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Note

Two bijections on weakly increasing trees

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ABSTRACT

Weakly increasing tree is a new kind of multiset-labeled tree introduced in a recent work of Lin-Ma-Ma-Zhou [9], which naturally unifies the classical concepts of plane trees and increasing trees on the set $\{1, 2, \dots, n\}$. Two bijections defined on weakly increasing trees are introduced. The first map has its roots in a bijection on plane trees due to Deutsch [4]. The second map is a closely related variant and turns out to be an involution. This involution amounts to give a combinatorial proof of certain equidistribution results for a sextuple of tree statistics over weakly increasing trees defined on any multiset M, extending a previous result of Lin-Ma-Ma-Zhou on restricted M.

1. Introduction

An *increasing tree* is a labeled rooted tree in which labels along any path from the root go in increasing order. A *plane tree* (also known as ordered tree), on the other hand, is a rooted tree where the order of all the subtrees of a given node is significant. For more information about increasing trees and plane trees, see [1,2,4,10]. The number of subtrees of a node, say v, is called the *degree* of v and denoted as $\deg(v)$.[1] A node of degree zero is called a *leaf*, a node of positive degree is called an *internal node*. The *level* of a node is the number of edges along the path from that node to the root. So in particular, the root itself is thought of as a level 0 node. A *permutation* w of $[n] := \{1, 2, \dots, n\}$ is defined as a linear ordering $w_1, w_2, \dots, w_n$ of the elements of $[n]$, written as $w = w_1 w_2 \dots w_n$. Define the *descent set* of w by $\mathrm{DES}(w) := \{i : w_i > w_{i+1}\}$, and its cardinality by $\mathrm{des}(w) = |\mathrm{DES}(w)|$, called the *descent number* of w. The set of all permutations of $[n]$ is denoted as $\mathfrak{S}_n$.

$$A_n(x) := \sum_{w \in \mathfrak{S}_n} x^{\mathrm{des}(w)} = \sum_{k=0}^{n-1} A(n,k) x^k$$

is the n-th *Eulerian polynomial*, where $A(n,k)$ is called *Eulerian number* and it can be interpreted alternatively as the number of increasing trees with $n+1$ nodes and $k+1$ leaves. Eulerian polynomials have always been an important research object of enumerative combinatorics. Many properties of Eulerian polynomials have been deeply studied, such as the γ-positivity of its coefficients, unimodality, log-convexity (concavity), asymptotic normality, etc. The *Narayana number* can be explicitly defined as

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[1] This is usually referred to as the *outdegree* in the literature; see for example [5–7].

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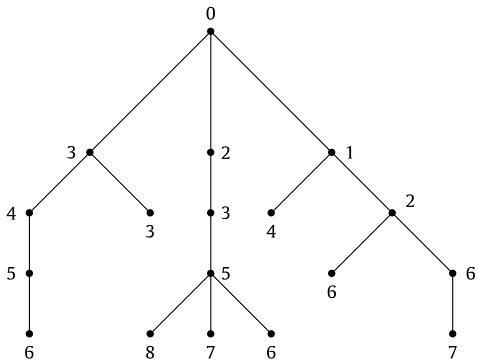


Fig. 1. A weakly increasing tree in Γ_M with $M = \{1, 2^2, 3^3, 4^2, 5^2, 6^4, 7^2, 8\}$.

$$N(n,k) = \frac{1}{n}\binom{n}{k+1}\binom{n}{k}.$$

Like Eulerian number, it can be given tree-related interpretation. Namely, $N(n,k)$ is the number of plane trees with $n+1$ nodes and $k+1$ leaves.

It is readily observed that one can superficially view plane trees as labeled trees using identical labels such as $\{1, 1, \ldots, 1\}$. This viewpoint hints at a way to unify Eulerian polynomials and Narayana polynomials using a notion of *weakly increasing tree*, which has been recently introduced and thoroughly studied by Lin-Ma-Ma-Zhou in their work [9].

The number of nodes of a tree T is denoted by $\#T$. Given any multiset M of positive integers, set $M_0 := M \cup \{0\}$. Then a *weakly increasing tree on M*, as defined in [9], is a plane tree such that

(i) the labels of the nodes form precisely the multiset M_0;
(ii) labels along a path from the root to any leaf are weakly increasing;
(iii) labels of the children of each node are in nonincreasing order from left to right.

