An Improved Approximation Algorithm for the Minimum k-Edge Connected Multi-Subgraph Problem

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Abstract

We give a randomized $1+\sqrt{\frac{8\ln k}{k}}$-approximation algorithm for the minimum k-edge connected spanning multi-subgraph problem, k-ECSM.

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1 Introduction

In an instance of the minimum k-edge connected subgraph problem, or k-ECSS, we are given an (undirected) graph $G = (V, E)$ with $n := |V|$ vertices and a cost function $c : E \to \mathbb{R}_{\geq 0}$, and we want to choose a minimum cost set of edges $F \subseteq E$ such that the subgraph (V, F) is k-edge connected. In its most general form, k-ECSS generalizes several extensively-studied problems in network design such as tree augmentation or cactus augmentation. The k-edge-connected *multi*-subgraph problem, k-ECSM, is a close variant of k-ECSS in which we want to choose a k-edge-connected *multi*-subgraph of G of minimum cost, i.e., we can choose an edge $e \in E$ multiple times. It turns out that one can assume without loss of generality that the cost function c in k-ECSM is a metric, i.e., for any three vertices $x, y, z \in V$, we have $c(x, z) \leq c(x, y) + c(y, z)$.

Around four decades ago, Fredrickson and Jájá [FJ81; FJ82] designed a 2-approximation algorithm for k-ECSS and a 3/2-approximation algorithm for k-ECSM. The latter essentially follows by a reduction to the well-known Christofides-Serdyukov approximation algorithm for the traveling salesperson problem (TSP). Over the last four decades, despite a number of papers on the problem [CT00; KR96; Kar99; Gab05; GG08; Gab+09; Pri11; LOS12], the aforementioned approximation factors were only improved in the cases where the underlying graph is unweighted or $k \gg \log n$. Most notably, Gabow, Goemans, Tardos and Williamson [Gab+09] showed that if the graph G is unweighted then k-ECSS and k-ECSM admit $1 + 2/k$ approximation algorithms, i.e., as $k \to \infty$ the approximation factor approaches 1. The special case of k-ECSM where $k = 2$ received significant attention and better than 3/2-approximation algorithms were designed for special cases [CR98; BFS16; SV14; Boy+20].

Motivated by [Gab+09], Pritchard posed the following conjecture:

Conjecture 1.1 ([Pri11]). *The k-ECSM problem admits a $1 + O(1)/k$ approximation algorithm.*

In other words, if true, the above conjecture implies that the 3/2-classical factor is not optimal for sufficiently large k, and moreover that it is possible to design an approximation algorithm whose factor gets arbitrarily close to 1 as $k \to \infty$. In this paper, we prove a weaker version of the above conjecture.

Theorem 1.2 (Main). *There is a randomized algorithm for (weighted) k-ECSM with approximation factor (at most) $1 + \sqrt{\frac{8 \ln k}{k}}$.*

We remark that our main theorem only improves the classical 3/2-approximation algorithm for k-ECSM only when $k \geq 164$ (although one can use the more precise expression given in the proof to, for example, improve upon 3/2 for even values of $k \geq 66$).

For a set $S \subseteq V$, let $\delta(S) = \{\{u, v\} : |\{u, v\} \cap S| = 1\}$ denote the set of edges leaving S. The following is the natural linear programming relaxation for k-ECSM.

$$\begin{aligned} \min \quad & \sum_{e \in E} x_e c(e) \\ \text{s.t.} \quad & x(\delta(v)) = k \quad \forall v \in V \\ & x(\delta(S)) \geq k \quad \forall S \subseteq V \\ & x_e \geq 0 \qquad\quad \forall e \in E. \end{aligned} \tag{1}$$

Note that while in an optimum solution of k-ECSM the degree of each vertex is not necessarily equal to k, since the cost function satisfies the triangle inequality we may assume that in any optimum fractional solution each vertex has (fractional) degree k. This follows from the parsimonious property [GB93].

We prove Theorem 1.2 by rounding an optimum solution to the above linear program. So, as a corollary we also upper-bound the integrality gap of the above linear program.

Corollary 1.3. *The integrality gap of LP (1) is at most* $1 + \sqrt{\frac{8 \ln k}{k}}$.

1.1 Proof Overview

Before explaining our algorithm, we recall a randomized rounding approach of Karger [Kar99]. Karger showed that if we choose every edge e independently with probability x_e, then the sample is $k - O(\sqrt{k \log n})$-edge connected with high probability. He then fixes the connectivity of the sample by adding $O(\sqrt{k \log n})$ copies of the minimum spanning tree of G. This gives a $1 + O(\sqrt{\log n / k})$ approximation algorithm for the problem.

