among others) sublinear algorithms:

Result 2 *There exists a randomized single-pass dynamic streaming algorithm for the $(\Delta+1)-$coloring problem using $\widetilde{\mathcal{O}}(n)$ space.*

In dynamic streaming model, both edge insertions and deletions are allowed.

Result 3 *There exists a randomized $\widetilde{\mathcal{O}}(n\sqrt{n})$ time algorithm for the $(\Delta + 1)-$coloring problem.*

They have also shown the above result is essentially tight upto $poly(\log n)$ factors, i.e., any algorithm for $(\Delta + 1)-$coloring problem requires $\Omega(n\sqrt{n})$ time.

Note 4 *For proving the **Palette-Sparsification Theorem**, they have used a modified version of the Harris-Schneider-Su (HSS) network decomposition [2], to partition the graph into "dense" and "sparse" subgraphs (as the palette sparsification works differently for "dense" and "sparse" graphs).*

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Presented paper by Guillermo Pineda-Villavicencio

A new proof of Balinski's theorem on the connectivity of polytopes

(https://www.sciencedirect.com/science/article/pii/S0012365X21001217)

Introduction

Balinski's theorem states that the graph of a d-dimensional convex polytope is d-connected. In this paper, the author shows a new proof of this theorem, using the notion of the link of a vertex.

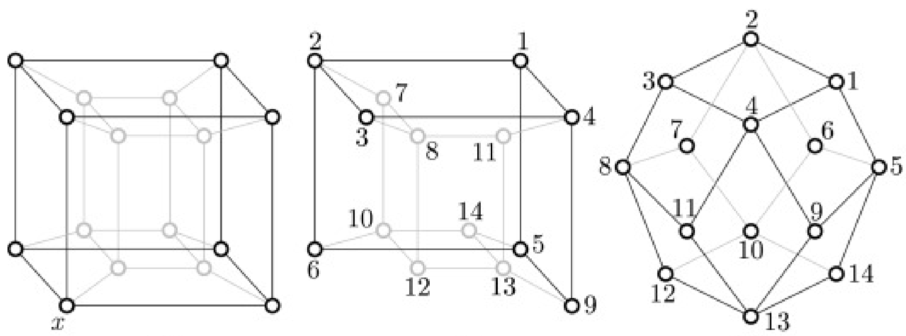
Definitions

Definition 1 *The boundary complex of a polytope P is the set of faces of P other than P itself.*

Definition 2 *The link of a vertex x in P, denoted lk(x), is the set of faces of P that do not contain x but lie in a facet of P that contains x.*

Definition 3 *An empty (d-1)-simplex in a d-polytope P is a set of d vertices of P that does not form a face of P but every proper subset does.*

Proposition 4 *(Ziegler, [1]) Let P be a d-polytope. Then the link of a vertex in P is combinatorially isomorphic to the boundary complex of a (d-1)-polytope. In particular, for each d $\geq$ 3, the graph of the link of a vertex is isomorphic to the graph of a (d-1)-polytope.*



(a) (b) (c)

Results

Theorem 5 *For d $\geq$ 1, the graph of d-polytope P is d-connected. Besides, for each d $\geq$ 3, each vertex x in a d-separator X of G(P) lies in the link of every other vertex of X, and the set $X \setminus \{x\}$ is a separator of the link of x.*

Corollary 6 *Let P be a simplicial d-polytope with d $\geq$ 2. A d-separator of G(P) forms an empty $(d-1)$ simplex of P.*

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Presented paper by Chaoliang Tang, Hehui Wu, Shengtong Zhang, Zeyu Zheng

Note on the Turán number of the 3-linear hypergraph C_{13}

(https://arxiv.org/abs/2109.10520)

Introduction

The crown C_{13} is the linear 3-graph with vertices $\{a, b, c, d, e, f, g, h, i\}$ and edges

$$\{a, b, c\}, \{a, d, e\}, \{b, f, g\}, \{c, h, i\}.$$

We will prove the following conjecture of Gyárfás et al.: Any C_{13}-free linear 3-graph G satisfies

$$|E(G)| \leq \frac{3(n-s)}{2}$$

where s is the number of vertices with degree at least 6. This combined with previous work essentially determines Turán number for linear 3-graphs with at most 4 edges.

We will first summarize previous work, then state and proof two similar theorems on upper bounds of $E(G)$ where G is a crown-free graph.

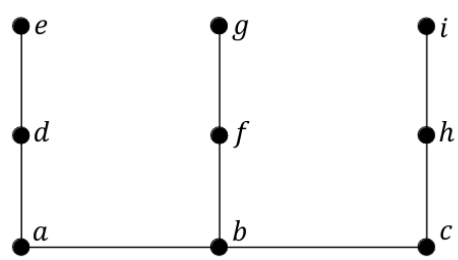

Figure 2: C_{13} as shown in the presented paper.

Important definitions

Definition 1 (Linear 3-graph) *A 3-graph $G = (V, E)$ consists of a finite set of vertices $V(G)$ and set of edges $E(G)$ where edges are 3-element subsets of V. A hypergraph is linear if any two edges share at most 1 vertex.*

Definition 2 (linear Turán number) *Linear Turán number $ex(n, F)$ is the maximum number of edges in any F-free linear 3-graph on n vertices.*

Definition 3 (F-free graph) *A graph is F-free if it does not contain F as a subgraph.*

Definition 4 (Minimal counter-example) *A minimal counter-example is a counter-example that contains no proper subgraph that is also a counter-example.*

Previous work

Gyárfás, Ruszinkó and Sárközy showed

$$6\left\lfloor\frac{n-3}{4}\right\rfloor + \varepsilon \leq ex(n, C_{13}) \leq 2n$$

where ε changes based on the numerator. It is 0 whenever $n-3 \equiv 0, 1 \mod 4$, it is 1 whenever $n-3 \equiv 2 \mod 4$ and 3 otherwise.