Denote by Γ_M the set of weakly increasing trees on M; see Fig. 1 for an example. Deutsch [4] concluded that for a plane tree, the parameters "number of nodes of degree q" and "number of odd-level nodes of degree $q-1$" are equidistributed for $q \geqslant 1$. Our first map f is motivated by Deutsch's bijection and serves to generalize his conclusion to weakly increasing trees. We note that independently, Lin and Ma [8] has obtained essentially the same bijection as our bijection f, and they have also studied a further bijection on weakly increasing trees that reveals interesting symmetries. Our second map Φ is a closely related variant and turns out to be an involution. This involution amounts to give a combinatorial proof of certain equidistribution results for a sextuple of tree statistics over weakly increasing trees defined on any multiset M, extending a previous result of Lin-Ma-Ma-Zhou [9].

Remark 1.1. Note that by definition, each weakly increasing tree must have a 0-labeled root, so strictly speaking, none of its proper subtrees could be a weakly increasing tree since it has a nonzero labeled root. This discrepancy causes us some trouble since both maps f and Φ will be defined recursively, assuming their validity on various subtrees first. To resolve this, we first point out that both maps f and Φ preserve the label of the root. Moreover, two trees T and R are said to be *congruent* to each other, denoted as $T \sim R$, if the only possible difference between these two trees is at their root labels. So we see two trees are identical if and only if they are congruent and have the same root label. In this context, Γ_M could be viewed as the set of representatives of congruence classes. Therefore, when a root-preserving mapping $g : \Gamma_M \to \Gamma_M$ is applied to a tree R with nonzero labeled root, $g(R)$ is tacitly understood to be the unique tree that is congruent to $g(R_0)$ and has the same root label as R. Here $R_0 \in \Gamma_M$ and $R_0 \sim R$.

For any multiset $M = \{1^{m_1}, 2^{m_2}, \ldots, n^{m_n}\}$ with $N := m_1 + m_2 + \cdots + m_n$, we introduce the six-variate generating function

$$F_M(x,y,p,b,a,c) = \sum_{T \in \Gamma_M} x^{\mathrm{int}(T)-1} y^{\mathrm{leaf}(T)-1} p^{\mathrm{par}(T)} b^{\mathrm{bro}(T)} a^{\mathrm{ole}(T)} c^{N-n-\mathrm{par}(T)-\mathrm{bro}(T)},$$

where the undefined statistics will be introduced in later sections. As an immediate application of our involution Φ, the following theorem shows the joint symmetry of four variables in $F_M(x,y,p,b,a,c)$, and it reduces to Theorem 2.11 (5) in [9] when we set $a = 1$ and restrict the multiset M to the case that $1 \leqslant m_i \leqslant 2$ for each i.

Theorem 1.2. *For any nonempty multiset M, we have $F_M(x,y,p,b,a,c) = F_M(y,x,b,p,a,c)$.*

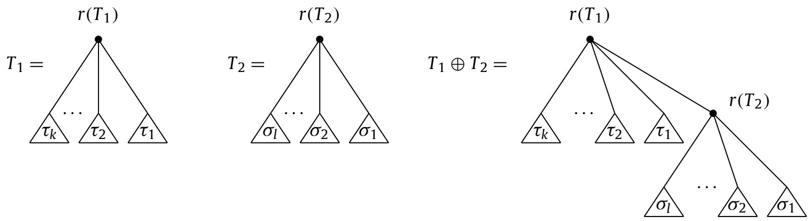


Fig. 2. Grafting of two trees.

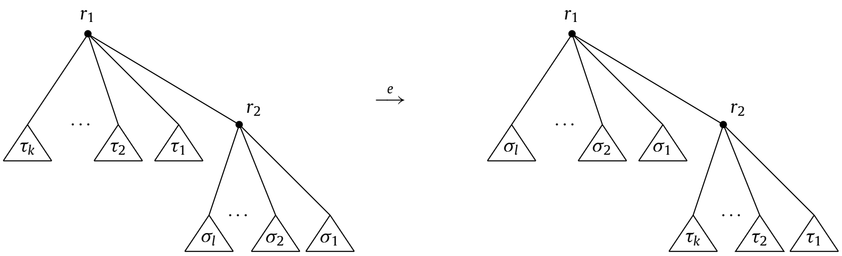


Fig. 3. The exchange map e.

The structure of the paper is as follows. In Section 2, we introduce two decompositions of weakly increasing trees and define the first map f. In Section 3, we construct the second map, namely the involution Φ and deduce some of its properties, which amount to give a proof of Theorem 1.2.

