Annual combinatorial Spring School ~~2020~~ ~~2021~~ 2022

Pavel Veselý, Tung Anh Vu (eds.)

Preface

Spring school on Combinatorics has been a traditional meeting organized for almost 40 years for faculty and students participating in the Combinatorial Seminar at Faculty of Mathematics and Physics of the Charles University. It is internationally known and regularly visited by students, postdocs and teachers from our cooperating institutions in the DIMATIA network. As it has been the case for several years, this Spring School is supported by Computer Science Institute (IÚUK) of Charles University, the Department of Applied Mathematics (KAM) and by some of our grants (SVV, Progres). This year we are glad we can also acknowledge generous support by the RSJ Foundation.

The Spring Schools are entirely organized and arranged by our students. The topics of talks are selected by supervisors from the Department of Applied Mathematics (KAM) and Computer Science Institute (IÚUK) of Charles University as well as from other participating institutions. In contrast, the talks themselves are almost exclusively given by students, both undergraduate and graduate. This leads to a unique atmosphere of the meeting, which helps the students in further studies and their scientific orientation.

This year the Spring School is organized in Lučany nad Nisou (in Jizera Mountains in northeastern Bohemia) with a great variety of possibilities for outdoor activities.

Michal Koucký, Martin Koutecký, Martin Tancer, Robert Šámal, Pavel Valtr, Pavel Veselý
Petr Chmel, Tung Anh Vu

Contents

Series Talks

Martin Černý

cerny@kam.mff.cuni.cz

Completely Positive Matrices *as part of series* Matrix Theory and Combinatorics Related to Optimization Problems

Introduction

In this talk, I present the class of *completely positive matrices* and its connection to copositive optimisation. I also discuss the properties, sufficient and necessary conditions of completely positive matrices which can be derived using graph theory.

Definition 1 *A matrix $A \in \mathbb{R}^{n\times n}$ is* completely positive *if there exists $B \in \mathbb{R}^{n\times m}$ such that $B \geq O_{n\times m}$ and $A = BB^T$.*

Definition 2 *A matrix $A \in \mathbb{R}^{n\times n}$ is* copositive *if $x^T Ax \geq 0$ for all $x \geq 0$.*

Notice that the class of copositive matrices is a generalisation of positive semidefinite matrices where we drop the condition of $x \geq 0$.

Both classes of matrices form convex cones, meaning they are closed for two matrices A, B from the classes on $A + B$ and αA for $\alpha \geq 0$. The connection of both classes is through *cone duality.* For the duality, we need the *Frobenius inner product*, i.e. $\langle A, B\rangle := \sum_{i,j} a_{ij} b_{ij}$ for $A, B \in \mathbb{R}^{n\times n}$.

Definition 3 *Let $\mathcal{C}$ be a cone of matrices. Its* dual cone *is then defined as*

$$\mathcal{C}^* := \{B \in \mathbb{R}^{n\times n} \mid \langle A, B\rangle \geq 0 \text{ for all } A \in \mathcal{C}\}.$$

When analysing completely positive matrices by the means of the graph theory, we employ *incidence* and *adjacency* matrices of a graph $G = (V, E)$. An *incidence matrix* $B \in \mathbb{R}^{|V|\times|E|}$ is defined as

$$b_{ve} := \begin{cases} 1 & \text{if } v \in e, \\ 0 & \text{if } v \notin e. \end{cases}$$

An *adjacency matrix* $A \in \mathbb{R}^{|V|\times|V|}$ is defined as

$$a_{uv} := \begin{cases} 1 & \text{if } \{u, v\} \in E, \\ 0 & \text{if } \{u, v\} \notin E. \end{cases}$$

Theorem 4 *Let A be a non-negative diagonally dominant symmetric matrix. Then A is completely positive.*

In our further analysis we talk about *graphs associated to matrices* and their counterparts, *matrix realisations of graph.* A graph $G(A) = (V, E)$ *associated to matrix* $A \in \mathbb{R}^{n\times n}$ is given by $V = \{1, \ldots, n\}$ and $E = \{\{i, j\} \mid a_{ij} > 0\}$. A matrix A is a *realisation* of G, if it holds $G(A) = G$.

Theorem 5 *For a graph G, its every realisation is completely positive if and only if it does not contain an odd cycle of length more than 3.*

Finally, we make use of *comparison matrices*. For a matrix $A \in \mathbb{R}^{n\times n}$, its *comparison matrix* $M(A) \in \mathbb{R}^{n\times n}$ is a matrix defined as

$$M(A)_{ij} \coloneqq \begin{cases} a_{ij} & \text{if } i = j, \\ -a_{ij} & \text{if } i \neq j. \end{cases}$$

Theorem 6 *For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, if $M(A)$ is positive semidefinite, then A is completely positive.*

Theorem 7 *For a triangle-free graph G it holds for its every matrix realisation A that if it is completely positive, than $M(A)$ is positive semidefinite.*

Bibliography

[1] Abraham Berman and Naomi Shaked-Monderer. *Completely positive matrices*, World Scientific, 2021.

[2] Natalia Kogan and Abraham Berman. *Characterisation of completely positive graphs*, Discrete mathematics, 1993.

[3] Immanuel M. Bomze. *Copositive optimization – Recent developments and applications*, European Journal of Operations Research, 2011.

Milan Hladík

hladik@kam.mff.cuni.cz

Absolute Value Equations *as part of series* Matrix Theory and Combinatorics Related to Optimization Problems

Introduction

The *absolute value equations (AVE)* is the algebraic problem of solving the system

$$Ax + |x| = b,$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$.

Properties

The *solution set* of the AVE reads

$$\Sigma = \{x \in \mathbb{R}^n;\ Ax + |x| = b\}.$$

When finite, it may possess up to 2^n isolated points.

Open problem: Is any value in $\{1, \ldots, 2^n\}$ attained as the number of solutions of certain AVE?

Observation 1 *Σ forms a convex polyhedron in each orthant.*

The linear complementarity problem (LCP) is an algebraic problem

$$y = Mz + q,\ y^T z = 0,\ y, z \geq 0.$$

Observation 2 *AVE $\Leftrightarrow$ LCP.*

Theorem 3 (Mangasarian, 2007) *Checking solvability of AVE is NP-complete.*

Theorem 4 (Wu & Li, 2018) *The AVE has a unique solution for each $b \in \mathbb{R}^n$ if and only if the interval matrix $[A - I_n, A + I_n]$ is regular.*

For the LCP, the analogous condition is P-matrix property (all principal minors are positive), which is co-NP-hard to check. Therefore unique solvability of AVE is co-NP-hard to check, too.

Sufficient conditions for the unique solvability (ρ = spectral radius, $\sigma_{\min}$ = minimal singular value):

$$\rho(|A^{-1}|) < 1 \quad \text{or} \quad \sigma_{\min}(A) > 1.$$

Open problem: Is AVE efficiently solvable if $[A - I_n, A + I_n]$ is regular? (It is the case if any of the sufficient conditions is fulfilled.)

Theorem 5 *The AVE is unsolvable if one of the following conditions holds:*

1. *$-y \leq A^T y \leq y,\ b^T y < 0$ is solvable, or*
2. *$\rho(|A|) < 1$ and $(I - |A|)^{-1} b$ is not nonnegative.*

Theorem 6 *If $2|A||b| < b$, then the AVE has 2^n solutions, lying in the interiors of the particular orthants.*

Bibliography

[1] Milan Hladík. *Absolute Value Programming.* chapter in Encyclopedia of Optimization (eds. P.M. Pardalos & O.A. Prokopyev), wait for the 2024 edition :-)

Matyáš Lorenc

lorenc@kam.mff.cuni.cz

From P-matrices to Eigenvalues *as part of series* Matrix Theory and Combinatorics Related to Optimization Problems

Introduction

In this handout we start by introducing some facts about eigenvalue approximation. Then we move to P-matrices, more precisely to its two subclasses, B-matrices and doubly B-matrices, and for each of those subclasses we state some properties and, more importantly, we show their application in eigenvalue estimation.

Definition 1 (Gershgorin circles) *Let $A \in \mathbb{C}^{n\times n}$ and for $i \in \{1,\ldots,n\}$ let $R_i = \sum_{j\neq i} |a_{ij}|$. Then by* Gershgorin circles *we understand a closed discs with radii R_i centered at a_{ii}, which we denote by $D(a_{ii}, R_i) \subseteq \mathbb{C}$.*

Note 2 *We will estimate just real eigenvalues and real parts of complex eigenvalues of real matrices, so for our purposes we may just use intervals $\left[a_{ii} - \sum_{j\neq i} |a_{ij}|, a_{ii} + \sum_{j\neq i} |a_{ij}|\right]$, which are intersections of Gershgorin circles (of real matrices) with the real axis.*

Definition 3 (Ovals of Cassini) *Let $A \in \mathbb{C}^{n\times n}$ and for $i \in \{1,\ldots,n\}$ let $R_i = \sum_{j\neq i} |a_{ij}|$. Then by* Ovals of Cassini*, sometimes called* Cassini ovals*, we understand $\forall i \neq j$ sets defined as follows:*

$$O_{ij} = \{z \in \mathbb{C} : |z - a_{ii}|\, |z - a_{jj}| \leq R_i R_j\}.$$

Theorem 4 (A. Brauer) *If $A \in \mathbb{C}^{n\times n}$, then all the eigenvalues of A lie inside the union of its $\binom{n}{2}$ ovals of Cassini, so $\forall\lambda$ eigenvalue of A: $\lambda \in \bigcup_{i\neq j} O_{ij}$.*

Definition 5 (*P*-matrix) *Let $A \in \mathbb{C}^{n\times n}$. We call A a* P-matrix *if all of its principal minors are positive.*

This class of matrices is computationally complex to recognize, the task of verifying given matrix on being a P-matrix is co-NP-complete. This leads us to try and define several classes of P-matrices that are easily recognizable. Such classes are e.g. B-matrices (introduced by Peña in [1]) or doubly B-matrices (introduced also by Peña in [2]).

B-matrices

Definition 6 (*B*-matrix) *We say that $A \in \mathbb{R}^{n\times n}$ is a* B-matrix*, if $\forall i \in \{1,\ldots,n\}$ the following holds:*

$$a)\quad \sum_{j=1}^{n} a_{ij} > 0 \qquad \wedge \qquad b)\quad \forall k \neq i : \quad \frac{1}{n}\sum_{j=1}^{n} a_{ij} > a_{ik}$$

Let $A \in \mathbb{R}^{n\times n}$, we define for each $i \in \{1,\ldots,n\}$:

$$r_i^+ := \max\{0, a_{ij} : j \neq i\} \qquad r_i^- := \min\{0, a_{ij} : j \neq i\} \qquad \text{and} \qquad r_i := \begin{cases} r_i^+ & \text{if } a_{ii} > 0, \\ r_i^- & \text{if } a_{ii} < 0. \end{cases}$$

We define c_i^+, c_i^-, and c_i in a similar way for the columns of the matrix instead of the rows.