In a different paper Gyárfás et. al. showed that every linear 3-graph with minimum degree 4 is not crown free.

Recently, Fletcher improved the previous upper bound to

$$ex(n, C_{13}) \leq \frac{5}{3}n.$$

Stronger bounds

In this talk, we will prove the following two upper bounds and try to connect them to the previous results.

Theorem 5 *Let G be any crown-free linear 3-graph G on n vertices. Then its number of edges satisfies*

$$|E(G)| \leq \frac{3(n-s)}{2}$$

where s is the number of vertices in G with degree at least 6.

Theorem 6 *Let G be any crown-free linear 3-graph G on n vertices, and let s be the number of vertices in G with degree at least 6. If $s \leq 2$, then the number of edges satisfies*

$$|E(G)| \leq \frac{10(n-s)}{7}.$$

Outline of proofs of the theorems

We will use the following notation $D(\{x,y,z\}) \geq \langle a,b,c \rangle$ to say that $d(x) \geq a$, $d(y) \geq b$ and $d(z) \geq c$ for positive integers a, b, c where $a \geq b \geq c$.

To do the proofs we also need the lemma stated below.

Lemma 7 *Let G be a crown-free graph and $e = \{x,y,z\} \in E(G)$ satisfy $D(e) \geq \langle 5,5,4 \rangle$. Then, the vertex set of all vertices sharing an edge with $\{x,y,z\}$*

$$S = \bigcup_{f \in E(G), f \cap \{x,y,z\} \neq \emptyset} f,$$

contains exactly 11 vertices and all vertices in S have degree at most 5. The set of edges that contain at least one vertex in S,

$$E_S = \{f : f \in E(G), f \cap S \neq \emptyset\}$$

contains at most 13 edges, and all elements of E_S are subsets of S.

Both theorems can be proved by contradiction. We assume a minimal counter-example. We always start by showing the existence of some special edge based on some equality on the degrees of its vertices. We then analyze $D(e)$ of this particular edge. This yields multiple cases. With most of them, it is simple to show that they lead to a contradiction. Only complicated case is the one when $D(e) \geq \langle 5,5,4 \rangle$. However, using the previous lemma we can show that the assumed minimal counterexample can be made even smaller.

Proofs of theorems are relatively straightforward as we will see. The hardest part is to prove the lemma which we might attempt at the end of the talk in case we have enough time left.

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Presented paper by Matt DeVos
Longer cycles in vertex transitive graphs
(http://arxiv.org/abs/2302.04255)

Introduction

Lászlo Babai in 1979 proved that in every vertex transitive graph on at least $n \geq 3$ vertices there exists a cycle of length at least $\sqrt{3n}$. This had been for a long time the only found general lower bound. In this talk we will take a look at the improvement by Matt DeVos who showed that such graph contains a cycle of length at least $(1 - o(1))n^{3/5}$.

Preliminaries

A graph G is *vertex transitive* if its automorphism group acts transitively on the vertex set $V(G)$. Thus for any two distinct vertices of graph G there is an automorphism mapping one to the other.

Theorem 1 (Babai [1]) *Every connected vertex transitive graph with $n \geq 3$ vertices contains a cycle of length at least $\sqrt{3n}$.*

Lemma 2 *Let G be a finite group acting transitively on the set V, let $B, C \subseteq V$ and let $k \geq 0$. If $|D \cap C^g| \geq k$ holds for every $g \in G$, then $|B||C| \geq k|V|$.*

Lemma 3 *Let X be a 2-connected graph and let C_1, C_2 be longest cycles in X. If $|V(C_1) \cap V(C_2)| = k$, then there exists a set of at most $k^2 + k$ vertices hitting all longest cycles.*

Main result

Theorem 4 *Every connected vertex transitive graph on $n \geq 3$ vertices contains a cycle of length at least $(1 - o(1))n^{3/5}$.*

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Presented paper by Juhana Laurinharju, Jukka Suomela

Introduction to LOCAL and Linial's Lower Bound Made Easy

(https://jukkasuomela.fi/doc/linial-easy.pdf)

The LOCAL model

Our goal is to model a distributed computation in a graph $G = (V, E)$:

- Every node of G corresponds to a computer with unbounded power.
- At the beginning, each node knows $n = |V|$ and its unique ID from the set $\{1, \dots, n\}$.
- The computation happens in rounds, each round consists the three following parts:
  1. All nodes do some computation according to the algorithm.
  2. All nodes send an arbitrarily large message to their neighbors.
  3. All nodes receive messages sent by their neighbors.
- Sometimes, the graph is directed, but the nodes can still communicate in both directions. (The direction is just additional information for the algorithm.)
- Each node is responsible for computing its part of the output. For example, if we were interested in graph colouring, we only expect each node to know its color (so that all its neighbors have a different color).
- The measure of complexity is the *number of rounds*.