2. A bijection on the set of weakly increasing trees

We introduce two operations for making bigger trees from smaller ones (i.e., trees with fewer nodes). If $T_1, T_2, \ldots, T_k$ are weakly increasing trees with root labels $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_k$, then take any integer $r \leqslant r_1$, we attach $T_1, T_2, \ldots, T_k$ from right to left, to a node labeled by r. This operation results in a new weakly increasing tree T called the *wedge sum of* $T_1, T_2, \ldots, T_k$ *at* r, and denoted as

$$T := \bigwedge_{i\in[k]}^{r} T_i.$$

From this definition, it is clear that each weakly increasing tree T can be written uniquely as the wedge sum of all the subtrees at its root. For instance, the tree T_1 in Fig. 2 is the wedge sum of $\tau_1, \ldots, \tau_k$ at the root $r(T_1)$.

Denote the tree (labeled or not) with a single node as ϵ. In what follows, we use $r(T)$ to denote the label of the root of T, and denote $s(T)$ the label of the rightmost child of the root of T (set $s(\epsilon) = \infty$). We speak of a node and its label interchangeably as long as doing so will not cause any confusion. Now, if for two weakly increasing trees T_1 and T_2, we have $r(T_1) \leqslant r(T_2) \leqslant s(T_1)$, then we can attach the root of T_2 to the root of T_1 as T_1's new rightmost child. This operation results in a new weakly increasing tree that we shall call the *grafting of* T_1 *and* T_2, and we denote it as $T = T_1 \oplus T_2$. See Fig. 2 for an example.

The following *exchange map* e is clearly seen to be an involution defined on Γ_M for any given multiset M, and it plays an important role in both of the bijections that we are going to introduce.

Definition 2.1. Suppose $T = (\bigwedge_{i\in[k]}^{r_1} \tau_i) \oplus (\bigwedge_{j\in[l]}^{r_2} \sigma_j)$, then let $e(T) = (\bigwedge_{j\in[l]}^{r_1} \sigma_j) \oplus (\bigwedge_{i\in[k]}^{r_2} \tau_i)$, as illustrated in Fig. 3.

Remark 2.2. It is worth mentioning that the exchange map on plane trees has already been applied by Eu, Seo, and Shin [6] to show a four-way equinumerosity between four sets of vertices among all plane trees with the same number of edges; see [7] for a follow-up work. Note that for any weakly increasing tree $T \neq \epsilon$, there is a unique way to write it as the grafting of two smaller trees, hence the exchange map e is well defined. Moreover, note that $r(T) = r(e(T)),\ s(T) = s(e(T))$.

Definition 2.3 *(1st bijection f).* We recursively define a mapping $f : \Gamma(M) \to \Gamma(M)$ such that $f(\epsilon) = \epsilon$, and for $T \neq \epsilon$, suppose $T = \bigwedge_{i\in[k]}^{r} \tau_i$, then we let

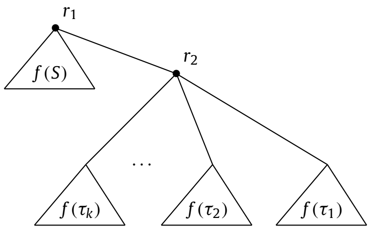
$f(T) =$ 

Fig. 4. The first map f.

$$f(T) := e(\bigwedge_{i\in[k]}^{r} f(\tau_i)).$$

See Fig. 4 for an illustration of f, where $T = T_1 \oplus T_2$ with $T_1 = \bigwedge_{i\in[k]}^{r_1} \tau_i$, $T_2 = \bigwedge_{j\in[l]}^{r_2} \sigma_j$, and $S = \bigwedge_{j\in[l]}^{r_1} \sigma_j$. More precisely, we have

$$\begin{aligned} f(T_1 \oplus T_2) &= f\left(\left(\bigwedge_{i\in[k]}^{r_1} \tau_i\right) \oplus T_2\right) = e\left(\left(\bigwedge_{i\in[k]}^{r_1} f(\tau_i)\right) \oplus f\left(\bigwedge_{j\in[l]}^{r_2} \sigma_j\right)\right) \\ &= f\left(\bigwedge_{j\in[l]}^{r_1} \sigma_j\right) \oplus \left(\bigwedge_{i\in[k]}^{r_2} f(\tau_i)\right) = f(S) \oplus \left(\bigwedge_{i\in[k]}^{r_2} f(\tau_i)\right). \end{aligned}$$

Remark 2.4. By using induction on the number of nodes, and the observation made in Remark 2.2 that $r(T) = r(e(T))$ and $s(T) = s(e(T))$, one sees that $r(T) = r(f(T))$ and $s(T) = s(f(T))$. Consequently, for the example in Fig. 4, we have

$$\begin{aligned} &r_2 \leqslant r(f(\tau_1)) \leqslant \cdots \leqslant r(f(\tau_k)), \\ &r(f(S)) = r_1 \leqslant r_2 \leqslant r(\sigma_1) = s(f(S)), \end{aligned}$$

so $f(T) \in \Gamma_M$ and f is well defined.