Proposition 7 *Let $A \in \mathbb{R}^{n\times n}$. It holds that A is a B-matrix if and only if $\forall i \in \{1,\ldots,n\}$:*

$$a_{ii} - r_i^+ > \sum_{j\neq i} \left(r_i^+ - a_{ij}\right)$$

Definition 8 ($\bar{B}$-matrix) *We say that $A \in \mathbb{R}^{n\times n}$ is a* $\bar{B}$-matrix*, if it is of a form DB, where D is a diagonal matrix with its diagonal entries from set $\{-1, 1\}$ and B is a B-matrix.*

Proposition 9 *Let $A \in \mathbb{R}^{n\times n}$. It holds that A is a $\bar{B}$-matrix if and only if $\forall i \in \{1, \ldots, n\}$:*

$$|a_{ii} - r_i| > \sum_{j\neq i} |r_i - a_{ij}|$$

Theorem 10 *Let $A \in \mathbb{R}^{n\times n}$ and let λ be a real eigenvalue of A. It holds that*

I) $\lambda \in S := \bigcup_{i=1}^{n} \left[a_{ii} - r_i^+ - \sum_{k\neq i} |r_i^+ - a_{ik}|, a_{ii} - r_i^- + \sum_{k\neq i} |r_i^- - a_{ik}|\right]$

II) Let $\mathcal{C}$ be a class of real matrices such that if $B \in \mathcal{C}$ then all eigenvalues of B are real and all matrices of for $B_t := D + t\,(B - D)\,, t \in [0, 1]$ belong to $\mathcal{C}$ and let us assume that $A \in \mathcal{C}$. If S' is the union of m intervals of S such that S' is disjoint from all other intervals, then S' contains precisely m eigenvalues (counting multiplicities) of A.

The intervals from part I) of the previous theorem will be called *row $\bar{B}$-intervals*. (Then of course, there exist even column $\bar{B}$-intervals.)

Theorem 11 *Let $A \in \mathbb{R}^{n\times n}$ and let λ be an eigenvalue of A. It holds that*

I) $Re(\lambda) \in S^* := \bigcup_{i=1}^{n} [\alpha_i, \beta_i]\,,$ where $\forall i \in \{1, \ldots, n\}$:

$$\begin{aligned}\alpha_i &:= \min\left\{a_{ii} - r_i^+ - \sum_{k\neq i} |r_i^+ - a_{ik}|, a_{ii} - c_i^+ - \sum_{k\neq i} |c_i^+ - a_{ik}|\right\},\\ \beta_i &:= \max\left\{a_{ii} - r_i^- + \sum_{k\neq i} |r_i^- - a_{ik}|, a_{ii} - c_i^- + \sum_{k\neq i} |c_i^- - a_{ik}|\right\};\end{aligned}$$

II) If S' is the union of m intervals of S^* such that S' is disjoint from all other intervals, then S' contains precisely the real part of m eigenvalues (counting multiplicities) of A.

Doubly B-matrices

Definition 12 (doubly B-matrix) *Let $A \in \mathbb{R}^{n\times n}$. We say that A is a* doubly B-matrix*, if $\forall i \in [n]$ the following holds:*

$$\begin{aligned}&a) \quad a_{ii} > r_i^+\\ &b) \quad \forall j \neq i : \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \left(\sum_{k\neq i}\left(r_i^+ - a_{ik}\right)\right)\left(\sum_{k\neq j}\left(r_j^+ - a_{jk}\right)\right)\end{aligned}$$

Definition 13 (doubly $\bar{B}$-matrix) *We say that $A \in \mathbb{R}^{n\times n}$ is a* doubly $\bar{B}$-matrix*, if it is of a form DB, where D is a diagonal matrix with its diagonal entries from set $\{-1, 1\}$ and B is a doubly B-matrix.*

Proposition 14 *Let $A \in \mathbb{R}^{n\times n}$. It holds that A is a doubly $\bar{B}$-matrix if and only if $\forall i \in \{1, \ldots, n\}$:*

$$|a_{ii} - r_i||a_{jj} - r_j| > \left(\sum_{k\neq i} |r_i - a_{ik}|\right)\left(\sum_{k\neq j} |r_j - a_{jk}|\right)$$

Theorem 15 *Let $A \in \mathbb{R}^{n\times n}$, let $C_i := \left[a_{ii} - r_i^+, a_{ii} - r_i^-\right]$ for $i \in \{1, \ldots, n\}$ and for any $i \neq j$ in $\{1, \ldots, n\}$, let us define (assuming without loss of generality that $a_{ii} \leq a_{jj}$)*

$$B_{ij} := B_{ij}^1 \cup B_{ij}^2 \cup B_{ij}^3,$$

where

$$B_{ij}^1 := \left\{ x \in (-\infty, a_{ii}) : |a_{ii} - r_i^+ - x||a_{jj} - r_j^+ - x| \leq \left(\sum_{k \neq i} |r_i^+ - a_{ik}| \right) \left(\sum_{k \neq j} |r_j^+ - a_{jk}| \right) \right\},$$

$$B_{ij}^2 := \left\{ x \in (a_{ii}, a_{jj}) : |a_{ii} - r_i^- - x||a_{jj} - r_j^+ - x| \leq \left(\sum_{k \neq i} |r_i^- - a_{ik}| \right) \left(\sum_{k \neq j} |r_j^+ - a_{jk}| \right) \right\},$$

$$B_{ij}^3 := \left\{ x \in (a_{jj}, \infty) : |a_{ii} - r_i^- - x||a_{jj} - r_j^- - x| \leq \left(\sum_{k \neq i} |r_i^- - a_{ik}| \right) \left(\sum_{k \neq j} |r_j^- - a_{jk}| \right) \right\}.$$

Then all real eigenvalues of A belong to set B which is defined as $B := \left(\bigcup_{i=1}^n C_i\right) \cup \left(\bigcup_{i \neq j} B_{ij}\right).$

Note 16 *It can be proved that the union of the Gerschgorin circles contains the union of ovals of Cassini. And in the same manner we can observe that the union of the row $\bar{B}$-intervals contains the set B of the previous theorem.*

Bibliography

[1] Peña, J. M. *A class of P-matrices with applications to the localization of the eigenvalues of a real matrix* SIAM Journal on Matrix Analysis and Applications, 2001.

[2] Peña, J. M. *On an alternative to Gerschgorin circles and ovals of Cassini* Numerische Mathematik, 2003.

Lukáš Folwarczný

folwarczny@math.cas.cz

Presented paper by P. Beame, S. Cook, J. Edmonds, R. Impagliazzo, T. Pitassi

The Relative Complexity of NP Search Problems *as part of series* Total Search Problems

(https://dl.acm.org/doi/10.1145/225058.225147)

Introduction

Consider two subclasses A and B of the class TFNP. If both these classes contain FP, then showing a separation, i.e. $\mathsf{A} \neq \mathsf{B}$, directly implies $\mathsf{P} \neq \mathsf{NP}$. The presented paper proves several separations of important classes inside TNFP with respect to generic oracles. In my talk, I will explain the model and prove one particular separation.

Definitions and preliminaries

We consider strings x over the binary alphabet $\{0,1\}$, functions α from strings to strings, and type-2 functions (operators) F taking a pair (α, x) to a string y. Such an F is polynomial-time computable if it is computable in deterministic time that is polynomial in $|x|$ with calls to α at unit cost.

A *type-2 search problem* Q is a type-2 function that associates with each (α, x) a set $Q(\alpha, x)$ of strings that are the allowable answers to the problem on inputs α and x.

The class FNP^2 is the set of all type-2 search problems Q that are polynomial-time checkable in the sense that $y \in Q(\alpha, x)$ is a type-2 polynomial-time computable predicate and all elements of $Q(\alpha, x)$ are of length polynomially bounded in $|x|$.

A problem Q is *total* if $Q(\alpha, x)$ is nonempty for all α and x. The class TFNP^2 is the subclass of total problems in FNP^2. An algorithm A solves a total search problem Q if and only if for each function α and string x we have $A(\alpha, x) \in Q(\alpha, x)$. The class FP^2 consists of those problems in TFNP^2 which can be solved by deterministic polynomial time algorithms.

In the problem $\textsc{Lonely} \in \mathsf{TFNP}^2$, an input pair (α, x) codes a graph $GM(\alpha, |x|)$ which is a partial matching. Let $n = |x|$. The nodes are the nonempty strings of length n or less, and there is an edge between nodes u and v iff (i) $u \neq v$, (ii) $\alpha(v) = u$, (iii) $\alpha(u) = v$, and (iv) neither u nor v is the standard node 0^n. Thus 0^n is always unmatched, and we are seeking a second unmatched (or lonely) node.

In the problem $\textsc{Pigeon} \in \mathsf{TFNP}^2$, an input pair (α, x) codes a function $f\colon \{0,1\}^{\leq n} \to \{0,1\}^{\leq n}$ where $n = |x|$. The value $f(a) = \alpha(a)$ if $\alpha(a)$ has length at most n; otherwise $f(a) = 0^n$. The solution is either a $c \in \{0,1\}^{\leq n}$ s.t. $f(c) = 0^n$ or a pair $c', c''' \in \{0,1\}^{\leq n}$ s.t. $f(c) = f(c'')$. (This corresponds to the pigeonhole principle in the following way: If there is no solution of the first type, the function f is a mapping from $\{0,1\}^{\leq n}$ to $\{0,1\}^{\leq n} \setminus \{0^n\}$. Then there is a collision by the pigeonhole principle.)

Observation 1 *Neither* $\textsc{Lonely}$ *nor* $\textsc{Pigeon}$ *is in* FP^2.

We say that a type-2 problem Q_1 is many-one reducible to a type-2 problem Q_2 if there exist type-2 polynomial-time computable functions F, G, and H, such that $H(\alpha, x, y)$ is a solution to Q_1 on input (α, x) for any y that is a solution to Q_2 on input $(G[\alpha, x], F(\alpha, x))$, where $G[\alpha, x] = \lambda z.G(\alpha, x, z)$.

I will define a more general notion of reductions (Turing reductions) during my talk.

Main presented result

Theorem 2 *Lonely is not Turing reducible to Pigeon.*

The consequence of this theorem is a separation of the important classes PPA and PPP in the oracle world.

František Kmječ

frantisek.kmjec@gmail.com

Presented paper by P. W. Goldberg and A. Hollender

The Hairy Ball Problem is PPAD-Complete
as part of series Total Search Problems

(https://arxiv.org/abs/1902.07657)

Introduction

The Hairy Ball Theorem states that given an even-dimensional k-sphere and a continuous tangent vector field f defined on that sphere, there exists a point x where $f(x)$ equals zero. As shown by Goldberg and Hollender, the problem of computing the (approximate) point x is PPAD complete. In the talk we will present the proof that Hairy Ball lies in PPAD. To do that, we will need to discuss two topics:

Firstly, we will show the conversion of the Hairy Ball problem to End-of-Line with multiple sources. After that, we will look at how the multiple-source End-of-Line is equivalent to the classic End-of-Line. From those two claims we will infer that Hairy Ball problem lies in PPAD.