Observation 1 *Every computable graph property can be calculated in* $\mathcal{O}(\text{diameter}(G))$ *rounds.*

The lower bound for 3-colouring a cycle

Theorem 2 *Any LOCAL algorithm for computing a 3-colouring of a directed n-vertex cycle must take at least* $\frac{1}{2}\log^*(n) - 1$ *rounds, where* $\log^*(n) = \begin{cases} 0 & \text{if } n \leq 1 \\ 1 + \log^*(\log_2(n)) & \text{otherwise.} \end{cases}$

Definition 3 (k-ary c-colouring function) $A : \{1, \dots, n\}^k \to \{1, \dots, c\}$ *is a k-ary c-colouring function if*

- *For all* $1 \leq x_1 < x_2 < \ldots < x_k \leq n$, $A(x_1, x_2, \dots, x_k) \in \{1, 2, \dots, c\}$
- *For all* $1 \leq x_1 < x_2 < \ldots < x_k < x_{k+1} \leq n$, $A(x_1, x_2, \dots, x_k) \neq A(x_2, x_3, \dots, x_{k+1})$

Observation 4 *Any algorithm computing a 3-colouring of a cycle is a k-ary 3-colouring function.*

Lemma 5 *If A is a 1-ary c-colouring function, we have $c \geq n$.*

Lemma 6 *If A is a k-ary c-colouring function, we can construct a $(k-1)$-ary 2^c-colouring function B.*

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LOCAL: Maximal Independent Set

(https://disco.ethz.ch/courses/fs21/podc/lecturenotes/chapter5.pdf)

Introduction

In this talk I will present a Luby's MIS Algorithm which computes an MIS and also I wil show how it is related to colouring problem of a graph. Below you can see the main topics of a talk.

Definition and Reductions

What is MIS?

Definition 1 *Given a graph $G = (V, E)$, a set of vertices $S \subseteq V$ is called a Maximal Independent Set (MIS for simplicity) if it is satisfies two properties:*

(1) the set S is an independent set meaning that no two vertices $v, u \in S$ are adjacent,

(2) the set S is maximal - with regard to independence - meaning that we cannot add any node $v \notin S$ to the set S, i.e., there exists a neighbor u of v such that $u \in S$.

Algorithms for MIS can be used to solve a number of other graph problems. We will see a simple and beautiful reduction that allows us to solve a $\Delta + 1$ coloring using an MIS algorithm, without any significant overhead in the round complexity.

Lemma 2 *Given a LOCAL algorithm $\mathcal{A}$ that computes an MIS on any N-node graph in $T(N)$ rounds, there is a local algorithm $\mathcal{B}$ that computes a $\Delta + 1$ coloring of any n-node graph with maximum degree Δ in $T(n(\Delta + 1))$ rounds.*

Luby's Algorithm: The algorithm is made of iterations, each of which has two rounds as follows:

• In the first round, each node v picks a random variable $r_v \in [0, 1]$ and sends it to its neighbors. Then node v joins the (eventual) MIS set S if and only if node v has a strict local maxima, that is, if $r_v > r_u$ for all neighbors u of v

• In the second round, if a node v joined the MIS, then it informs its neighbors and then, node v and all of its neighbors get removed from the problem. That is, they will not participate in the future iterations

Analysis: It is easy to see that the algorithm always produces an independent set, and eventually, this set is maximal. The main question is, how long does it take for the algorithm to reach a maximal independent set?

Theorem 3 *Luby's Algorithm computes a maximal independent set in $O(\log n)$ rounds, with high probability*

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Presented paper by Sepehr Assadi, Aditi Dudeja
A Simple Semi-Streaming Algorithm for Global Minimum Cuts
(https://kam.mff.cuni.cz/~spring/media/papers/3/1.9781611976496.19.pdf)

Definitions

Definition 1 *A* cut *is a set $S \subset V, S \neq \emptyset$ of undirected graph $G = (V, E)$. The size of cut is the number of edges between S and $\overline{S} = V \setminus S$.*

Definition 2 *In* semi-streaming model *the data stream is given in arbitrary order, we are allowed to make one or few passes over the stream.*

Definition 3 *In the* M-FOLD-GREATER-THAN *problem for any integers $M, N \geq 1$, Alice and Bob are each given M separate N-bit numbers, $X = \{x^1, \ldots, x^M\}$ and $Y = \{y^1, \ldots, y^M\}$ (each $x^i, y^i \in \{0,1\}^N$). The goal is to determine the value of $GT_N^M(X, Y)$ defined as follows:*

- $GT_N^M(X, Y) = 1$ *if* $\exists x^i, y^i : y^i > x^i$.
- $GT_N^M(X, Y) = 0$ *otherwise.*

Proposition 4 *For any integers $M, N, r \geq 1$, any r round communication protocol for the* M-FOLD-GREATER-THAN *problem, wherein Alice and Bob send only r messages to each other, requires $\Omega_r\left(M \cdot N^{\frac{1}{r}}\right)$ bits to succeed with constant probability more than half.*

Main results

Theorem 5 *There is a semi-streaming algorithm that with high constant probability outputs an exact minimum cut of given n-vertex graph in two passes and space of $\mathcal{O}(n \log n)$ bits.*

Theorem 6 *Any streaming algorithm that outputs the exact value of minimum cut of given n-vertex graph with probability more than half a $p > 1$ passes requires $\Omega\left(n \cdot (\log n)^{\frac{1}{2p-1}}\right)$ space.*

Algorithm

First pass: For every vertex $v \in V$ sample two edges incident to v uniformly at random (with repetition). Let $G^{(2)}$ be the resulting graph with connected components $V_1, \ldots, V_t$. Compute the minimum degree $\deg_{\min}$ of G. If $t > 100 \cdot n / \deg_{\min}$, terminate and output FAIL.