The following two theorems reveal the main properties of f.

Theorem 2.5. *f is a bijection.*

Proof. We show this by constructing the inverse of f. Define the inverse f^{-1} of f inductively as follows: first off, $f^{-1}(\epsilon) = \epsilon$, then take $T \neq \epsilon$, suppose $e(T) = \bigwedge_{i\in[k]}^{r} \tau_i$, then set

$$f^{-1}(T) := \bigwedge_{i\in[k]}^{r} f^{-1}(\tau_i).$$

One checks that indeed $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}_{\Gamma_M}$. $\square$

Theorem 2.6 *(Theorem 1.2 in [8]). Given a weakly increasing tree $T \in \Gamma_M$, we have*

(1) For an integer $q \geqslant 1$, the number of nodes of T having degree q is equal to the number of odd-level nodes of $f(T)$ having degree $q-1$.
(2) The number of leaves of T is equal to the number of even-level nodes of $f(T)$.

Proof. We outline a proof that is similar to the proof of Deutsch [4]. First note that since $\#T = \#f(T)$, we can deduce (2) from (1) by combining all nodes that meet the conditions in (1) for possible values of q and then taking complement. So it suffices to prove (1) only. We use induction on the cardinality of M. For $|M| = 0$ and $|M| = 1$, we see $\Gamma_M = \{\epsilon\}$ and $\Gamma_M = \epsilon \oplus \epsilon$, respectively. In either case f is the identity map and the claim about the equinumerousity of two types of nodes holds true trivially. Now for $|M| \geqslant 2$, we assume the theorem is true for multisets with smaller cardinality than M.

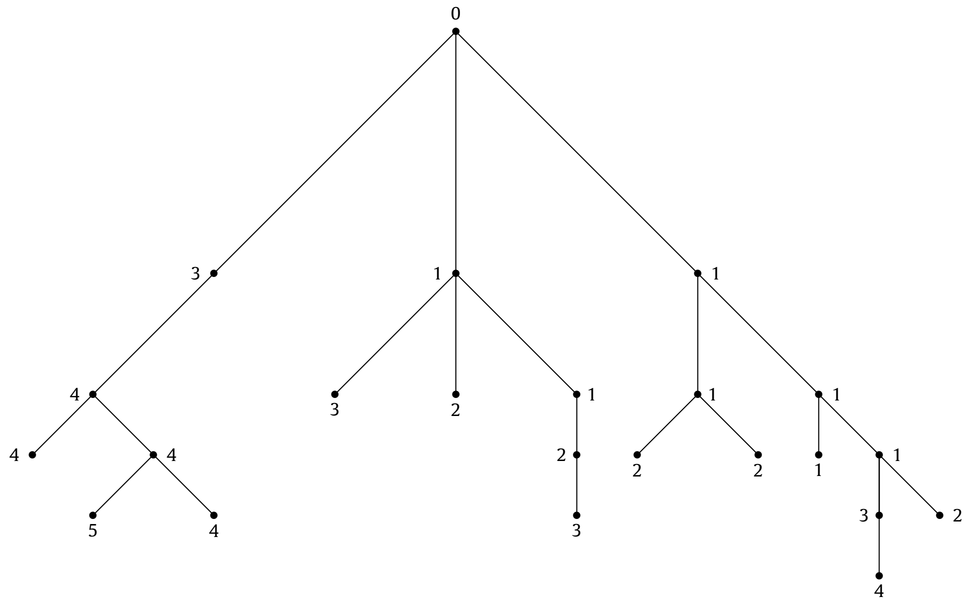


Fig. 5. The weakly increasing tree T.

Suppose $T = T_1 \oplus T_2 \in \Gamma_M$ with $T_1 = \bigwedge_{i\in[k]}^{r_1} \tau_i$, $T_2 = \bigwedge_{j\in[l]}^{r_2} \sigma_j$, and set $S = \bigwedge_{j\in[l]}^{r_1} \sigma_j$. Then by definition, $f(T) = f(S) \oplus \bigwedge_{i\in[k]}^{r_2} f(\tau_i)$. Note that the only difference between T_2 and S is the labels of their roots. Consequently, the only difference between $f(T_2)$ and $f(S)$ is the labels of their roots. Now let v be a vertex of T, we consider the following three cases.