Definitions

Definition 1 (kD-Hairy-Ball problem) *Let $k \geq 2$ be even. The kD-Hairy-Ball problem is defined as: given $\varepsilon > 0$ and an arithmetic circuit F with $k+1$ inputs and outputs, find $x \in S^k$ such that $P_x[F(x)] < \varepsilon$. Here S^k denotes a k-dimensional sphere, F represents the continuous vector field on S_k and $P_x[\cdot]$ denotes the projection onto the tangent space of the sphere at point x.*

Definition 2 (End-of-Line) *Given boolean circuits S, P with n input bits and n output bits and such that $P(0) = 0 \neq S(0)$, find x such that $P(S(x)) \neq x$ or $S(P(x)) \neq x \neq 0$.*

In plain language, the two circuits S and P define a graph whose vertices have in and out-degree of at most one. We are given a source and are tasked with finding a sink or another source.

Definition 3 (PPAD) *Problem A is in PPAD if there exists a polynomial time reduction from A to End-of-Line.*

Nikolaj Schwartzbach

nis@cs.au.dk

Presented paper by M. Göös, P. Kamath, K. Sotiraki, and M. Zampetakis

On the Complexity of Modulo-q Arguments and the Chevalley–Warning Theorem *as part of series* Total Search Problems

(https://dl.acm.org/doi/abs/10.4230/LIPIcs.CCC.2020.19)

Introduction

Let $q \geq 2$ be an integer. We define PPA_q as the subset of TFNP reducible to a certain problem on bipartite graphs. The problem is based on the observation that a bipartite graph with a node with a non-multiple-of-q degree has at least one additional such node. For convenience, we let the distinguished node be 0^n.

Problem $\textsc{Bipartite}_q$**:**

- **Input**: Bipartite graph $(V \cup U, E)$, with $V = \{0\} \times \{0,1\}^{n-1}$, and $U = \{1\} \times \{0,1\}^{n-1}$, and a circuit $C : \{0,1\}^n \to (\{0,1\}^n)^k$. There is an edge $(u, v) \in E$ iff $u \in C(v)$ and $v \in C(u)$.
- **Output**: 0^n, if $\deg(0^n) \equiv 0 \pmod q$; otherwise, a node $v \neq 0^n$ with $\deg(v) \not\equiv 0 \pmod q$.

Note that we get $\mathsf{PPA} = \mathsf{PPA}_2$, as $\textsc{Bipartite}_2$ is a canonical problem for PPA.

Definition 1 *Let S_0, S_1 be two problems, then we define $S_0 \,\&\, S_1$ as the problem that on input $(x, b) \in \{0,1\}^* \times \{0,1\}$ should give the solution to x interpreted as an instance of S_b. If $\mathsf{M}_0, \mathsf{M}_1$ are complexity classes, we define $\mathsf{M}_0 \,\&\, \mathsf{M}_1$ as the complexity class containing all problems reducible to $S_0 \,\&\, S_1$.*

Theorem 2 (Structural properties of PPA_q) *The following holds for every $q \geq 2$.*

1. *$\mathsf{PPAD} \subset \mathsf{PPA}_q$.*
2. *$\mathsf{PPA}_q = \&_{p \text{ prime divisor of } q}\, \mathsf{PPA}_p$.*
3. *If q is prime, PPA_q is closed under Turing reductions.*
4. *PPA_q is oracle separated from $\mathsf{PPAD}, \mathsf{PPADS}, \mathsf{PPP}$, and PLS.*

We now state a theorem from number theory that can be used to define a so-called 'natural problem'[1] for PPA_q.

Theorem 3 (Chevalley-Warning) *Let p be prime, and let $\mathbf{f} = \langle f_1, f_2, \ldots, f_n \rangle \in \mathbb{F}_p[\mathbf{x}]^n$ be a set of polynomials, such that $\sum_{i=1}^{n} \deg(f_i) < n$. Let*

$$\mathcal{V}_{\mathbf{f}} = \{\mathbf{x} \in \mathbb{F}_p^n \mid \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$$

be the set of solutions to $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. Then $|\mathcal{V}_{\mathbf{f}}| \equiv 0 \pmod p$.

[1] According to the authors, a 'natural problem' is a problem that does not explicitly have a circuit/Turing machine as part of its description. Whether this constitutes a 'natural' problem is debatable. In particular, $\textsc{Chevalley}_q$ has a system of polynomials as part of its description that can be explicitly used to encode a circuit.

This theorem can be used to define a problem, $\textsc{Chevalley}_q$, related to finding solutions in a system of polynomials. The observation is that by letting all constant terms be zero (a *zecote* polynomial), we have a trivial solution $\mathbf{0} = (0, 0, \ldots, 0)$, and hence by Theorem 3 there must be at least one additional solution, so find it.

Problem $\textsc{Chevalley}_q$:

- **Input**: System of zecote polynomials $\mathbf{f} \in \mathbb{F}_p[\mathbf{x}]^n$ with $\sum_{i=1}^n \deg(f_i) < n$.
- **Output**: $\mathbf{x} \in \mathbb{F}_p^n \setminus \{\mathbf{0}\}$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

It can be established that $\textsc{Chevalley}_q \in \mathsf{PPA}_q$, but it is not known if the problem is also hard; it is conjectured by the authors this is not the case because the condition that ensures there is a non-trivial solution needs to be 'syntactically refutable'. However, if we add more structure to the problem, it does become PPA_q-complete.

First, let $\sigma \in S_n$ be a permutation, and let $\langle\sigma\rangle$ be the subgroup generated by σ, and let $|\sigma|$ be the order of $\langle\sigma\rangle$. If $\mathbf{x} \in \mathbb{F}_p^n$, we denote by $\sigma(\mathbf{x}) \in \mathbb{F}_p^n$ the vector resulting from permuting the rows according to σ. Now let $\mathcal{V} \subseteq \mathbb{F}_p^n$, we say that $\langle\sigma\rangle$ *acts freely on* $\mathcal{V}$ if for every $\mathbf{x} \in \mathcal{V}$, it holds that $\sigma(\mathbf{x}) \in \mathcal{V} \setminus \{\mathbf{x}\}$. Also denote $\overline{\mathcal{V}} = \mathbb{F}_p^n \setminus \mathcal{V}$.

Theorem 4 (Chevalley-Warning with Symmetry) *Let p be prime, and let $\mathbf{g} \in \mathbb{F}_p[\mathbf{x}]^n, \mathbf{h} \in \mathbb{F}_p[\mathbf{x}]^m$ be two systems of polynomials, and let $\mathbf{f} = \langle \mathbf{g}, \mathbf{h} \rangle$. If there is a permutation $\sigma \in S_{n+m}$ with $|\sigma| = p$ such that 1) the degree of $\mathbf{f}$ is 'not too large'*[2]*, and 2) $\langle\sigma\rangle$ acts freely on $\mathcal{V}_{\mathbf{g}} \cap \overline{\mathcal{V}_{\mathbf{h}}}$, then $|\mathcal{V}_{\mathbf{f}}| \equiv 0 \pmod{p}$.*

Problem $\textsc{ChevalleyWithSymmetry}_q$:

- **Input**: System of zecote polynomials $\mathbf{f} = \langle \mathbf{g}, \mathbf{h} \rangle \in \mathbb{F}_p[\mathbf{x}]^n$, and a permutation $\sigma \in S_n$.
- **Output**: Any of the following:
  1. A proof that the degree of $\mathbf{f}$ is 'too large' (see [2]).
  2. A proof that σ does not act freely, i.e. an $\mathbf{x} \in \mathcal{V}_{\mathbf{g}} \cap \overline{\mathcal{V}_{\mathbf{h}}}$, s.t. $\sigma(\mathbf{x}) \notin \mathcal{V}_{\mathbf{g}} \cap \overline{\mathcal{V}_{\mathbf{h}}} \setminus \{\mathbf{x}\}$.
  3. An element $\mathbf{x} \in \mathcal{V}_{\mathbf{f}} \setminus \{\mathbf{0}\}$.

Theorem 5 (Main result) *For any prime p, $\textsc{ChevalleyWithSymmetry}_p$ is PPA_p-complete.*

[2] Formally, this property relates to the symbolic monomial expansion of a certain polynomial, $\mathsf{CW}_{\mathbf{f}}$, that can be deduced from the polynomial system; we require that there are no 'maximum degree' monomials in the resulting symbolic expansion.

Standalone Talks

Brage Isak Keiserås Bakkane

brage@bakkane.net

Mim-width and $W[1]$-hardness of (σ, ρ)-dominatation Problems Parameterized by Mim-width

Introduction

If we accept that some problems are hard (i.e. $P \neq NP$) then we would still like to be able to solve hard problems fast. One way to do this is by either knowing more about the input, or knowing more about what solution we want to find. This can often be expressed by some number known as a parameter which is given along with the other input for each instance of a problem.

For instance, instead of the hard problem of finding the smallest vertex cover we could instead have the problem of finding a vertex cover of size at most k. This problem can be solved in $O(2^k n)$ time by a method known as branching.

When a problem can be solved in $f(k) \cdot n^c$ time, for some constant c and some computable function f the problem is fixed parameter tractable or FPT. This class can be seen as the "P" of parameterized complexity. Note that $f(k) \cdot n^c$ is still exponential, and if one wanted to find the smallest vertex cover one would have to check k for all values $\{1, 2, ..., n\}$. However $f(k) \cdot n^c$ can be considerably faster than for instance $n^{f(k)}$ which is known as XP time.

Another type of parameter are width parameters. These are parameters for graph problems which describe some sort of complexity of the graph. For instance, tree-width describes how "tree-like" the graph is, clique-width generalises tree-width and is bounded on several simply structured dense graphs. And the width parameter which will be described in this handout is mim-width, short for maximum induced matching width.

The central question of a parameter is of course how useful is it, does it indeed make hard algorithms easy? And the answer is of course it depends. For mim-width known hard problems, like maximum independent set, can be shown to be XP (solvable in $n^{f(k)}$ time, where $f(.)$ is some computable function) when parameterized by mim-width when given its branch decomposition along with the input graph. In addition it can be shown that certain graph classes have bounded mim-width, which in particular implies polynomial time solvability for XP or FPT algorithms on those classes.

Mim-width

Recall that a matching is a set of edges whose end points are distinct, see Figure 1. An induced matching is matching for which all edges that cannot be in the matching is not in the graph, i.e. if M is a matching, and E the edges of some graph, where $ab \in M$ and $cd \in M$, then $ac \notin E$, $ad \notin E$, $bc \notin E$, and $bd \notin E$, see Figure 1.

The mim-width measures the largest induced matching, but where? This is a described by a branch decomposition. A branch decomposition of a graph G is a pair $(T, \mathcal{L})$, where T is a tree where its vertices have degree at most 3, which in general is not rooted but for simplicity we will assume that it is, and $\mathcal{L}$ a bijection mapping the vertices of the graph $V(G)$ to the leaves of the tree T. For an example of a branch decomposition see Figure 2.

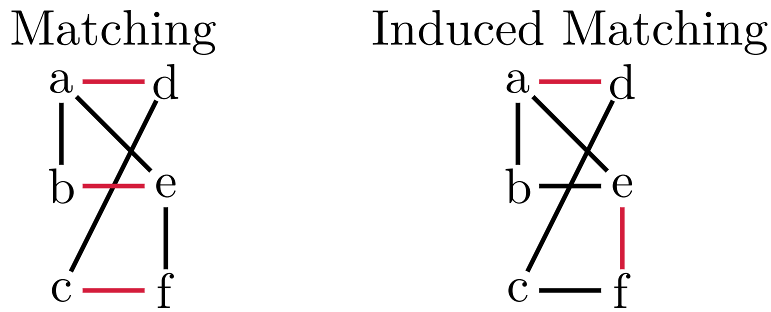

Figure 1: Example of a matching and an induced matching.