Second pass: Consider the multi-graph H obtained from G by contracting each component V_i into a single vertex and removing the self-loops. Let $F_1, \ldots, F_{\deg_{\min}}$ be initially empty. For each arriving edge e in stream include e in F_i, where i is the smallest index such $\{e\} \cup F_i$ contains no cycle as an induced sub-graph of H; skip e if no such i exists.

Post processing: Compute minimum cut of $F = F_1 \cup \cdots \cup F_{\deg_{\min}}$; if it contains less then $\deg_{\min}$ edges, return this cut. Otherwise return any vertex v with $\deg(v) = \deg_{\min}$.

Tools for Theorem 5

Let $C := \{e_1, \ldots, e_\lambda\}$ denote the minimum non-singleton cut of G, where λ denotes the size of the cut.

Lemma 7 *With probability $\Omega(1)$ in the graph $G^{(2)}$ at the end of the first pass:*

1. *number of connected components is at most $100 \cdot n/\deg_{\min}$.*
2. *if $\lambda < \deg_{\min}$ the no edge $e \in C$ has both endpoints in one connected component.*

Lemma 8 *Conditioned on event in Lemma 7 the cut output by the algorithm after post-processing is a minimum cut in G.*

Tools for Theorem 6

Lemma 9 *Let M, N be positive integers such that $M = 8 \cdot 2^N$, let $\mathcal{A}$ be a δ-error p-pass streaming algorithm that determines if the global min-cut of $(2M)$-vertex graph is $\geq 2^N$ or $< 2^N$. Then there is a δ-error $(2p-1)$-round protocol π that solves GT_N^M with communication cost $\mathcal{O}(p \cdot s)$, where s is the space needed by $\mathcal{A}$.*

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The Ramsey numbers of squares of paths and cycles

(https://arxiv.org/pdf/2212.14860.pdf)

Introduction

The study of unavoidable regularities has a long history in mathematics and has given interesting results in multiple areas. One of the first (and better known) examples is a lemma used to prove the Bolzano-Weierstrass theorem, and it states that every sequence in $\mathbb{R}$ admits a monotone subsequence. This lemma is a good example of a wide array of results that try to study what kind of regular substructures cannot be avoided in large structures.

The first results of this kind in graph theory are due to Ramsey. In his seminal paper [5], Ramsey showed that for any graph H it is possible to find a monochromatic copy of H in any two-colouring of K_n, provided n is large enough. This result started an active and influential area of graph theory which is now called Ramsey Theory.

Definition 1 *Let H be a graph. We denote by $R(H,H)$ the Ramsey number of H, which is the smallest $n \in \mathbb{N}$ such that any {red, blue}-edge colouring of K_n admits a monochromatic copy of H.*

In his 1930 paper [5], Ramsey showed that the value $R(H,H)$ is well defined for any graph H. Determining or approximating the value of $R(H,H)$ for any given H has been the driving question of Ramsey theory since then.

One of the first families of graph for which the Ramsey number was determined is the family of paths. By path P_n we mean the graph over the vertex set $\{1,\dots,n\}$ and with edge set $\{12, 23, \dots, (n-1)n\}$. The result, due to Gerencsér and Gyárfás [4], reads as follows.

Theorem 2 (Gerencsér and Gyárfás, [4]) *Let $n \geq 2$ be a natural number. Then $R(P_{2n}, P_{2n}) = 3n-1$.*

More recently, Chvátal, Rödl, Szemerédi and Trotter proved that if a graph H has bounded degree, then its Ramsey number is linear in the number of vertices of H.

Theorem 3 (Chvátal, Rödl, Szemerédi and Trotter, [3]) *Let H be a graph over n vertices and with maximum degree Δ. The Ramsey number $R(H,H)$ is bounded above by $c_\Delta \cdot n$ for some constant c_Δ depending only on Δ.*

However, this result is still very far from giving us more precise estimates for $R(H,H)$.

For $k,n \in \mathbb{N}^+$, we denote by P_n^k the k-th power of the path P_n, which is the graph obtained from P_n by adding an edge between any two vertices at distance at most k in the path P_n.

There are at least two reasons why studying the Ramsey number of power of paths is an important question in Ramsey theory. Firstly, it is a natural next step in the strengthening of the result of Gerencsér and Gyárfás. Secondly, powers of paths are of particular importance in the study of Ramsey problems because they are related to a measure of complexity of graphs (the bandwidth) that has been proved to be relevant in the area.

More in detail, we say that a graph H over n vertices has bandwidth k if k is the smallest integer such that H is a subgraph of P_n^k. A result by Allen, Brightwell and Skokan [2] shows a better upper bound for the Ramsey number of graphs with bounded maximum degree if in addition we assume that the graph has sublinear bandwidth.