(1) $v = r_1$ is the root of T_1. $w = r_2 = s(f(T))$ is an odd-level (level 1 to be precise) node of $f(T)$. $\deg(v) = q$ if and only if $\deg(w) = q - 1$.
(2) v is a vertex of T_2. Since $\#T_2 < \#T$, by induction we know there are as many v with $\deg(v) = q$, as odd-level nodes w of $f(T_2)$ (or equivalently of $f(S)$) with $\deg(w) = q - 1$.
(3) v is a non-root vertext of T_1, say contained in τ_i for some $1 \leqslant i \leqslant k$. Since $\#\tau_i < \#T$, by induction we know there are as many v with $\deg(v) = q$, as odd-level nodes w of $f(\tau_i)$ with $\deg(w) = q - 1$. Note that such a node w is still at an odd-level when viewed as a node in $f(T)$.

In either case, we see a node v in T having degree q corresponds uniquely to a node w in $f(T)$ having degree $q - 1$. And for both v and its counterpart w, the cases listed above are mutually exclusive and cover all the possibilities. The proof is thus completed by induction. □

Gathering all nodes of degree p from each tree in Γ_M for all possible values of p no less than a given positive integer q, we get an immediate corollary.

Corollary 2.7. *For each positive integer q, on the set Γ_M of all weakly increasing trees on a multiset M, the number of nodes of degree $\geqslant q$ is equal to the number of odd-level nodes of degree $\geqslant q - 1$. In particular, the number of non-leaf nodes is equal to the number of odd-level nodes.*

3. An involution on the set of weakly increasing trees

The first bijection is to apply the mapping f to each summand in the wedge sum decomposition of T, and then apply the exchange map. It's natural to consider the mapping defined recursively via the direct sum decomposition of T instead. More precisely, we give the following definition.

Definition 3.1 *(2nd bijection Φ).* We recursively define a mapping $\Phi\colon \Gamma_M \mapsto \Gamma_M$, such that $\Phi(\epsilon) = \epsilon$, and for any $\epsilon \neq T = T_1 \oplus T_2 \in \Gamma_M$, $\Phi(T) = e(\Phi(T_1) \oplus \Phi(T_2))$.

Example 3.2. Let T be a weakly increasing tree on multiset $M = \{1^7, 2^5, 3^4, 4^5, 5\}$ as shown in Fig. 5. Its image $\Phi(T)$ is shown in Fig. 6.

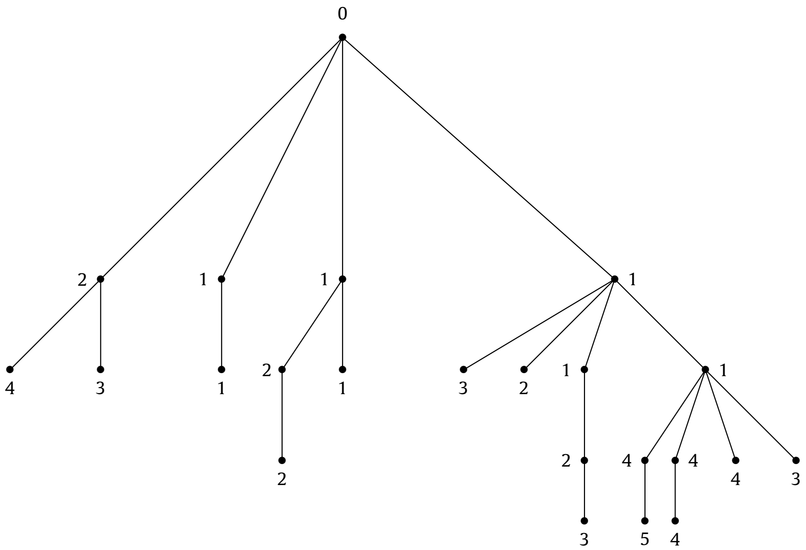


Fig. 6. The weakly increasing tree $\Phi(T)$.

Note that for each $T \in \Gamma_M$, the exchange map satisfies $r(T) = r(e(T))$ and $s(T) = s(e(T))$. Using this and induction on $\#T$, we deduce that

$$r(T) = r(\Phi(T)), \tag{3.1}$$

$$s(T) = s(\Phi(T)). \tag{3.2}$$

The notion of congruence (see Remark 1.1) is compatible with the exchange map e and the newly introduced map Φ in the following sense.