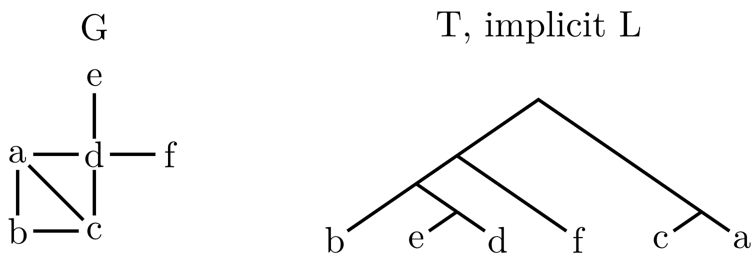

Figure 2: Example of a branch decomposition.

Let T_t be the subtree of T rooted at t, $V_t = \mathcal{L}^{-1}(L(T_t))$, $G_t = G[V_t]$, and $\overline{V}_t = V(G) \setminus V_t$. Then the mim-width of $(T, \mathcal{L})$ is $\max_{t \in V(T)} mim_G(V_t)$, where

$$mim_G(A) = \max\{|M| \mid M \text{ is an induced matching in } G[A, V(G) \setminus A]\}$$

Furthermore, we define the mim-width of a graph G to be the minimum mim-width over all branch decompositions of G.

A simpler type of branch decomposition is a linear branch decomposition, which is equivalent to a linear ordering on the vertex set of the graph. If we have a graph with the ordering: $a < b < c < d < ... < z$, then the mim-width of the ordering is

$$\max\{\max\{|M| \mid M \text{ is an induced matching in } G[L_a, V(G) \setminus L_a] = G[L_a, R_a]\} \mid a \in V(G)\}$$

where $L_a = \{\alpha \mid \alpha \leq a\}$ and $R_a = \{\alpha \mid \alpha > a\}$, and we call (L_a, R_a) a cut along the ordering. For an example see Figure 3.

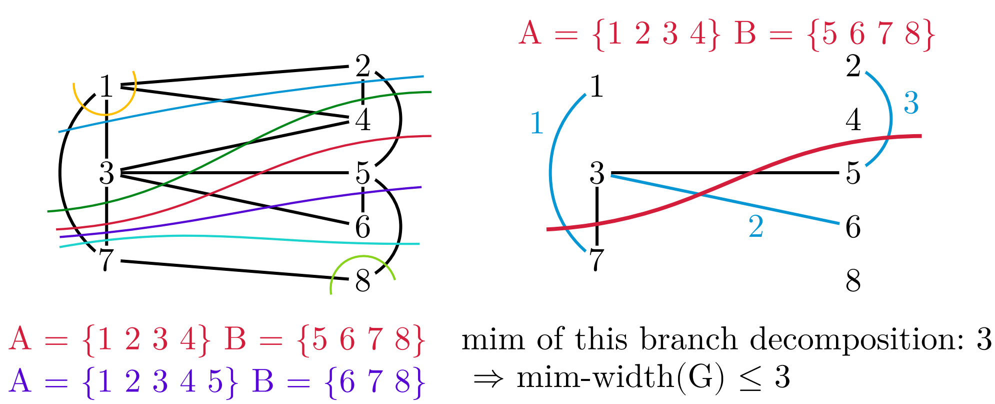

Figure 3: Example of linear mim-width along with ordering.

Graph classes and their mim-width

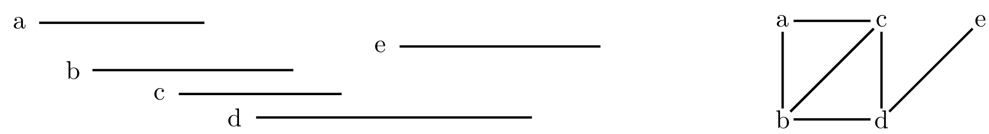

Figure 4: Example of an interval graph.

Many graph classes have bounded mim-width, the graph class with bounded mim-width we will consider here are Interval graphs.

An interval graph G has a set of intervals $I = \{I_v = (s_v, e_v) \mid s_v \leq e_v, v \in V(G)\} \subset \mathbb{R} \times \mathbb{R}$ where $uv \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$. For an example see Figure 4.

We will then prove that any interval graph G has bounded mim-width by giving an ordering on the vertices (equivalent with a linear branch decomposition) of mim-width 1.

Order the vertices of $V(G)$ by the start point of the interval, i.e. by s_v. Then $u < v$ if $s_u < s_v$. Then suppose that for some cut along the ordering (L_i, G_i) there is a matching of size > 1, then there must be a pair of edges $ab, cd \in M$ with $a, c \in L_i$ and $b, d \in G_i$. As $ab \in M$ $I_a \cap I_b \neq \emptyset$ i.e. $s_b < e_a$ and similarly $s_d < e_c$. Furthermore we can assume without loss of generality that $e_a > e_c$. Then $s_d < e_c < e_a$ and therefore $ad \in E(G)$. But then M is not induced. Therefore there is no induced matching of size > 1 in (L_i, G_i) and therefore mim-width of G is 1. And in general the mim-width of any interval graph is 1. See Figure 5.

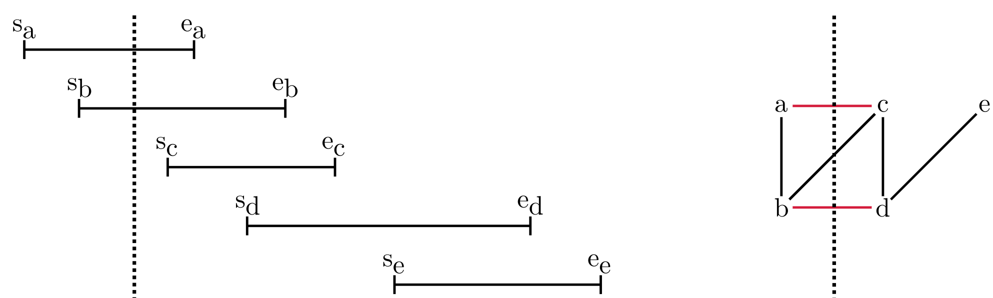

Figure 5: The same interval graph as in Figure 4, but ordered and with a cut. Note that the red edges in the graph cannot be an induced matching.

Algorithmic applications of branch decompositions and mim-width

As mentioned maximum independent set, when given its branch decomposition, parameterized by its mim-width is in XP and we will show this by giving an algorithm. The algorithm uses the branch decomposition, the mim-width itself is simply used for an upper bound on the run time. Let G be a graph with the rooted branch decomposition $(T, \mathcal{L})$, and let r be the root of T.

Then we define an equivalence relation over 2^{V_t} such that for any set X equivalent with Y X and Y have the same neighbourhoods outside V_t. We say for two sets $X, Y \subseteq V_t$ $X \equiv_t Y$ if $N(X) \cap \overline{V}_t = N(Y) \cap \overline{V}_t$. Then for each equivalence class Q_t we only need to store the largest independent set of G_t contained in Q_t, as the vertex sets in Q_t all have the same neighbourhoods outside of V_t. Therefore in G_t the independent set in Q_t can always be replaced by any other independent set in Q_t. For an example of a branch decomposition along with examples of the equivalence relation see Figure 6.

Therefore for each $t \in V(T)$ and each equivalence class $Q_t \in 2^{V_t}/\equiv_t$ we store the maximum independent set of G_t contained in Q_t. However equivalence classes themselves can be exponentially

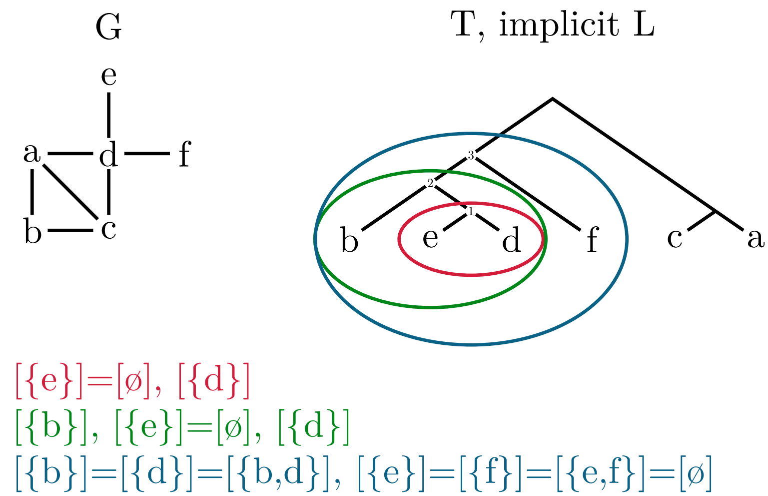

Figure 6: A branch decomposition highlighting some examples of the equivalence relation $\equiv_t$.

large, therefore we instead store a description of the equivalence class, this description we call $desc_t(Q_t) = \{S \subseteq \overline{V}_t \mid \forall X \in Q_t \;\; N(X) \cap \overline{V}_t = S\}$.

Then we define a table: tab, where $\mathsf{tab}[t, desc_t(Q_t)] = argmax_{S \in Q_t}\{|S| \mid S \text{ is an independent set in } G_t\}$. A problem does arise here, as $desc_t(Q_t)$ is a subset of $V(G)$ and therefore one would require $\geq 2^{|V(G)|}$ many table entries. However "most" of these are uninteresting, as we only care about the subsets of V_t which actually describe / correspond to some equivalence class of $\equiv_t$, and as we will see later the number of equivalence classes is bounded by mim-width. We therefore assume that if the table entry is not filled then it holds $\emptyset$.

Let t be a leaf of T, then recall that t is "equivalent" with some $v \in V(G)$. Therefore $\equiv_t$ has two equivalence classes: $\{\emptyset\}$ and $\{\{v\}\}$, with the following descriptions $\emptyset$ and $N(v)$. We then fill the table entries: $\mathsf{tab}[t, desc_t\{\emptyset\}] = \emptyset$ and $\mathsf{tab}[t, desc_t(\{\{v\}\})] = \{v\}$.

Now let t be an internal node of T with the children a and b. Observe that if $X \equiv_a Y$ or $X \equiv_b Y$ then $X \equiv_t Y$, therefore the equivalence classes of $\equiv_a$ and $\equiv_b$ do not have to be split up, only merged, in order to obtain the equivalence classes of $\equiv_t$. We can therefore enumerate all descriptions of the equivalence classes of $\equiv_t$ by considering at most $|2^{V_a}/\equiv_a| \times |2^{V_b}/\equiv_b|$ subsets of $\overline{V}_t$.

```
foreach Q_a ∈ 2^{V_a}/≡_a and Q_b ∈ 2^{V_b}/≡_b do
    X_a ← tab[a, desc_a(Q_a)], X_b ← tab[b, desc_b(Q_b)];
    if no edges from X_a to X_b then
        S = (desc_a(Q_a) ∪ desc_b(Q_b)) ∩ V̄_t ;
        if |X_a ∪ X_b| ≥ |tab[t, S]| then
            tab[t, S] ← X_a ∪ X_b ;
```

Algorithm 1: Maximum independent set table.