Theorem 4 (Allen, Brightwell and Skokan, [2]) *For any Δ positive integer, there exist $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that the following holds for any $n \geq n_0$. Let H be a graph over n vertices with maximum degree at most Δ and with bandwidth at most εn, then we have that $R(H, H) \leq 2(\chi(H) + 2)n$.*

The proof of this theorem relies on good estimates of the value of $R(P^k_{(k+1)n}, P^k_{(k+1)n})$ and an improvement in the approximation of the Ramsey number for the square of paths is likely to lead to better upper bounds for the Ramsey number of graphs with sublinear bandwidth. In particular, Allen, Brightwell and Skokan conjectured the following:

Conjecture 5 (Allen, Brightwell and Skokan, [2]) *For any Δ positive integer, there exist $n_0 \in \mathbb{N}$ and $c, \varepsilon > 0$ such that the following holds for any $n \geq n_0$. Let H be a graph over n vertices with maximum degree at most Δ and with bandwidth at most εn, then we have that $R(H, H) \leq (\chi(H) + c)n$.*

For the nature of the proof of Theorem 4, it seems that determining the value of $R(P^k_{(k+1)n}, P^k_{(k+1)n})$ would be of a big step forward in proving Conjecture 5.

The aim of this paper is to prove the following result.

Theorem 6 *There exists an $n_0 \in \mathbb{N}$ such that for all integers $n \geq n_0$ we have*

$$R(P^2_{3n}, P^2_{3n}) = 9n - 3.$$

Let us point out that the n_0 of this theorem is given us by the Regularity Lemma, and we did no effort to try to minimise n_0. Even if this result answers a natural question in the Ramsey theory setting and it might be of help in improving the result of Theorem 4, additional study will be required to extend Theorem 6 to higher powers of k and to determine the value of $R(P^k_{(k+1)n}, P^k_{(k+1)n})$ for other values of k.

Lower bound

In order to prove Theorem 6, we first show that there exists a {red, blue}-edge colouring of K_{9n-4} without monochromatic copied of P^2_{3n}. The construction follows the recipe drawn in Figure 3.

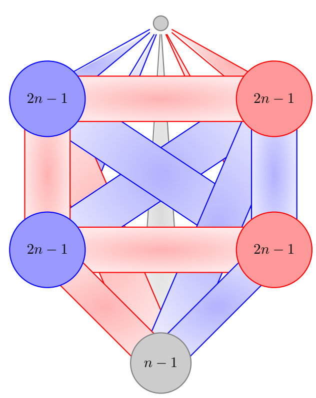

Figure 3: Our extremal colouring

We partition our $9n - 4$ vertices in six sets. The sets B_1, B_2, R_1, R_2 of size $2n - 1$, the set Z of size $n - 1$ and an additional single vertex r. We colour all the edges in $B_1, B_2, (B_1 \cup B_2, \{r\}), (R_1, R_2, Z)$

by blue and all the edges in $R_1, R_2, (R_1 \cup R_2, \{r\}), (B_1, B_2, Z)$ by red. We then arbitrarily colour the rest of the edges.

It is evident that in our construction there is no monochromatic P^2_{3n}

Proof strategy for the upper bound

At the base of our strategy for the upper bound of $R(P^2_{3n}, P^2_{3n})$ there are the regularity method of Szemerédi [6] and an embedding lemma due to Allen, Böttcher and Hladký [1].

We show that any two-edge-colouring of a large clique not containing a monochromatic copy of P^2_{3n} must have a very specific structure. Which is, any such colouring must be similar in structure to the colouring in Figure 3. Some of the details follow.

Theorem 7 (Szemerédi, [6]) *For every $\varepsilon > 0$ there exist natural numbers M and n_0 such that for any $n \geq n_0$ and any two-colouring G of K_n we can partition the vertex set of G in at most M sets $V_0, \ldots, V_m$ such that $m \geq \frac{1}{\varepsilon}$ and $|V_0| \leq \varepsilon n$ and $|V_1| = \ldots = |V_m|$. Moreover, all but at most $\varepsilon\binom{m}{2}$ of the pairs (V_i, V_j) are ε-regular in both colours.*

Here, by (V_i, V_j) being ε-regular we mean that whenever $A \subseteq V_i$ and $B \subseteq V_j$ are such that $|A| \geq \varepsilon |V_i|$ and $|B| \geq \varepsilon |V_j|$, the density of edges (both in blue and in red) between A and B is the same (up to an error ε) of the density of the same colour between V_i and V_j.

Therefore, given n sufficiently large and a two edge colouring of K_n, we can partition the vertex set of K_n in a bounded number of subsets such that between most pairs of subsets we see some strong regularity property. In particular, we can build a support graph R, called an ε-reduced graph for G, over the parts $V_1, \ldots, V_m$ such that we have the edge V_iV_j if and only if (V_i, V_j) is ε-regular in both colours. We can colour each edge V_iV_j of the majority colour in the set of edges $E(V_i, V_j)$. Notice that R is an almost complete two-edge-coloured graph.

The use of Szemerédi regularity lemma has been used to embed substructures in large graphs, and it is of fundamental importance here because it allows us to apply a result introduced by Allen, Böttcher and Hladký [1].

We first need a definition.

Definition 8 *Let R be a {red, blue}-edge-coloured graph. Let T and T' be monochromatic (wlog blue) triangles. We say that T and T' are triangle-connected if there exists a sequence of blue triangles $T = T_0, \ldots, T_\ell = T'$ such that for every $i = 0, \ldots, \ell - 1$ we have that T_i and T_{i+1} share an edge.*

A triangle factor is a set of vertex disjoint triangles. It is natural to define as monochromatic triangle-connected triangle factor a set of pairwise vertex disjoint monochromatic triangles of the same colour that are pairwise triangle connected.

The following embedding lemma allows us to reduce the problem of finding a monochromatic copy of P^2_{3n} in a two-colouring of K_n to the problem of finding a monochromatic triangle-connected triangle factor in the reduced graph R.