Lemma 3.3. *Suppose $T = T_1 \oplus T_2$ and $e(T) = T_3 \oplus T_4$, then we see that*

$$T_1 \sim T_4, \text{ and } T_2 \sim T_3. \tag{3.3}$$

If R is another tree such that $T \sim R$, then we have

$$\Phi(T) \sim \Phi(R). \tag{3.4}$$

The proof follows directly from the definitions of the exchange map and congruence, so we decide to omit the details. This lemma will facilitate our proof of the following proposition.

Proposition 3.4. *Φ is an involution, that is, $\Phi^2(T) = T$, for any $T \in \Gamma_M$.*

Proof. We use induction on $m := \#T$. For $m \leqslant 2$, $\Phi(T) = T$ so $\Phi^2(T) = T$ as well. Now suppose $T = T_1 \oplus T_2$ with $m \geqslant 3$, and assume the claim holds true for all trees in Γ_M with fewer vertices than T. Suppose further that $\Phi(T) = T_3 \oplus T_4$ and $\Phi^2(T) = T_5 \oplus T_6$. Applying (3.3) on $\Phi(T)$ and $\Phi^2(T)$ gives us

$$T_3 \sim \Phi(T_2), \; T_4 \sim \Phi(T_1), \; T_5 \sim \Phi(T_4), \text{ and } T_6 \sim \Phi(T_3).$$

Combining these with (3.4) from Lemma 3.3 and using induction hypothesis, we get

$$T_5 \sim \Phi(T_4) \sim \Phi^2(T_1) = T_1, \text{ and } T_6 \sim \Phi(T_3) \sim \Phi^2(T_2) = T_2.$$

Moreover, note that $r(T_6) = s(\Phi^2(T)) \overset{\text{by (3.2)}}{=} s(T) = r(T_2)$, so actually $T_6 = T_2$. Similarly, $r(T_5) = r(\Phi^2(T)) \overset{\text{by (3.1)}}{=} r(T) = r(T_1)$, which implies that $T_5 = T_1$. We conclude that $\Phi^2(T) = T_5 \oplus T_6 = T_1 \oplus T_2 = T$, as desired. $\square$

Next proposition tells us the pair of statistics $(\mathrm{int}, \mathrm{leaf})$ is swapped by the mapping Φ.

Proposition 3.5. *For $T \in \Gamma_M$ with $\#T \geqslant 2$, we have $\mathrm{int}(T) = \mathrm{leaf}(\Phi(T))$ and $\mathrm{leaf}(T) = \mathrm{int}(\Phi(T))$.*

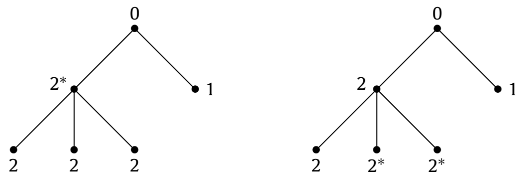


Fig. 7. A weakly increasing tree T with $\mathrm{Par}(T)=\{2\}$ and $\mathrm{Bro}(T)=\{2,2\}$.

Proof. First note that $\mathrm{int}(T)+\mathrm{leaf}(T)=\#T=\#\Phi(T)=\mathrm{int}(\Phi(T))+\mathrm{leaf}(\Phi(T))$, so it suffices to prove $\mathrm{int}(T)=\mathrm{leaf}(\Phi(T))$. We proceed by induction on $m:=\#T$. Clearly, for $m=2$, $\Phi(T)=T$, and $\mathrm{int}(T)=\mathrm{leaf}(\Phi(T))=1$. For $T=T_1\oplus T_2$ with $m\geqslant 3$, suppose the conclusion is true for trees whose number of vertices is fewer than m. We consider the following three cases.

Case 1 If $\#T_1=1$, then $\#T_2=m-1\geqslant 2$. By induction hypothesis, $\mathrm{int}(T_2)=\mathrm{leaf}(\Phi(T_2))$, thus $\mathrm{int}(T)=1+\mathrm{int}(T_2)=1+\mathrm{leaf}(\Phi(T_2))=\mathrm{leaf}(\Phi(T))$.

Case 2 If $\#T_2=1$, then $\#T_1=m-1\geqslant 2$. By induction hypothesis, $\mathrm{int}(T_1)=\mathrm{leaf}(\Phi(T_1))$, thus $\mathrm{int}(T)=\mathrm{int}(T_1)=\mathrm{leaf}(\Phi(T_1))=\mathrm{leaf}(\Phi(T))$.