Algorithm 1 solves $\mathsf{tab}[t, desc_t(Q_t)]$, which needs $\mathsf{tab}[a, desc_a(Q_a)]$ and $\mathsf{tab}[b, desc_b(Q_b)]$ to be computed for all $Q_a \in 2^{V_a}/\equiv_a$ and $Q_b \in 2^{V_b}/\equiv_b$:

Then the root node of the tree r holds the value $\mathsf{tab}[r, Q_r]$ with $Q_r = 2^{V_r}$ which holds the maximum independent set of the whole graph. In order to find this we first fill in the values of all leaves, then gradually traverse upwards in the tree computing $\mathsf{tab}[.,.]$ as we go.

We now describe the running time of the algorithm. Let the number of equivalence classes in the branch decomposition $(T, \mathcal{L})$ be $nec(T, \mathcal{L}) = \max_{t \in V(T)} |2^{V_t} / \equiv_t|$. The leaf nodes take constant time to compute, internal nodes we need to consider $\leq nec(T, \mathcal{L})^2$ pairs of equivalence classes, and for each equivalence class pair we do operations taking at most $O(n^2)$ time. Furthermore there are $O(n)$ nodes in $V(T)$ for which we need to compute $\texttt{tab}[.,.]$. Therefore the total runtime is $O(nec(T, \mathcal{L})^2 \cdot n^3)$.

Finally $nec(T, \mathcal{L}) \leq n^{mim(V_t)}$. This is as if the size of a maximum induced matching over $(V_t, \overline{V}_t)$ is at most k then each equivalence class of $\equiv_t$ contains a set of size at most k, as we show below. Therefore if we look at all equivalence class of size $\leq k$ then all equivalence classes will have been enumerated by the time we reach size $= k$.

We will show that $mim(V_t) \leq k$ if and only if for all $S \subset V_t$ there is a set $R \subseteq S$ where $|R| \leq k$ and $R \equiv_t S$. To do so we will show that $mim(V_t) > k$ if and only if there is some $S \subseteq V_t$ where for every $R \subseteq S$, either $|R| > k$ or $N(R) \cap \overline{V}_t \neq N(S) \cap \overline{V}_t$ i.e. $S \not\equiv_t R$.

Let M be an induced matching in $G[V_t, \overline{V}_t]$ of size $> k$, and let $S = V(M) \cap V_t$. Furthermore let R be any subset of S. Then if $R = S$ then $|R| = |S| > k$. If $R \subset S$ then let $u \in S \setminus R$, and let $v \in \overline{V}_t$ be the vertex matched to u (i.e. $uv \in M$). M is an induced matching therefore v is not adjacent to any other vertices in S and therefore $v \in N(S) \cap \overline{V}_t$ but $v \notin N(R) \cap \overline{V}_t$. Therefore $S \not\equiv_t R$.

For the other direction let $S \subseteq V_t$ be an inclusion-wise minimal set such that for all $R \subseteq S$ either $|R| > k$ or $R \not\equiv_t S$. Then $|S| > k$ as if not R cannot always be $> k$, and there does not exist a vertex $v \in S$ such that $N(S \setminus \{v\}) \cap \overline{V}_t = N(S) \cap \overline{V}_t$ as if there were S would not be minimal.

Therefore for all $v \in S$ trivially $N(S \setminus \{v\}) \cap \overline{V}_t \subseteq N(S) \cap \overline{V}_t$, combined with $N(S \setminus \{v\}) \cap \overline{V}_t \neq N(S) \cap \overline{V}_t$ gives $N(S \setminus \{v\}) \cap \overline{V}_t \subset N(S) \cap \overline{V}_t$.

Therefore for all $v \in S$ there must exist an unique vertex $\overline{v} \in \overline{V}_t$ adjacent to v, but not adjacent to any other vertex in S. Therefore $M = \{v\overline{v} \mid v \in S\}$ is an induced matching in $G[V_t, \overline{V}_t]$ of size $> k$.

Then as described above if we enumerate all subsets of size $\leq k$ we will have enumerated at least one element of each equivalence class. Therefore $\equiv_t$ has at most $|V_t|^{mim(V_t)} \leq n^{mim(V_t)}$. And if w is the mim-width of $(T, \mathcal{L})$ we can then solve maximum independent set in time $O(n^{2w+3})$.

Generalised domination problems

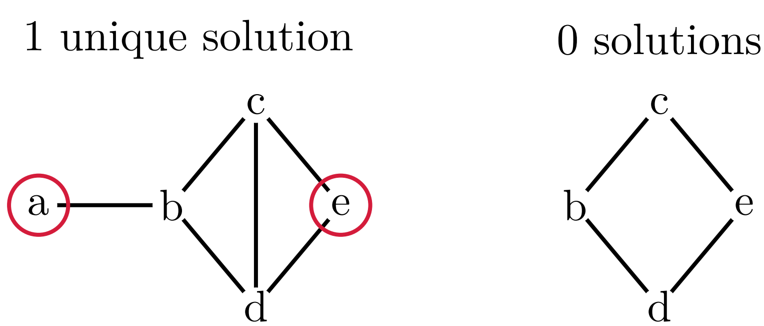

Figure 7: Example of the perfect code problem, or the $(\{0\}, \{1\})$-dominating set problem. Circled vertices are vertices in the solution set.

The dominating set problem asks if there is some set of a certain size such that every vertex not in the set is adjacent to at least one vertex in the set. There also exist problems asking if there exists some set such that every vertex not in the set is adjacent to a given amount of vertices in

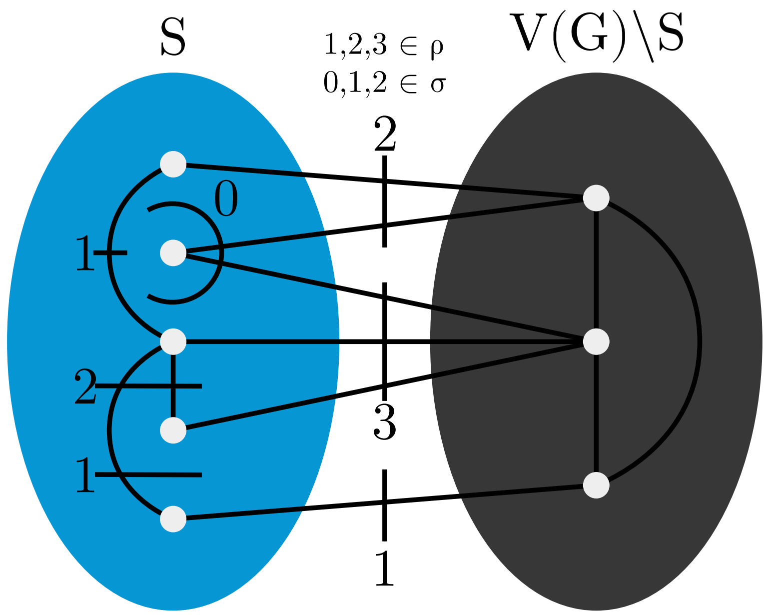

Figure 8: Example of the (σ, ρ)-dominating set problem, where S is the (σ, ρ)-dominating set. More specifically this is the $(\{0, 1, 2\}, \{1, 2, 3\})$-dominating set problem or any $(\sigma_{012}, \rho_{123})$ where $0, 1, 2 \in \sigma_{012}$ and $1, 2, 3 \in \rho_{123}$.

the set. One such example is the perfect dominating set problem where every vertex not in the set is adjacent to exactly one vertex in the set.

Other dominating set-like problems place restrictions on how the dominating set should interact with itself. One example of which is the dominating independent set problem, which asks for a dominating set which is also an independent set. Finally both of these "generalised" dominating set problems can be combined into a problem imposing restrictions on both the dominating set and on how many vertices a vertex should be dominated by. One example of this is the perfect code problem, which asks for an independent perfect dominating set. See Figure 7 for an example of the perfect code problem.

All of these problems can be generalised into one class of problems: (σ, ρ)-dominating set problem, where $\sigma, \rho \subseteq \mathbb{N}$. These are problems asking whether there is a set of a certain size with the following property: $\forall v \in V(G)$ if $v \in D$ then $|N(v) \cap D| \in \sigma$. If $v \notin D$ then $|N(v) \cap D| \in \rho$. So σ describes how vertices in the set should interact and ρ describe by how many vertices a vertex not in the set should be dominated. See Figure 8 for an example of a (σ, ρ)-dominating set problem.

The dominating set problem is only difficult ($W[1]$-hard) if we ask for if there is a dominating set of size $\leq k$ for some k. And the independent set problem is only difficult if we ask for sets of size $\geq k$. So when are (σ, ρ)-dominating set problems hard? The answer is that it depends. For some problems it might be hard for minimisation i.e. $\leq k$, for others for maximisation i.e. $\geq k$, and for some problems both are hard. The problems where both are hard are when we ask for a set with "difficult" constraints on both σ and ρ, for instance the perfect code problem and the dominating induced matching problem.

The "normal" dominating set problem is then the $(\mathbb{N}, \mathbb{N}\setminus\{0\})$-dominating set, the perfect dominating set problem is the $(\mathbb{N}, \{1\})$-dominating set, the dominating induced matching problem is the $(\{1\}, \mathbb{N}\setminus\{0\})$-dominating set, and perfect code is the $(\{0\}, \{1\})$-dominating set.

$W[1]$-hardness of the (σ, ρ)-dominating set problem parameterized by mim-width.

We will show the $W[1]$-hardness of the (σ, ρ)-dominating set problem parameterized by mim-width by a parameterized reduction from the multicoloured clique problem. Formally a parameterized reduction is an algorithm $\mathcal{A}$ taking in a instance (G, k) of problem A and turning it into the instance of problem B: (H, l), where (H, l) is a yes-instance of B if and only if (G, k) is a yes-instance of A. Furthermore $l < g(k)$, and $\mathcal{A}$ runs in time $f(k) \cdot |G|^{O(1)}$ for some computable functions $f(.)$ and $g(.)$.

We will start with the multicoloured clique problem which asks if for some instance: $G, V_1, ..., V_k$, where $V_1, ..., V_k$ is a partition of $V(G)$, if there is a clique in G $c_1, c_2, ..., c_k$ such that $c_1 \in V_1, c_2 \in V_2, ..., c_k \in V_k$. First observe that we can assume that $|V_1| = |V_2| = \cdots = |V_k|$ as if they are not we can add isolated vertices to the sets such that they all are the same size. And isolated vertices clearly do not affect whether G is has a multicoloured clique. Let $p = |V_1| = \cdots |V_k|$. We will also label the vertices of V_i by $\{v_1^i, v_2^i, ..., v_p^i\}$.

Then we construct the instance $(\mathcal{H}, k')$ of the perfect code problem parameterized by mim-width, where $k' = \binom{k}{2}$. See Figure 9.

First let $i < j$ for some $i, j \in [k]$, and $a, b \in [p]$, with the notation that $[k] = \{1, 2, ..., k\}$.

We first add both x_a^{ij} and x_b^{ji} to $\mathcal{H}$.