Theorem 9 (Allen, Böttcher and Hladký, [1]) *For all positive $\delta, \lambda < 1$ there exists $\varepsilon > 0$ and $M, n_0 \in \mathbb{N}$ such that whenever $n > n_0$ the following holds. Let G be a two-colouring of K_n, and let R be an ε-reduced graph of G with $|R| = m \leq M$ vertices. If R contains a monochromatic triangle-connected triangle factor over $3(1 + \delta)\lambda m$ vertices, we can find a monochromatic copy of $P^2_{3\lambda n}$ in G.*

We can show that whenever R is an almost complete two-coloured graph over m vertices, either R contains a triangle-connected triangle factor of the right size or the colouring of R is close to the lower bound construction. A careful analysis of the possible extremal structure finishes the proof of the upper bound.

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LOCAL: Deterministic Coloring of General Graphs

(https://disco.ethz.ch/courses/fs21/podc/lecturenotes/chapter3.pdf)

Introduction

We present a deterministic algorithm in the LOCAL model that colors a graph G using $\Delta(G)+1$ colors in $O(\Delta(G)\log\Delta(G)+\log^*|V(G)|)$ rounds. The model definition is as follows.

Definition 1 (the LOCAL model; [1, 2]) *Let $G=([n],E)$ be a graph, where $[n]:=\{1,2,\dots,n\}$. In the LOCAL model, there is one process on each node $v\in[n]$ (which is named the same). At the beginning of an algorithm, every process $v\in[n]$ knows only its name, neighbors, and n. Then the algorithm works in synchronous rounds: per round, each process (in this exact order)*

(i) performs some computation based on its knowledge,

(ii) sends a message to all of its neighbors,

(iii) receives the messages sent to it by its neighbors in that round.

Moreover, we require that each process learns its own part of the output.

Observation 2 *In the LOCAL model, every graph problem can be solved in $O(n)$ rounds.*

Later, in the proof of Lemma 6, we use the following notion.

Definition 3 (Δ-cover free family) *A family of sets $S_1,S_2,\dots,S_k\subseteq\{1,2,\dots,k'\}$ is called a* Δ-cover free family *if no set in the family is a subset of the union of Δ other sets.*

Warm up: coloring rooted trees

Theorem 4 *There is a deterministic algorithm in the LOCAL model that colors any rooted tree T using 3 colors in $\log^* n+O(1)$ rounds.*

Remark: The number of rounds is optimal up to an additive constant.

The main result: coloring arbitrary graphs

Let $G=([n],E)$ be a fixed graph with maximum degree $\Delta:=\Delta(G)$.

Theorem 5 (The main result) *There is a deterministic algorithm in the LOCAL model that colors the graph G using $\Delta+1$ colors in $O(\Delta\log\Delta+\log^* n)$ rounds.*

The algorithm starts with the trivial n-coloring and, in the first $O(\log^* n)$ rounds, transforms the coloring into a $O(\Delta^2\log\Delta)$-coloring. In one additional round, the algorithm reduces the number of colors to $O(\Delta^2)$. This part of the algorithm is based on a repeat application of the subsequent lemma.

Lemma 6 ([1, 2]) *Given a k-coloring $\varphi_{\texttt{old}}$, in a single round, we can compute a k'-coloring $\varphi_{\texttt{new}}$ for $k'=O(\Delta^2\log k)$. In addition, if $k=O(\Delta^3)$, then the bound can be improved to $k'=O(\Delta^2)$.*

Finally, using the following lemma, the algorithms computes a desired $(\Delta+1)$-coloring.

Lemma 7 ([3]) *Let $k \geq \Delta + 2$. Given a k-coloring $\varphi_{\texttt{old}}$, in $O(\Delta \log \frac{k}{\Delta+1})$ rounds, we can compute a $(\Delta + 1)$-coloring $\varphi_{\texttt{new}}$.*

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5-List-Coloring Graphs on the Torus: A Computational Approach

Introduction

In this talk, we present a computer-aided approach to some problems related to list-coloring graphs, developed with the goal of finding the finitely many 6-list-critical graphs embeddable on the torus. The basic ideas of this approach were originally published in [1] (§2.1), and are motivated by the results in Postle's PhD thesis ([2]).

Background

List-coloring (for vertices) is a concept similar to regular (vertex) coloring in which each vertex has a list of possible color. Let G be a graph.

Definition 1 *A* list assignment *is a function $L : V(G) \to 2^{\mathbb{N}}$. A k-*list-assignment *is a list assignment with $|L(v)| \geq k$ $\forall v \in V(G)$. An L-*coloring *is a (proper vertex) coloring f for which $f(v) \in L(v)$ $\forall v \in V(G)$. A graph is k-*list-colorable *or k-*choosable *if there exists an L-coloring for all k-list-assignments L. The* list chromatic number *or* choosability $\chi_\ell(G)$ *is the least integer so that G is $\chi_\ell(G)$-list-colorable.*

A natural question is whether there is an analogue of the four color theorem for list-colorability,

Theorem 2 (Voigt, 1993) *There exists a planar graph with $\chi_\ell(G) = 5$.*

Theorem 3 (Thomassen, 1994) *For all planar graphs, $\chi_\ell(G) \leq 5$.*

The previous theorem has a simple proof based on proving the following stronger statement.