Case 3 If $\#T_1\geqslant 2$, $\#T_2\geqslant 2$, by induction hypothesis, $\mathrm{int}(T_1)=\mathrm{leaf}(\Phi(T_1))$, and $\mathrm{int}(T_2)=\mathrm{leaf}(\Phi(T_2))$. Thus $\mathrm{int}(T)=\mathrm{int}(T_1)+\mathrm{int}(T_2)=\mathrm{leaf}(\Phi(T_1))+\mathrm{leaf}(\Phi(T_2))=\mathrm{leaf}(\Phi(T))$.

In all three cases we get $\mathrm{int}(T)=\mathrm{leaf}(\Phi(T))$, so the proof is now completed by induction. □

For each tree $T\in\Gamma_M$, we trace out two sequences of weakly increasingly labeled vertices. Starting with $l_1:=s(T)$, the level 1 node that is to the immediate left of l_1 is denoted as l_2, then l_2's left neighbor is denoted as l_3, so on and so force, until we arrive at the leftmost level 1 node, say l_s. This sequence is called the *level* 1 *section* of T and denoted as $\mathbf{l}_T=(l_1,\dots,l_s)$. In another direction, we also start with $r_1:=s(T)$, its rightmost child is denoted as r_2, then r_2's rightmost child is denoted as r_3, so on and so forth, until we reach a leaf, say r_t. This sequence is called the *right profile* of T and denoted as $\mathbf{r}_T=(r_1,\dots,r_t)$.

The following result strengthens (3.2).

Proposition 3.6. *For $T\in\Gamma_M$ with $\#T\geqslant 2$, we have $\mathbf{l}_T=\mathbf{r}_{\Phi(T)}$ and $\mathbf{r}_T=\mathbf{l}_{\Phi(T)}$.*

Proof. First note that the level 1 sections and right profiles are the same for two congruent trees. Now suppose $\mathbf{l}_T=(l_1,\dots,l_p)$, $\mathbf{r}_T=(r_1,\dots,r_q)$, $\mathbf{l}_{\Phi(T)}=(\hat{l}_1,\dots,\hat{l}_s)$, and $\mathbf{r}_{\Phi(T)}=(\hat{r}_1,\dots,\hat{r}_t)$. Decompose $T=T_1\oplus T_2$, $\Phi(T)=T_3\oplus T_4$. Applying (3.3) again we have $T_3\sim\Phi(T_2)$ and $T_4\sim\Phi(T_1)$. By induction on $\#T$, the proof now reduces to the following verification, where the equal signs marked with "$*$" use induction hypothesis.

$$
\begin{aligned}
&l_1=r_1=s(T)=s(\Phi(T))=\hat{l}_1=\hat{r}_1,\\
&(l_2,\dots,l_p)=\mathbf{l}_{T_1}\overset{*}{=}\mathbf{r}_{\Phi(T_1)}=\mathbf{r}_{T_4}=(\hat{r}_2,\dots,\hat{r}_t),\\
&(r_2,\dots,r_q)=\mathbf{r}_{T_2}\overset{*}{=}\mathbf{l}_{\Phi(T_2)}=\mathbf{l}_{T_3}=(\hat{l}_2,\dots,\hat{l}_s). \quad \square
\end{aligned}
$$

For each tree T that is congruent to a weakly increasing tree in Γ_M (i.e., the label of T's root may be nonzero), we introduce two multiset-valued statistics and show in the next proposition that they are also interchanged under the mapping Φ. Two vertices sharing the same parent are said to be *brothers* of each other, and between two adjacent brothers, the one on the left is viewed as the elder one. We call every vertex that has an elder brother with the same label a *repeated brother*. The labels of all repeated brothers in T form a multiset that we denote as $\mathrm{Bro}(T)$. We call every non-root vertex[2] that has at least one child with the same label a *repeated parent*. The labels of all repeated parents in T form another multiset that we denote as $\mathrm{Par}(T)$. The cardinalities of these two multisets are denoted as $\mathrm{bro}(T)$ and $\mathrm{par}(T)$, respectively. See Fig. 7 for an example, where each label of repeated brothers and repeated parent has been marked with a "$*$".

Proposition 3.7. *For any $T\in\Gamma_M$, we have* $\mathrm{Bro}(T)=\mathrm{Par}(\Phi(T))$ *and* $\mathrm{Par}(T)=\mathrm{Bro}(\Phi(T))$. *Consequently,* $\mathrm{bro}(T)=\mathrm{par}(\Phi(T))$ *and* $\mathrm{par}(T)=\mathrm{bro}(\Phi(T))$.