We then add s_a^j to $\mathcal{H}$. We connect the vertices s_a^j and $x_a^{jj'}$ for all $j' \neq j$.

If $v_a^i v_b^j$ is an edge in G we add the vertex r_{ab}^{ij} to $\mathcal{H}$. This vertex will be connected to all the vertices in $\{x_{a'}^{ij} \in V(\mathcal{H}) \mid a' \neq a\}$ and to all the vertices in $\{x_{b'}^{ji} \in V(\mathcal{H}) \mid b' \neq b\}$.

Let $S^j = \{s_a^j \mid a \in [p]\}$, $R^{ij} = \{r_{ab}^{ij} \in V(\mathcal{H}) \mid a, b \in [p]\}$, and $X^{ij} = \{x_a^{ij} \mid a \in [p]\}$, and $X^{ji} = \{x_a^{ji} \mid a \in [p]\}$. We make all of these sets into cliques. Furthermore we make $X = \{x_a^{ij}, x_a^{ji} \mid i, j \in [k], a \in [p]\}$ into a clique.

Furthermore let $X^j = \bigcup_{j' \neq j} X^{jj'}$

Notice that for all $i', j' \in [p]$ where $i' \neq j'$, $R^{i'j'}$ exists but $R^{j'i'}$ does not. However both $X^{i'j'}$ and $X^{j'i'}$ exist.

Then if $v_{c_1}^1, v_{c_2}^2 ..., v_{c_k}^k$ is a multicoloured clique in G, then $\{r_{c_i c_j}^{ij}, s_{c_j}^j \mid i < j, i, j \in [k]\}$ is a perfect code of size $\leq k'$ of $\mathcal{H}$.

We will give a rough sketch for the proof of the other direction. We can assume that $p > 2$, either C intersects a clique or all the vertices of the clique has to be dominated by some vertex outside the clique, which can only be in X. The presence of a vertex in $X \cap C$ would then imply that all cliques, except for X, must have no vertices in C, or that more than 1 vertex has to be in X, neither of which can be the case. Therefore any perfect code C can intersect each clique exactly once, and $X \cap C = \emptyset$.

Furthermore, if some $s_{c_i}^i \in C$ then $r_{c_i a''}^{ij''} \in C$ $r_{a' c_i}^{j'i} \in C$ for all $i, j', j'' \in k$ such that $j' < i < j''$. This ensures that $v_{c_1}^1, ..., v_{c_k}^k$ is a multicoloured clique in G.

This graph $\mathcal{H}$ is then the supergraph of the graph in the general reduction from multicoloured clique to the minimisation or maximisation (σ, ρ)-dominating set problem. More vertices are added depending on σ and ρ. See the Figures 10 and 11.

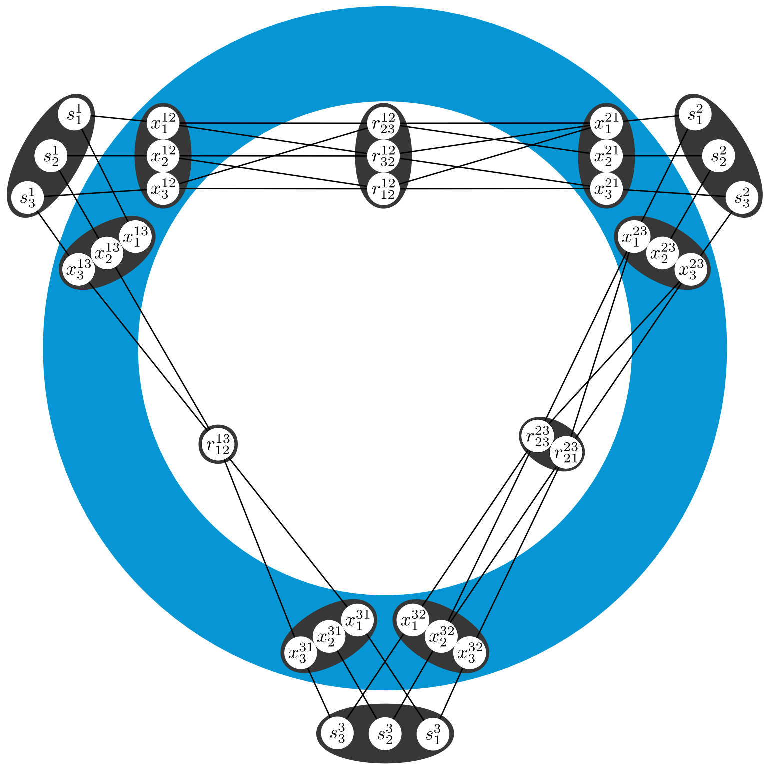

Figure 9: Example of $\mathcal{H}$.

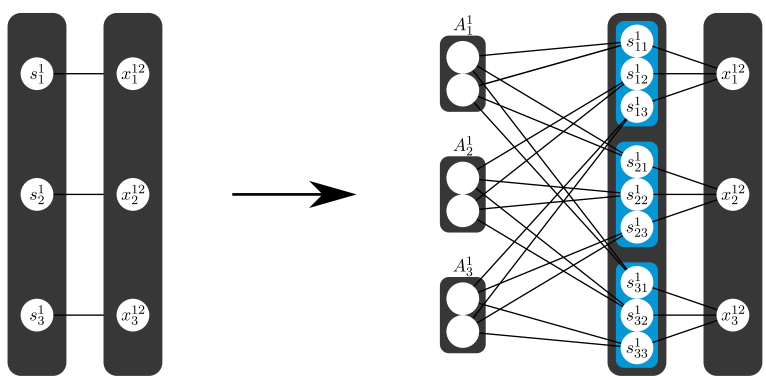

Figure 10: Example of $S^1 \cup X^{12}$ for $p = 3$, $\varrho = 6$, $\varsigma = 3$ where $\varsigma = \min(\sigma)$ and $\varrho = \min(\rho)$. Circles indicate vertices, and grey coloured regions indicate cliques. The blue regions indicate vertices not adjacent even though they should be according to the grey colouring. This is the construction used for minimisation problems if $\varrho - 1 > \varsigma$, and for the maximisation problems if we instead let $\varsigma = \max(\sigma)$ and $\varrho = \max(\rho)$ and $\varrho > \varsigma$.

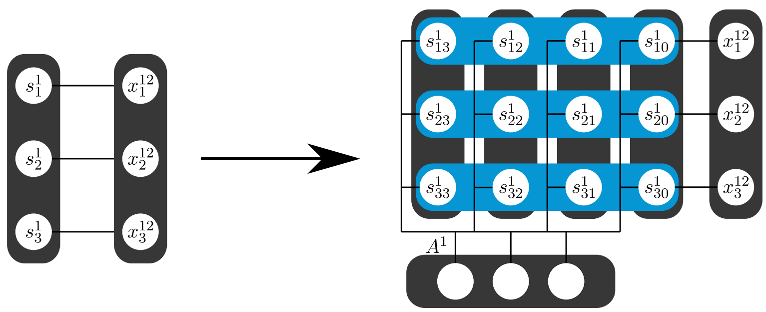

Figure 11: Example of $S^1 \cup X^{12}$ for $p = 3$, $\varsigma = 6$, $\varrho = 3$ where $\varsigma = \min(\sigma)$ and $\varrho = \min(\rho)$. This is the construction used for minimisation problems if $\varsigma \geq \varrho$, and for the maximisation problems if we instead let $\varsigma = \max(\sigma)$ and $\varrho = \max(\rho)$ and $\varsigma \geq \varrho$.

Petr Chmel
petr@chmel.net
Presented paper by Marcus Schaefer

A New Algorithm for Embedding Plane Graphs at Fixed Vertex Locations

(https://doi.org/10.37236/10106)

Introduction

We investigate the problem of attempting to draw a plane graph with preassigned locations of vertices while drawing the edges as polygonal chains and attempting to minimize the number of bends of the polygonal chains per edge.

The main result

Theorem 1 *Given a plane graph G on n vertices $v_1, \dots, v_n$ and n distinct points $p_1, \dots, p_n$ in the plane, we can find an isomorphic embedding of G in which v_i is located at p_i for all $1 \leq i \leq n$, and every edge is a polygonal chain with at most $2.5n + 1$ bends. Moreover, the embedding can be found in quadratic time.*

The tools

Lemma 2 *Any plane Hamiltonian graph on n vertices has an isomorphic embedding at fixed vertex locations in which every edge is drawn as a polygonal chain with at most $2n - 1$ bends. Moreover, the embedding can be found in quadratic time given the Hamiltonian cycle.*

Definition 3 (Synchronized covering) *Polygonal chains $P_1, \dots, P_n$ form a* synchronized covering *of points $p_1, \dots, p_n$ if the chains are pairwise disjoint, chain P_i contains the point p_i for every $i \in [n]$, the i-th points of all chains lie on the same line ℓ_j (we label the lines so that $p_j \in \ell_j$ and we call them* bend-lines*) and the polygonal chains do not cross the bend-lines except at their vertices.*

Lemma 4 *For every set of n distinct points in the plane, we can find a synchronized covering such that all bend-lines are parallel.*

Lemma 5 *Every plane graph on n vertices can be made Hamiltonian by subdividing each edge at most once, and adding some edges. The added vertices have degree at most four. A Hamiltonian cycle can be found in linear time. Moreover, whenever the Hamiltonian cycle passes through a subdivision vertex, the two parts of the subdivided edge lie on the opposite sides of the cycle in the embedding.*

Tools from other literature

Theorem 6 (Whitney [1]) *Every plane triangulation without a separating triangle has a Hamiltonian cycle.*

Theorem 7 (Finding Hamiltonian cycles in plane triangulations [2]) *There exists a linear time algorithm for finding Hamiltonian cycles in 4-connected maximal planar graphs.*

Bibliography

[1] Hassler Whitney, A Theorem on Graphs, Annals of Mathematics, https://doi.org/10.2307/1968197.

[2] Takao Asano, Shunji Kikuchi, Nobuji Saito, A linear algorithm for finding Hamiltonian cycles in 4-connected maximal planar graphs, Discrete Applied Mathematics, https://doi.org/10.1016/0166-218X(84)90109-4.

Bartłomiej Kielak

bartlomiej.kielak@doctoral.uj.edu.pl

Presented paper by A. P. Bharathi, S. A. Choudum

Colouring of $(P_3 \cup P_2)$-free Graphs

(https://link.springer.com/article/10.1007/s00373-017-1870-8)

Introduction

It is a well-known fact that there exist triangle-free graphs with an arbitrarily high chromatic number. However, for many classes of graphs we can bound the chromatic number by a certain function of the clique number — we call such classes of graphs χ-bounded. Our main object of interest will be the class of $(P_3 \cup P_2)$-free graphs, for which we will show that the chromatic number is bounded by the cubic function of the clique number.

Preliminaries

For a graph G, we will denote its chromatic number by $\chi(G)$ and its clique number (i.e. the size of any largest clique in G) by $\omega(G)$.

