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Note

A new proof of Balinski's theorem on the connectivity of polytopes

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ABSTRACT

Balinski (1961) proved that the graph of a d-dimensional convex polytope is d-connected. We provide a new proof of this result. Our proof provides details on the nature of a separating set with exactly d vertices; some of which appear to be new.

1. Introduction

A (convex) polytope is the convex hull of a finite set X of points in $\mathbb{R}^d$; the *convex hull* of X is the smallest convex set containing X. The *dimension* of a polytope in $\mathbb{R}^d$ is one less than the maximum number of affinely independent points in the polytope; a set of points $\vec{p}_1, \ldots, \vec{p}_k$ in $\mathbb{R}^d$ is *affinely independent* if the $k-1$ vectors $\vec{p}_1 - \vec{p}_k, \ldots, \vec{p}_{k-1} - \vec{p}_k$ are linearly independent. A polytope of dimension d is referred to as a *d-polytope*.

A polytope is structured around other polytopes, its faces. A *face* of a polytope P in $\mathbb{R}^d$ is P itself, or the intersection of P with a hyperplane in $\mathbb{R}^d$ that contains P in one of its closed halfspaces. A face of dimension 0, 1, and $d-1$ in a d-polytope is a *vertex*, an *edge*, and a *facet*, respectively. The set of vertices and edges of a polytope or a graph are denoted by V and E, respectively. The *graph* $G(P)$ of a polytope P is the abstract graph with vertex set $V(P)$ and edge set $E(P)$.

A graph with at least $d+1$ vertices is *d-connected* if removing any $d-1$ vertices leaves a connected subgraph. Balinski [1] showed that the graph of a d-polytope is d-connected. His proof considers a hyperplane in $\mathbb{R}^d$ passing through a set of $d-1$ vertices of a d-polytope, and so do the proofs of Grünbaum [9, Thm. 11.3.2], Ziegler [12, Thm. 3.14], and Brøndsted [4, Thm. 15.6]. Such proofs yield a geometric structure of separators in the graph of the polytope (Lemma 1). A set X of vertices in a graph G *separates* two vertices x, y if every path in G between x and y contains an element of X, and $x, y \notin X$. And X *separates* G if it separates two vertices of G. A separating set of vertices is a *separator* and a separator of cardinality r is an *r-separator*.

Lemma 1. *Let P be a d-polytope in $\mathbb{R}^d$ and let H be a hyperplane in $\mathbb{R}^d$. If X is a proper subset of $H \cap V(P)$, then removing X does not disconnect $G(P)$. In particular, a separator of $G(P)$ with exactly d vertices must form an affinely independent set in $\mathbb{R}^d$.*

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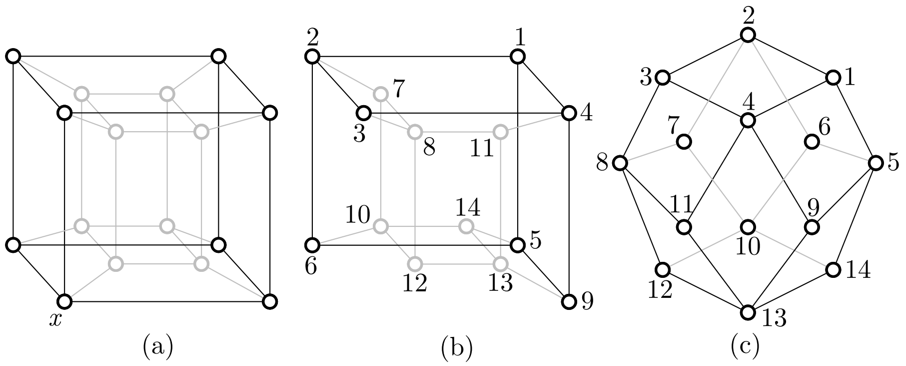

Fig. 1. The link of a vertex in the four-dimensional cube, the convex hull of the 2^4 vectors $(\pm1, \pm1, \pm1, \pm1)$ in $\mathbb{R}^4$. (a) The four-dimensional cube with a vertex x highlighted. (b) The link of the vertex x in the cube. (c) The link of the vertex x as the boundary complex of the rhombic dodecahedron (Proposition 2).

Other proofs with a geometric flavour were given by Brøndsted and Maxwell [5] and Barnette [3]. Our proof has a more combinatorial nature, relying on certain polytopal complexes in a polytope. Another combinatorial proof, based on a different idea, can be found in [2].

The *boundary complex* of a polytope P is the set of faces of P other than P itself. And the *link* of a vertex x in P, denoted $\mathrm{lk}(x)$, is the set of faces of P that do not contain x but lie in a facet of P that contains x (Fig. 1(b)). We require a result from Ziegler [12].

Proposition 2 (*Ziegler [12, Ex. 8.6]*)**.** *Let P be a d-polytope. Then the link of a vertex in P is combinatorially isomorphic to the boundary complex of a $(d-1)$-polytope. In particular, for each $d \geqslant 3$, the graph of the link of a vertex is isomorphic to the graph of a $(d-1)$-polytope.*

We proved Proposition 2 in Bui et al. [6, Prop. 12] and exemplified it in Fig. 1. In this paper, we prove the following. The part about links appears to be new.