Proof. Recall from Proposition 3.4 that Φ is an involution, so it suffices to show that $\mathrm{Bro}(T)=\mathrm{Par}(\Phi(T))$. By induction on $\#T$, we claim that there is a one-to-one correspondence, say ζ_T, between the set of repeated brothers of T and the set

[2] Here the requirement that "the vertex is not the root" is not redundant since the root of the tree may be nonzero, thus could be repeated.

of repeated parents of $\Phi(T)$. This claim then implies that $\mathrm{Bro}(T) = \mathrm{Par}(\Phi(T))$. When $\#T \leqslant 2$, both sets are empty sets so the correspondence trivially exists. We next assume $\#T \geqslant 3$ and the correspondence ζ_R exists for all trees R (possibly with nonzero root) satisfying $\#R < \#T$. Write $T = T_1 \oplus T_2$ and $\Phi(T) = T_3 \oplus T_4$. For each repeated brother, say $v \in T$, we find its image $w := \zeta_T(v)$, a repeated parent in $\Phi(T)$, according to the following three cases.

- v is the root of T_2. Let w be the root of T_4. Denote w' the rightmost child of w and v' the adjacent elder brother of v, respectively. We see that the labels of w and w' are the first two integers of the right profile sequence $\mathbf{r}_{\Phi(T)}$, which equals $\mathbf{l}_T$ by Proposition 3.6. And the first two integers of $\mathbf{l}_T$ are respectively the labels of v and v'. Hence w and w' have the same label, making w a repeated parent in $\Phi(T)$, as desired.
- v is a non-root vertex of T_2. Since $\#T_2 < \#T$, by induction hypothesis we see that $\zeta_{T_2}(v)$ is a repeated parent in $\Phi(T_2)$. Recall that $\Phi(T_2) \sim T_3$, so $\zeta_{T_2}(v)$ has a unique copy in T_3, which we take to be w. Note that w is a repeated parent in T_3 (also in $\Phi(T)$).
- v is a vertex of T_1. Since $\#T_1 < \#T$, by induction hypothesis we see that $\zeta_{T_1}(v)$ is a repeated parent in $\Phi(T_1)$. Recall that $\Phi(T_1) \sim T_4$, so this parent has a unique copy in T_4, which we take to be w. Note that w is a repeated parent in T_4 (also in $\Phi(T)$).

The three cases above are mutually exclusive and cover all possibilities of a repeated brother v in T. The same can be said about its uniquely determined image w in $\Phi(T)$. Thus we see $\zeta_T : v \mapsto w$ is indeed a one-to-one correspondence as claimed. □

In their study of bijections between plane trees and 2-Motzkin paths, Chen, Deutsch and Elizalde [3] introduced the notion of young and old leaves. These concepts naturally generalize to the weakly increasing trees. A leaf node of a weakly increasing tree is *old* if it is the leftmost child of its parent; otherwise, it is *young*. Denote $\mathrm{Ole}(T)$ the multiset of labels of all old leaves in T, and let $\mathrm{ole}(T)$ be its cardinality. Take the two trees T in Fig. 5 and $\Phi(T)$ in Fig. 6 for example, we have $\mathrm{Ole}(T) = \mathrm{Ole}(\Phi(T)) = \{1, 2, 3, 3, 4, 4, 5\}$, and $\mathrm{ole}(T) = \mathrm{ole}(\Phi(T)) = 7$.

Proposition 3.8. *For each $T \in \Gamma_M$, $\mathrm{Ole}(T) = \mathrm{Ole}(\Phi(T))$, and thus $\mathrm{ole}(T) = \mathrm{ole}(\Phi(T))$.*

Proof. The proof is analogous to the proofs of previous propositions by using inducton on $\#T$, so we omit the details. Note that for $T = T_1 \oplus T_2$ with $\#T \geqslant 3$, the node $s(T) = r(T_2)$ can never be an old leaf, so there are only two cases to consider. Namely, when the old leaf is contained in T_1, or when the old leaf is contained in T_2. In both cases, one uses induction to see that this node keeps its label and remains an old leaf in $\Phi(T)$. □

We are now in a position to prove the main result of this paper.

Proof of Theorem 1.2. Relying on the involution Φ, this theorem is just the generating function version of Propositions 3.5, 3.7 and 3.8 combined. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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