Definition 1 *A class $\mathcal{C}$ of graphs is χ-bounded if there exists a function $f : \mathbb{N} \to \mathbb{N}$ with the following property — for every graph $G \in \mathcal{C}$, $\chi(G) \leq f(\omega(G))$. If the function f is polynomial, we also say that $\mathcal{C}$ is polynomially χ-bounded.*

Definition 2 *A graph G is H-free if any of its induced subgraph is not isomorphic to H.*

Main results

Theorem 3 *For any $(P_3 \cup P_2)$-free graph G we have $\chi(G) \leq \binom{\omega(G)+2}{3}$. In particular, a class of $(P_3 \cup P_2)$-free graphs is polynomially χ-bounded.*

It turns out that one can obtain the same bound for the $(P_4 \cup P_2)$-free graphs.

Theorem 4 *For any $(P_4 \cup P_2)$-free graph G we have $\chi(G) \leq \binom{\omega(G)+2}{3}$. In particular, a class of $(P_4 \cup P_2)$-free graphs is polynomially χ-bounded.*

Matyáš Křišťan
kristja6@fit.cvut.cz

Efficient attack sequences in m-eternal domination

Introduction

We study the m-eternal domination problem from the perspective of the attacker. For many graph classes, the minimum required number of guards to defend eternally is known. By definition, if the defender has less than the required number of guards, then there exists a sequence of attacks that ensures the attacker's victory. Little is known about such sequences of attacks, in particular, no bound on its length is known.

We show that if the game is played on a tree T on n vertices and the defender has less than the necessary number of guards, then the attacker can win in at most n turns. Furthermore, we present an efficient procedure that produces such an attacking strategy.

Definition

Consider the following game of two players, an attacker and a defender, played on a simple graph. First, the defender places guards on some vertices. In each turn, the attacker attacks a vertex. The defender responds by moving some (all) guards along adjacent edges, so that the attacked vertex is occupied, thus defending against the attack. The defender wins if he is able to defend indefinitely.

Definition 1 *A set of vertices is* m-eternally dominating*, if used as a starting configuration of guards, it is possible to defend agains any sequence of attacks indefinitely.*

Definition 2 *The m-eternal domination number of G is the minimum number of guards required to defend G indefinitely and is denoted by $\gamma_m^\infty(G)$.*

Definition 3 *A* neo-colonization *of a graph G is a partition $\{V_1, \ldots, V_t\}$ of the vertex set of G such that each $G[V_i]$ is connected. Let $\gamma_c(G)$ denote the size of a minimum connected dominating set. Each part V_i is assigned a weight $\omega(V_i)$ as follows.*

$$\omega(V_i) = \begin{cases} 1 & \textit{if } G[V_i] \textit{ is a clique} \\ \gamma_c(G[V_i]) + 1 & \textit{otherwise} \end{cases}$$

Definition 4 *A* shrubbery *is a tree with no vertices of degree 2.*

Result

Theorem 5 ([2]) *If the number of guards on a tree T is less than $\gamma_m^\infty(G)$, then the attacker can win in at most d turns, where d is the diameter of the tree. Furthermore, the sequence of attacks can be generated in polynomial time.*

Bibliography

[1] William F. Klostermeyer and Christina M. Mynhardt *Protecting a Graph with Mobile Guards*. Applicable Analysis and Discrete Mathematics, 2014

[2] Efficient attack sequences in m-eternal domination
`https://arxiv.org/abs/2204.02720`, 2022

Gaurav Kucheriya
gaurav@kam.mff.cuni.cz
Presented paper by D. Conlon, A. Ferber

Lower Bounds on Multicolor Ramsey Numbers

(https://www.sciencedirect.com/science/article/pii/S0001870820305569)

Introduction

Definition 1 *The Ramsey number $r(t;\ell)$ is the smallest natural number n such that every ℓ-coloring of the edges of the complete graph K_n contains a monochromatic K_t.*

Some simple bounds known since the 1940's: $\sqrt{2}^t \leq r(t,2) \leq 4^t$.

Observation 2 (Lefmann) $r(t;\ell_1+\ell_2)-1 \geq (r(t;\ell_1)-1)(r(t;\ell_2)-1)$

Theorem 3 (D. Conlon, A. Ferber) *[1] For any prime q, $r(t;q+1) > 2^{t/2}q^{3t/8+o(t)}$. In particular, $r(t;3) > 2^{7t/8+o(t)}$.*

In the special case of $q=2$, we will see the Conlon-Ferber construction. Let t be even and let $V \subset \mathcal{F}_2^t$ denote the set of vectors of even Hamming weight, so that $|V| = 2^{t-1}$. Define a graph G_0 with vertex set V by letting $\{u,v\} \in E(G_0)$ if and only if $u \cdot v = 1 \mod 2$.

Lemma 4 *G_0 has no clique of order t.*

Lemma 5 *G_0 has at most $2^{5t^2/8+o(t^2)}$ independent sets of order at most t.*

Further results:

Theorem 6 (Y. Widgerson) *[2]* $r(t;\ell+2) \geq 2^{3\ell t/8+t/2-o(t)}$.

Theorem 7 (W. Sawin) *[3]* $r(t;\ell+2) \geq 2^{.383796\ell t+t/2+o(t)}$.

Bibliography

[1] David Conlon and Asaf Ferber. *Lower bounds for multicolor Ramsey numbers.* Advances in Mathematics, Volume 378, 2021, 107528.

[2] Yuval Widgerson. *An improved lower bound on multicolor Ramsey numbers* Proc. Amer. Math. Soc. 149 (2021), 2371-2374

[3] Will Sawin. *An improved lower bound for multicolor Ramsey numbers and the half-multiplicity Ramsey number problem* https://arxiv.org/pdf/2105.08850.pdf

Félix Moreno Peñarrubia
felix.moreno.penarrubia@estudiantat.upc.edu
Presented paper by L. Postle, R. Thomas

Five-list-coloring Graphs on Surfaces I. Two Lists of Size Two in Planar Graphs

(https://arxiv.org/abs/1402.1813)

Abstract

The paper we are presenting concerns itself with proving the following result:

Theorem 1 (Two lists of size two) *Let G a plane graph with outer cycle C, let $v_1, v_2 \in C$ and let L be a list assignment with $|L(v_1)| = |L(v_2)| = 2$, $|L(v)| \geq 3$ $\forall v \in C \setminus \{v_1, v_2\}$, and $|L(v)| \geq 5$ $\forall v \in V(G) \setminus V(C)$. Then G has an L-coloring.*

This is a variation of Thomassen's notable theorem on the five-list-colorability of planar graphs.

In this talk, we will explain the proof presented in the paper, as well as provide some background for this result.

Background

List-coloring (for vertices) is a concept similar to regular (vertex) coloring in which each vertex has a list of possible color. Let G be a graph.

Definition 2 *A* list assignment *is a function $L : V(G) \to 2^{\mathbb{N}}$. A k-*list-assignment *is a list assignment with $|L(v)| \geq k$ $\forall v \in V(G)$. An L-*coloring *is a (proper vertex) coloring f for which $f(v) \in L(v)$ $\forall v \in V(G)$. A graph is k-*list-colorable *or k-*choosable *if there exists an L-coloring for all k-list-assignments L. The* list chromatic number *or* choosability *$\chi_\ell(G)$ is the least integer so that G is $\chi_\ell(G)$-list-colorable.*

A natural question is whether there is an analogue of the four color theorem for list-colorability,

Theorem 3 *There exists a planar graph (with 238 vertices) with $\chi_\ell(G) = 5$.*

Theorem 4 (Thomassen) *For all planar graphs, $\chi_\ell(G) \leq 5$.*

The previous theorem has a simple proof based on proving the following stronger statement.

Theorem 5 *Let G be a plane (embedded) graph whose faces are all triangles except for possibly the outer face C, and let L be a list assignment satisfying: $|L(v)| \geq 5$ for all internal vertices, $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{x, y\}$ where x, y are a pair of adjacent vertices, $|L(x)| = |L(y)| = 1$, $L(x) \neq L(y)$. Then G has an L-coloring.*

Let us consider list-coloring of graph embedded in general surfaces, not just the plane. We have the following result, mirroring analogous results for regular coloring.

Definition 6 *A graph G is L-*critical *for some list assignment L if G has no L-coloring but every proper subgraph G' has. A graph G is k-*list-critical *if there exists a $(k-1)$-list assignment L such that G is L-critical.*

Theorem 7 *For $k \geq 6$ there exist only finitely many k-list-critical graphs embeddable in a given surface Σ.*

The two lists of size two result is used as a lemma to prove the above theorem.

Proof of the two lists of size two theorem

The following definitions are used in the paper:

Definition 8 *(G, S, L) is a* canvas *if G is a plane graph with outer face boundary C, S is a subgraph of C and L is a list assignment with $|L(v)| \geq 5$ $\forall v \in V(G) \setminus V(C)$, $L(v) \geq 3$ $\forall v \in V(C) \setminus V(S)$, and there exists a L-coloring of S.*

Definition 9 *A canvas (G, S, L) is* critical *if there does not exist an L-coloring of G but for all edges $e \in E(G) \setminus E(S)$ there exists an L-coloring of $G \setminus e$.*

Definition 10 *A cutvertex $v \in V(G)$ of a canvas (G, S, L) is* essential *if for the decomposition $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \{v\}$, then $V(S) \nsubseteq V(G_i)$. A chord $u, v \in V(C)$ of a canvas (G, S, L) is* essential *if for the decomposition $G_1 \cup G_2 = G$, $G_1 \cap G_2 = \{u, v\}$, then $V(S) \nsubseteq V(G_i)$.*

Additionally, we use the following lemmas about critical canvases.

Lemma 11 *If (G, S, L) is a critical canvas, then every cutvertex and every chord of the outer walk of G is essential.*

Lemma 12 *If (G, S, L) is a critical canvas, then every cycle of G of length at most four bounds an open disk containing no vertex of G.*

In a similar way to the proof of the five-list-colorability of planar graphs, the main idea is to find a suitable strengthening of the statement:

Theorem 13 *Let (G, S, L) be a canvas, where S has two components: a path P and an isolated vertex u with $|L(u)| \geq 2$. Assume that if $|V(P)| \geq 2$, then G is 2-connected, u is not adjacent to an internal vertex of P and there does not exist a chord of the outer walk of G with an end in P which separates a vertex of P from u. Let L_0 be a set of size two. If $L(v) = L_0$ for all $v \in V(P)$, then G has an L-coloring, unless $L(u) = L_0$ and $V(S)$ induces and odd cycle in G.*

The proof assumes a minimum counterexample and successively reduces it to more particular cases. To finish the proof, we need the following lemma following from results of [2].

Lemma 14 *Let (G, P, L) be a canvas with outer cycle C and $P = p_1p_2p_3$ such that there is no path Q contained in C with ends p_1 and p_3 so that every vertex of Q is adjacent to p_2. Then there is at most one L-coloring of P that does not extend to an L-coloring of G.*

Bibliography

[1] L. Postle, R. Thomas, *Five-list-coloring graphs on surfaces I. Two lists of size two in planar graphs.* Journal of Combinatorial Theory, Series B, Volume 111, Issue C.