First, we observe that where x is a solution to the LP (1), the vector $2x/k$ is in the spanning tree polytope (after modifying x slightly, see Fact 2.1 for more details). Following a recent line of works on the traveling salesperson problem [OSS11; KKO20] we write $2x/k$ as a λ-uniform spanning tree distribution, μ_λ. Then, we independently sample $k/2$ spanning trees[1] $T_1, \ldots, T_{k/2}$. It follows that $T^* = T_1 \uplus T_2 \uplus \cdots \uplus T_{k/2}$ has the same expectation across every cut as x, and due to properties of λ-uniform spanning tree distributions it is concentrated around its mean. Unlike the independent rounding procedure, T^* has at least $k/2$ edges across each cut with probability 1. This implies that the number of "bad" cuts of T^*, i.e. those of size strictly less than k, is at most $(n-1)k/2$ (with probability 1). This is because any tree T_i has strictly less than 2 edges in exactly $n-1$-"tree cuts," and a cut lying on no tree cuts must have at least k edges in T^*.

We divide these potentially $O(nk)$ bad cuts into two types: (i) Cuts S such that $\delta(S)_{T^*} \geq k - \alpha\sqrt{k/2 - 1}$ and (ii) Cuts S where $\delta(S)_{T^*} < k - \alpha\sqrt{k/2-1}$, for some $\alpha = \Theta(\sqrt{\ln k})$. We fix all cuts of type (i) by adding $\alpha\sqrt{k/2-1}$ copies of the minimum spanning tree of G. To fix cuts S of type (ii), we employ the following procedure: for any tree T_i where $\delta(S)_{T_i} = 1$ and S is of type (ii), we add one extra copy of the unique edge of T_i in $\delta(S)$. To bound the expected cost of our rounded solution, we use the concentration property of λ-uniform trees on edges of T^* to show that the probability any fixed cut $\delta(S)$ is of type (ii) is exponentially small in α, $\leq e^{-\alpha^2/2}$, even if we condition on $\delta(S)_{T_i} = 1$ for a single tree T_i.

1.2 Algorithm

For two sets of edges $F, F' \subseteq E$, we write $F \uplus F'$ to denote the multi-set union of F and F' allowing multiple edges. Note that we always have $|F \uplus F'| = |F| + |F'|$.

Let x be an optimal solution of LP (1). We expand the graph $G = (V, E)$ to a graph G^0 by picking an arbitrary vertex $u \in V$, splitting it into two nodes u_0 and v_0, and then, for every edge $e = (u, w)$ incident to u, assigning fraction $\frac{x(e)}{2}$ to each of the two edges (u_0, w) and (v_0, w) in G^0. Call this expanded graph G^0, its edge set E^0, and the resulting fractional solution x^0, where $x^0(e)$

[1] If k is odd, we sample $\lceil k/2 \rceil$ trees. The bound remains unchanged relative to the analysis we give below as the potential cost of one extra tree is $O(OPT/k)$.

and $x(e)$ are identical on all other edges. (Note that each of u_0 and v_0 now have fractional degree $k/2$ in x^0.) In Fact 2.1 below, we show that $\frac{2}{k} \cdot x^0$ is in the spanning tree polytope for the graph G^0. For ease of exposition, the algorithm is described as running on G^0 (and spanning trees[2] of G^0), which has the same edge set as G (when u_0 and v_0 are identified).

Our algorithm is as follows:

Algorithm 1 An Approximation Algorithm for k-ECSM

Let x^0 be an optimum solution of (1) extended to the graph G^0 as described above.
Find weights $\lambda : E^0 \to \mathbb{R}_{\geq 0}$ such that for any $e \in E^0$, $\mathbb{P}_{\mu_\lambda}[e] \leq \frac{2}{k} x_e^0 (1 + 2^{-n})$. ▷ By Theorem 2.2
Sample $k/2$ spanning trees $T_1, \cdots, T_{k/2} \sim \mu_\lambda$ (in G^0) independently and let $T^* \leftarrow T_1 \uplus \cdots \uplus T_{k/2}$.
Let B be $\alpha\sqrt{k/2-1}$ copies of the MST of G^0. ▷ $\alpha = \Theta(\sqrt{\ln k})$ is a parameter we choose later.
for $i \in [\frac{k}{2}]$ and $e \in T_i$ **do**
    if $C_{T_i}(e)_{T^*} < k - \alpha\sqrt{k/2-1}$ and $(u_0, v_0) \notin C_{T_i}(e)$ **then**
        $F \leftarrow F \uplus \{e\}$.
    end if
end for
Return $T^* \uplus B \uplus F$.