Theorem 3. *For $d \geqslant 1$, the graph of d-polytope P is d-connected. Besides, for each $d \geqslant 3$, each vertex x in a d-separator X of $G(P)$ lies in the link of every other vertex of X, and the set $X \setminus \{x\}$ is a separator of the link of x.*

As a corollary, we get a known result on d-separators in simplicial d-polytopes [8, p. 509]; see Corollary 4. A polytope is *simplicial* if all its facets are simplices, and a *d-simplex* is a d-polytope whose $d+1$ vertices form an affinely independent set in $\mathbb{R}^d$. An *empty* $(d-1)$*-simplex* in a d-polytope P is a set of d vertices of P that does not form a face of P but every proper subset does. An empty $(d-1)$-simplex is also called a *missing* $(d-1)$*-simplex*.

Corollary 4. *Let P be a simplicial d-polytope with $d \geqslant 2$. A d-separator of $G(P)$ forms an empty $(d-1)$-simplex of P.*

We remark that the paragraph after Balinski's theorem in Goodman et al. [8, p. 509] is meant to concern only simplicial d-polytopes, and not d-polytopes in general. While it is true that a d-separator of the graph of a d-polytope must form an affinely independent set in $\mathbb{R}^d$, it is not true that it must form an empty simplex. Take, for instance, the neighbours of a vertex in a d-dimensional cube (Fig. 1(a)).

We follow [7] for the graph theoretical terminology that we have not defined.

2. Proofs of Theorem 3 and Corollary 4

A path between vertices x and y in a graph is an *$x-y$ path*, and two $x-y$ paths are *independent* if they share no inner vertex. For a path $L := x_0 \ldots x_n$ and for $0 \leqslant i \leqslant j \leqslant n$, we write x_iLx_j to denote the subpath $x_i \ldots x_j$. We require a theorem of Whitney [11] and one of Menger [10].

Theorem 5 (*Whitney [11]*)**.** *Let G be a graph with at least one pair of nonadjacent vertices. Then there is a minimum separator of G disconnecting two nonadjacent vertices.*

Theorem 6 (*Menger [10]*)**.** *Let G be a graph, and let x and y be two nonadjacent vertices. Then the minimum number of vertices separating x from y in G equals the maximum number of independent $x-y$ paths in G.*

Proof of Theorem 3. Let P be a d-polytope and let G be its graph. Then G has at least $d+1$ vertices. If G is a complete graph, there is nothing to prove, and suppose otherwise. In this case, G has at least one pair of nonadjacent vertices. For

$d = 2$, G is d-connected. And so induct on d, assuming that $d \geqslant 3$ and that the theorem is true for $d-1$. Let X be a separator in G of minimum cardinality, and let y and z be vertices separated by X. Then $y, z \notin X$. According to Whitney's theorem (Theorem 5), there is a minimum separator of G disconnecting two nonadjacent vertices. Hence we may assume that y and z are nonadjacent, and by Menger's theorem (Theorem 6), that there are $|X|$ independent $y-z$ paths in G, each containing precisely one vertex from X. Let L be one such $y-z$ paths and let x be the vertex in $X \cap V(L)$; say that $L = u_1 \dots u_m$ such that $y = u_1$, $u_j = x$, and $u_m = z$.

The graph G_x of the link of x in P is isomorphic to the graph of a $(d-1)$-polytope (Proposition 2), and by the induction hypothesis it is $(d-1)$-connected. The neighbours of x are all part of $\mathrm{lk}(x)$, and so $u_{j-1}, u_{j+1} \in G_x$. Again, from Menger's theorem follows the existence of at least $d-1$ independent $u_{j-1} - u_{j+1}$ paths in G_x. We must have that $X \setminus \{x\}$ separates u_{j-1} from u_{j+1} in G_x, since X separates y from z. Hence $|X \setminus \{x\}| \geqslant d-1$, which establishes that G is d-connected.

Finally, let $d \geqslant 3$ and suppose X is a d-separator of G. As stated above, the set $X \setminus \{x\}$, of cardinality $d-1$, separates G_x, implying that $X \setminus \{x\} \subseteq V(G_x)$. The aforementioned path L was arbitrary among the $y-z$ paths separated by X, and each such path contains a unique vertex of X. It follows that every vertex in X is in the link of every other vertex of X, which concludes the proof of the theorem. □

Proof of Corollary 4. Let P be a simplicial d-polytope and let G be its graph. Suppose that X is a d-separator of G, that x is a vertex of X, and that G_x is the graph of the link of x in P.

A simplicial 2-polytope is a polygon and a 2-separator in it satisfies the corollary. So assume that $d \geqslant 3$. From Theorem 3, it follows that every vertex in X is in the link of every other vertex of X, and that $X \setminus \{x\}$ is a $(d-1)$-separator of G_x. Consequently, the subgraph $G[X]$ of G induced by X is a complete graph, as the set of neighbours of each vertex in P coincides with the vertex set of the link of the vertex.

If $d = 3$, then, from $G[X]$ being a complete graph, it follows that it is an empty 2-simplex. And so an inductive argument on d can start. Assume that $d \geqslant 4$. From the definition of a link and Proposition 2, we obtain that $\mathrm{lk}(x)$ is combinatorially isomorphic to the boundary complex of a simplicial $(d-1)$-polytope.

By the induction hypothesis on $\mathrm{lk}(x)$, every proper subset of $X \setminus \{x\}$ forms a face F of $\mathrm{lk}(x)$. And from the definition of $\mathrm{lk}(x)$, that face F lies in a facet of P containing x, a $(d-1)$-simplex containing x. As a consequence, if F is a face of dimension k, then the set $\mathrm{conv}(F \cup \{x\})$ is a face of P of dimension $k+1$. Since the vertex x of X was taken arbitrarily, the corollary ensues. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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