[2] C. Thomassen, *Exponentially many 5-list-colorings of planar graphs.* Journal of Combinatorial Theory, Series B Volume 97, Issue 4, July 2007, Pages 571-583

Jędrzej Olkowski
jo417777@students.mimuw.edu
Presented paper by M. Bonamy, C. Groenland, C. Muller, J. Narboni, J. Pekárek, A. Wesolek

A Note on Connected Greedy Edge Colouring

(https://arxiv.org/abs/2012.13916)

Introduction

Following a given ordering of the edges of a graph G, the greedy edge colouring procedure assigns to each edge the smallest available colour. The minimum number of colours thus involved is the chromatic index $\chi'(G)$. Here, we are interested in the restricted case where the ordering of the edges builds the graph in a connected fashion. Let $\chi'_c(G)$ be the minimum number of colours involved following such an ordering.

Main Results

Theorem 1 *For all $\Delta \geq 4$, it is NP-hard to decide whether $\chi'(G) = \chi'_c(G)$ on the class of graphs with chromatic index δ.*

Theorem 2 *If G is a bipartite connected graph, then $\chi'(G) = \chi'_c(G)$.*

Theorem 3 *If G has maximum degree 3, then $\chi'_c(G) \leq 4$.*

Open Problem

Conjecture 4 *Is it true that $\chi'_c(G) \leq \Delta + 1$ for every graph G of maximum degree Δ?*

Bibliography

[1] D. Leven and Z. Galil. *NP completeness of finding the chromatic index of regular graphs.* Journal of Algorithms 4.1 (1983): 35-44.

Olga Pribytkova
oisho@zavidnyi.com
Presented paper by A. Kostochka & M. Yancey

Ore's Conjecture for k = 4 and Grotzsch Theorem

(http://arxiv.org/abs/1209.1173v1)

Definitions

D1. A *proper k-coloring* or simply *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, \ldots, k\}$, such that for each $uv \in E, f(u) \neq f(v)$. A graph G is *k-colorable* if there exists a k-coloring of G.

D2. The *chromatic number*, $\chi(G)$, of a graph G is the smallest k such that G is k-colorable.

D3. A graph G is *k-critical* if G is not $(k-1)$-colorable, but every proper subgraph of G is $(k-1)$-colorable.

D4. For $R \subseteq V(G)$, define the potential of R to be $\rho_G(R) = 5|R| - 3|E(G[R])|$. When there is no chance for confusion, we will use $\rho(R)$. Let $P(G) = \min_{\emptyset \neq R \subseteq V(G)} \rho(R)$.

D5. For a graph G, a set $R \subset V(G)$ and a 3-coloring φ of $G[R]$, the graph $Y(G, R, \varphi)$ is constructed as follows. First, for $i = 1, 2, 3$ let R'_i denote the set of vertices in $V(G) - R$ adjacent to at least one vertex $v \in R$ with $\varphi(v) = i$. Second, let $X = \{x_1, x_2, x_3\}$ be a set of new vertices disjoint from $V(G)$. Now, let $Y = Y(G, R, \varphi)$ be the graph with vertex set $(V(G) - R) \cup X$, such that $Y[V(G) - R] = G - R$ and $N(x_i) = R'_i \cup (X - x_i)$ for $i = 1, 2, 3$

D6. A *charge* is a small positive number that is assigned to each face and each vertex of a graph. *Discharging phase* is a process of redistribution of the charges to nearby vertices and faces in accordance to a custom set of discharging rules.

Auxilary theorems

A1. If $k \geq 4$ and $k + 2 \leq n \leq 2k - 1$, then

$$f_k(n) = \frac{1}{2}((k-1)n + (n-k)(2k-n)) - 1$$

A2. Every triangle-free planar graph is 3-colorable

A3. If $k \geq 4$ and G is k-critical, then $|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)} \right\rceil$. In other words if $k \geq 4$ and $n \geq k$, $n \neq k+1$, then

$$f_k(b) \geq F(k, n) := \left\lceil \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)} \right\rceil$$

Main theorem

MT. If G is 4-critical, then $|E(G)| \geq \left\lceil \frac{5|V(G)|-2}{3} \right\rceil$

Step 1. Suppose $R \subset V(G)$, and φ is 3-coloring of $G[R]$. Then $\chi(Y(G, R, \varphi)) \geq 4$

Step 2. There is no $R \subsetneq V(G)$ with $|R| \geq 2$ and $\rho_G(R) \leq 5$

Step 3. If $R \subsetneq V(G)$, $|R| \geq 2$ and $\rho(R) \leq 6$, then R is a K_3

Step 4. G does not contan $K_4 - e$

Step 5. Each triangle in G contains at most one vertex of degree 3

Step 6. Let $xy \in E(G)$ and $d(x) = d(y) = 3$. Then both, x and y are in triangles.

Bibliography

[1] T.Gallai, Kritische Graphen II, *Publ. Math. Inst. Hungar.*. Acad. Sci. (1963), 373-395

[2] A.V. Kostochka and M.Yancey, Ore's conjecture is almost true, submitted.

Jakub Svoboda
jakub.svoboda@ist.ac.at
Presented paper by T. Korhonen
A Single-Exponential Time 2-Approximation Algorithm for Treewidth
(https://arxiv.org/abs/2104.07463)

We give an algorithm, that given an n-vertex graph G and an integer k, in time $2^{\mathcal{O}(k)}n$ either outputs a tree decomposition of G of width at most $2k+1$ or determines that the treewidth of G is larger than k.

Definitions

The vertices of a graph G are denoted by $V(G)$ and edges by $E(G)$. The subgraph induced by a vertex set $X \subseteq V(G)$ is denoted by $G[X]$ and the subgraph induced by a vertex set $V(G) \setminus X$ is denoted by $G \setminus X$.

A balanced separator of W is a vertex set X such that for each connected component C_i of $G \setminus X$ it holds that $|W \cap C_i| \leq |W|/2$.

A vertex set $W \subseteq V(G)$ is splittable if $V(G)$ can be partitioned into (C_1, C_2, C_3, X) such that there are no edges between C_i and C_j and $|(W \cap C_i) \cup X| < |W|$ for all i. We refer to such 4-tuple as a split of W.

Let W be a root bag of a tree decomposition T . A split (C_1, C_2, C_3, X) is a minimum split of W if the split minimizes $|X|$ among all splits of W , and among splits minimizing $|X|$ the split minimizes $\sum_{x \in X} d(x)$ where $d(x)$ is the distance from the home bag B_x of x to W in T.

Exercises

Problem 1 *Let G be a graph of treewidth k and $W \subseteq V(G)$ a vertex set of G. There is a balanced separator X of W of size $|X| \leq k+1$.*

Problem 2 *A graph is outerplanar if it can be embedded in the plane in such manner that all its vertices are on one face. What values can treewidth of an outerplanar graph have?*

Problem 3 *Prove that the treewidth of a simple graph cannot increase after subdividing any of its edges. Show that in the case of multigraphs the same holds, with the exception that the treewidth can possibly increase from 1 to 2.*

Problem 4 *For a graph G with tree decomposition of with t construct in time $t^{\mathcal{O}(1)}n$ time a structure that in time $\mathcal{O}(t)$ time aswers if vertices u and v are adjacent.*

Problem 5 *Solve MAX-CUT in time $2^{\mathcal{O}(t)} \cdot t^{\mathcal{O}(1)}n$*

Problem 6 *Solve q-coloring in time $q^{\mathcal{O}(t)} \cdot t^{\mathcal{O}(1)}n$*

Problem 7 *Solve Hamiltonian cycle in time $t^{\mathcal{O}(t)}n$*

Problem 8 *Find a set X, such that $G \setminus X$ does not contain any four-cycles (time $2^{\mathcal{O}(t^2)}n^{\mathcal{O}(1)}$).*

Bibliography

[1] Marek Cygan et al. *Parametrized Algorithms*. Springer, 2016

Ilia Zavidnyi
hello@zavidnyi.com
Presented paper by M. Ghaffari
An Improved Distributed Algorithm for Maximal Independent Set
(http://arxiv.org/abs/1506.05093v2)

Introductions

MIS Is an independent set which is maximal with respect to the independent set property, meaning that you cannot add any vertex to it.

LOCAL The network is abstracted as a graph $G = (V, E)$ where $|V| = n$; initially each node only knows its neighbors; communications occur in synchronous rounds, where in each round nodes can exchange information only with their graph neighbors.

Old local complexity

$$O(\log^2 \Delta + \log 1/\varepsilon) \quad O(\log \Delta \log\log \Delta + \log \Delta \log 1/\varepsilon)$$

Cool new local complexity

$$O(\log^2 \Delta + \log 1/\varepsilon)$$

Getting there

The Algorithm

In each round t, each node v has a desire-level $p_t(v)$ for joining MIS, which initially is set to $p_0(v) = 1/2$. We call the total sum of the desire-levels of neighbors of v it's effective-degree $d_t(v)$, i.e., $d_t(v) = \sum_{u \in N(v)} p_t(u)$. The desire-levels change over time as follows:

$$p_{t+1}(v) = \begin{cases} p_t(v)/2 & \text{if } d_t(v) \geq 2 \\ \min\{2p_t(v), 1/2\} & \text{if } d_t < 2 \end{cases}$$

The desire-levels are used as follows: In each round, node v gets marked with probability $p_t(v)$ and if no neighbor of v is marked, v joins the MIS and gets removed along with its neighbors.

Theorem For each node v, the probability that v has not made its decision within the first $\beta(\log \mathsf{deg} + \log 1/\varepsilon)$ rounds, for a large enough constant β and where deg denotes v's degree at the start of the algorithm, is at most ε. Furthermore, this holds even if the outcome of the coin tosses outside $N_2^+(v)$ are determined adversarially.

Golden rounds Let us say that a node u is low-degree if $d_t(u) < 2$, and high-degree otherwise. Considering the intuition discussed above, we define two types of golden rounds for a node v:

(1) rounds in which $d_t(v) < 2$ and $p_t(v) = 1/2$,

(2) rounds in which $d_v(t) \geq 1$ and at least $d_t(v)/10$ of it is contributed by low-degree neighbors.

Lemma 1 By the end of round $\beta(\log \mathsf{deg} + \log 1/\varepsilon)$, either v has joined, or has a neighbor in, the (computed) MIS, or at least one of its golden round counts reached $100(\log \mathsf{deg} + \log 1/\varepsilon)$.

Lemma 2 In each type-1 golden round, with probability at least $1/100$, v joins the MIS. Moreover, in each type-2 golden round, with probability at least $1/100$, a neighbor of v joins the MIS. Hence, the probability that v has not been removed (due to joining or having a neighbor in MIS) during the first $\beta(\log \mathsf{deg} + \log 1/\varepsilon)$ rounds is at most ε. These statements hold even if the coin tosses outside $N_2^+(v)$ are determined adversarially.

List of participants

Brage Bakkane

Samuel Braunfeld

Martin Černý

Petr Chmel

Lukáš Folwarczný

Milan Hladik

Pavel Hubáček

Bartłomiej Kielak

František Kmječ

Michal Koucký

Matyáš Křišťan

Gaurav Kucheriya

Matyáš Lorenc

Félix Moreno Peñarrubia

Jędrzej Olkowski

Olga Pribytkova

Robert Šámal

Nikolaj Schwartzbach

Jiří Sgall

Jakub Svoboda

Martin Tancer

Mykhaylo Tyomkyn

Pavel Valtr

Pavel Veselý

Ilia Zavidnyi