Theorem 4 *Let G be a plane (embedded) graph whose faces are all triangles except for possibly the outer face C, and let L be a list assignment satisfying: $|L(v)| \geq 5$ for all internal vertices, $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{x, y\}$ where x, y are a pair of adjacent vertices, $|L(x)| = |L(y)| = 1$, $L(x) \neq L(y)$. Then G has an L-coloring.*

Definition 5 *A graph G is L-*critical *for some list assignment L if G has no L-coloring but every proper subgraph G' has. A graph G is k-*list-critical *if there exists a $(k-1)$-list assignment L such that G is L-critical.*

Let us consider list-coloring of graph embedded in general surfaces, not just the plane. Postle proved in [2] the following result, mirroring analogous results for regular coloring.

Theorem 6 *For $k \geq 6$ there exist only finitely many k-list-critical graphs embeddable in a given surface Σ.*

List-Coloring Graphs on the Torus

Our goal is to find all the 6-list-critical graphs on the torus. Thomassen did this for regular vertex coloring:

Theorem 7 *A graph G embeddable on the torus is 5-colorable if and only if it does not contain the following subgraphs:*

- K_6.
- $C_3 + C_5$.

- K_2+H_7, *where* H_7 *is the* Moser spindle*, the graph obtained by applying the Hajós construction to a pair of* K_4*.*

- T_{11}*, where* T_{11} *is a triangulation of the torus with* 11 *vertices.*

Where + denotes the join of two graphs: their disjoint union with all pairs of vertices from different graphs joined by edges.

We conjecture that there are no other *minimal* 6-list-critical graphs:

Conjecture 8 *A graph* G *embeddable on the torus is* 5*-list-colorable if and only if it does not contain the following subgraphs:* K_6*,* C_3+C_5*,* K_2+H_7*,* T_{11}*.*

How can we prove this? Postle's general result for arbitrary surfaces involves carefully studying when precolorings of certain subgraphs extend to the entire graph.

Definition 9 (Canvas) *We say that* (G,S,L) *is a* canvas *if* G *is a connected plane graph with outer walk* C*,* S *is a subgraph of* C*, and* L *is a list assignment such that* $|L(v)| \geq 5\,\forall v \in V(G)\setminus V(C)$ *and* $|L(v)| \geq 3\,\forall v \in V(C)\setminus V(S)$*. If* $S=C$ *and* C *is a cycle, then* (G,C,L) *is a* cycle-canvas*.*

A canvas is critical *if* $S \neq G$ *and for every proper subgraph* $H \supseteq S$ *of* G*, there exists an* L*-coloring of* S *which extends to* H *but not to* G*.*

One example of such a result is:

Theorem 10 *If* (G,C,L) *is a critical cycle-canvas, then* $|V(G)| \leq 19|V(C)|$*.*

Having a explicit list of all the critical cycle-canvases for small cycle sizes can be helpful to obtain all the 6-list-critical graphs on the torus.

Critical Graphs Generation

We use the following result from Postle:

Theorem 11 (Cycle Chord or Tripod Theorem) *If* (G,C,L) *is a critical cycle-canvas, then either*

1. C *has a chord in* G*, or*

2. *there exists a vertex* $v \in V(G)\ V(C)$ *with at least three neighbors on* C *such that at most one of the faces of* $G[\{v\}\cup V(C)]$ *includes a vertex or edge of* G*.*

This means that we can generate critical cycle-canvases of size ℓ with the following algorithm (assuming we have already generated those with size $< \ell$):

1. Generate all possible canvases with a chord iterating a from 3 to $\ell-1$ and fusing together two canvases of size a and $\ell+2-a$ in all possible orientations. Put all the canvases that are critical in a queue.

2. Generate all possible canvases with a tripod with biggest face of size at most $\ell-1$. Again, put those canvases that turn out to be critical in the queue.

3. While the queue is not empty, dequeue the first canvas, add a tripod with three consecutive neighbours to it in all the possible ways, and enqueue the canvases that turn out to be critical.

Critical Graphs Testing

How do we determine if there exists some L so that a cycle-canvas is L-critical? Even if L were fixed, it would be a computationally hard problem. We can use some coloring heuristics, as well as results such as the following:

Observation 12 *In a L-critical graph, $d(v) \geq |L(v)|\ \forall v \in V(G)$.*

Theorem 13 (Gallai) *Let G be a L-critical graph and let H be the subgraph of H induced by the vertices with $d(v) = |L(v)|$. Then each 2-connected component of H is a complete graph or an odd cycle.*

Theorem 14 (Alon, Tarsi) *Let G be a directed graph and L a list assignment with $|L(v)| \geq d^+(v) + 1$. If the number of even spanning eulerian subgraphs of G is different than the number of odd spanning eulerian subgraphs of G, then G is L-colorable.*

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Union-closed families with small average overlap densitites

(https://doi.org/10.37236/10121)

Introduction

Definition 1 *If X is a set, a family $\mathcal{F}$ of subsets of X is said to be* union-closed *if the union of any two sets in $\mathcal{F}$ is also in $\mathcal{F}$.*

A famous open problem in Combinatorics posed by Péter Frankl in 1979 states:

Conjecture 2 ***(Union-Closed Conjecture)*** *If X is a finite set and $\mathcal{F}$ is a union-closed family of subsets of X (with $\mathcal{F} \neq \{\emptyset\}$, then there exists $x \in X$ such that x is contained in at least half of the sets in $\mathcal{F}$.*

One approach to tackle this problem (studied during a Polymath project in 2016) is based on the definitions below.

Definitions

Definition 3 *If X is a finite set and $\mathcal{F} \subset \mathcal{P}(X)$ with $\mathcal{F} \neq \{\emptyset\}$, we define* the abundance of x *(with respect to $\mathcal{F}$) as the probability that a uniformly random element of $\mathcal{F}$ contains x. We denote it as $\gamma_x := |\{A \in \mathcal{F}\colon x \in A\}|/|\mathcal{F}|$*

Note 4 *If the average abundance of a uniformly random element of the ground set X were always at least 1/2, the Union-Closed Conjecture would immediately follow. However, there's a simple counterexample with average abundance of 4/9. Moreover, for any $n \in \mathbb{N}$ there exists a family with average abundance of $\Theta(1/\sqrt{n})$.*

Definition 5 *The* average overlap density *AOD($\mathcal{F}$) of $\mathcal{F}$ is the expected value of γ_x, where x is a uniformly random element of a uniformly random nonempty member of $\mathcal{F}$:*

$$AOD(\mathcal{F}) := \frac{1}{|\mathcal{F} \setminus \{\emptyset\}|} \sum_{A \in \mathcal{F} \setminus \{\emptyset\}} \frac{1}{|A|} \sum_{x \in A} \gamma_x$$

Note 6 *If the average overlap density of $\mathcal{F}$ is at least 1/2, then the Union-Closed Conjecture easily follows. Unfortunately, there exists infinitely many families with AOD($\mathcal{F}$) = 7/15 + o(1) as $n \to \infty$.*

The last note brought the following conjecture:

Conjecture 7 *There exists a constant $c > 0$ such that the following holds. Let $n \in \mathbb{N}$ and let $\mathcal{F} \subset \mathcal{P}(\{1, 2, ..., n\})$ be union-closed with $\mathcal{F} \neq \{\emptyset\}$. Then the average overlap density of $\mathcal{F}$ is at least c.*

Main result

The main result disproves Conjecture 7:

Theorem 8 *For infinitely many positive integers n, there exists a union-closed family $\mathcal{F}$ of subsets of $\{1, 2, ..., n\}$ with average overlap density of $\Theta(\frac{log_2(log_2(|\mathcal{F}|))}{log_2(|\mathcal{F}|)})$.*

We will construct these families.

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Presented paper by Mikolaj Lewandowski, Joanna Polcyn & Christian Reiher
Two disjoint cycles in digraphs
(https://arxiv.org/pdf/2205.10826.pdf)

Definitions

D1. For a nonempty and nondecreasing sequence $(d_1, ..., d_n)$ of nonnegative integers and a positive integer k the relation $(d_1, ..., d_n) \to k$ means that every directed graph on n vertices with outdegree sequence $(d_1, ..., d_n)$ contains k vertex disjoint cycles.

D2. Let integers $1 \le r \le s \le n$ be given. A sequence $(d_1, ..., d_n)$ satisfying
(a) $d_r \ge r, d_s \ge s+1$, and
(b) if $n \ge 2s-r+2$ and $d_{2s-r+2} = s+1$, then there is an integer $j \in [2s-r+3, n]$ such that $d_j \ge j$ is called $(r,s)-large$. We say that $(d_1, ..., d_n)$ is *large* if it is $(r,s)-large$ for some two integers $r \le s$ in $[n]$.

D3. *Transitive tournament* T_n is a digraph whose vertex set can be enumerated such that $V(T_n) = \{v_1, v_2, ..., v_n\}$ and $E(T_n) = \{v_i \to v_j : v_i, v_j \in V(T_n) \text{ and } i > j\}$

Auxilary statements

Conjecture. For every positive integer k every digraph with minimum outdegree at least $2k-1$ contains k vertex disjoint cycles.

Lemma. The statement $(d_1, ..., d_n) \to 1$ is true if and only if for some $j \in [n]$ the inequality $d_j \ge j$ holds.

Fact. Let integers $1 \le r \le s < n$ be given. If a nondecreasing sequence $d' = (d_1, ..., d_n)$ with $d_n < n$ is (r,s)-large, then every sequence $e' = (e_1, ..., e_{n-1})$ obtained from d' by deleting one arbitrary term is also (r,s)-large.

Main theorem

MT. Let $(d_1, ..., d_n)$ be a nonempty, nondecreasing sequence of nonnegative integers. The relation $(d_1, ..., d_n) \to 2$ holds if and only if the sequence $(d_1, ..., d_n)$ is large.

Step 1. We have $r = 1$.

Step 2. There are no loops in D. In particular, $d_n < n$.

Step 3. Every 2-cycle of D is dominated by a vertex of outdegree $s+1$.

Step 4. Suppose that an arc $x \to y$ of D does not appear in a 2-cycle.
(1) There is some vertex $a \notin \{x, y\}$ dominating x and y.
(2) If the outdegree of x is 1, then at least $s+1$ vertices distinct from y and having outdegree $s+1$ dominate $\{x, y\}$.

Step 5. The inneighbourhood of every vertex of D contains a cycle.

Step 6. There is a 2-cycle in D.

Step 7. In D there are a directed cycle C all of whose vertices have outdegree $s+1$ and a vertex $x \notin V(C)$ connected to every vertex of C by a 2-cycle.

Step 8. We have $s > 1$.

Step 9. If $n \geq 2s+1$, then $d_{2s+1} \geq s+2$. In particular, D has at most $s+1$ vertices of outdegree $s+1$.

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