2 Preliminaries

For any set of edges $F \subseteq E$ and a set of edges $T \subseteq E$, we write

$$F_T := |F \cap T|.$$

Also, for any edge weight function $x : E \to \mathbb{R}$, we write $x(F) := \sum_{e \in F} x(e)$.

For any spanning tree T of G^0, and any edge $e \in T$, we write $C_T(e) \subseteq E$ to denote the set of edges in the unique cut obtained by deleting e from T. Of particular interest to us below will be $C_{T_i}(e)_{T^*} = |C_{T_i}(e) \cap T^*|$ where e is an edge in T_i.

We will also use the notation $(u_0, v_0) \notin C$ to indicate that u_0 and v_0 are on the same side of the cut C.

2.1 Random Spanning Trees

Edmonds [Edm70] gave the following description for the convex hull of the spanning trees of any graph $G = (V, E)$, known as the *spanning tree polytope*.

$$\begin{aligned} & z(E) = |V| - 1 & \\ & z(E(S)) \leq |S| - 1 & \forall S \subseteq V \\ & z_e \geq 0 & \forall e \in E. \end{aligned} \tag{2}$$

Edmonds also [Edm70] proved that the extreme point solutions of this polytope are the characteristic vectors of the spanning trees of G.

[2] A spanning tree in G^0 is a 1-tree in G, that is, a tree plus an edge.

Fact 2.1 ([KKO20]). *Let x be the optimal solution of LP (1) and x^0 its extension to G^0 as described above. Then $\frac{2}{k} \cdot x^0$ is in the spanning tree polytope (2) of G^0.*

Proof. For any set $S \subseteq V(G^0)$ with $u_0, v_0 \notin S$, $x^0(E(S)) = \frac{k|S| - x(\delta(S))}{2} \leq \frac{k}{2}(|S| - 1)$. If $u_0 \in S, v_0 \notin S$, then $x^0(\delta(S)) \geq k/2$, so $x^0(E(S)) \leq \frac{k|S| - k/2 - x^0(\delta(S))}{2} \leq \frac{k}{2}(|S| - 1)$. Finally, if $u_0, v_0 \in S$, then $x^0(\delta(S)) \geq k$. Thus, $x^0(E(S)) = \frac{k|S| - k - x^0(\delta(S))}{2} \leq \frac{k}{2}(|S| - 2)$. The claim follows because $x^0(E) = \frac{k|V(G)|}{2} = \frac{k}{2}(|V(G^0)| - 1)$. □

Given nonnegative edge weights $\lambda : E \to \mathbb{R}_{\geq 0}$, we say a distribution μ_λ over spanning trees of G is λ-*uniform*, if for any spanning tree T,

$$\mathbb{P}_{T \sim \mu_\lambda}[T] \propto \prod_{e \in T} \lambda(e).$$

Theorem 2.2 ([Asa+17]). *There is a polynomial-time algorithm that, given a connected graph $G = (V, E)$, and a point $z \in \mathbb{R}^{|E|}$ in the spanning tree polytope (2) of $G = (V, E)$, returns $\lambda : E \to \mathbb{R}_{\geq 0}$ such that the corresponding λ-uniform spanning tree distribution μ_λ satisfies*

$$\sum_{T \in \mathcal{T} : e \in T} \mu_\lambda(T) \leq (1 + 2^{-n}) z_e, \ \forall e \in E,$$

i.e., the marginals are approximately preserved. In the above $\mathcal{T}$ is the set of all spanning trees of G.

2.2 Bernoulli-Sum Random Variables

Definition 2.3 (Bernoulli-Sum Random Variable). *We say $BS(q)$ is a* Bernoulli-Sum *random variable if it has the law of a sum of independent Bernoullis, say $B_1 + B_2 + \cdots + B_t$ for some $t \geq 1$, with $\mathbb{E}[B_1 + \cdots + B_t] = q$.*

Fact 2.4. *If $X = BS(q_1)$ and $Y = BS(q_2)$ are two independent Bernoulli-sum random variables then $X + Y = BS(q_1 + q_2)$.*

Lemma 2.5 ([BBL09; Pit97]). *Given $G = (V, E)$ and $\lambda \in E \to \mathbb{R}_{\geq 0}$, let μ_λ be the λ-uniform spanning tree distribution of G. Let T be a sample from μ_λ. Then for any fixed $F \subseteq E$, the random variable F_T is distributed as $BS(\mathbb{E}[F_T])$.*

Theorem 2.6 (Multiplicative Chernoff-Hoeffding Bound for BS Random Variables). *Let $X = BS(q)$ be a Bernoulli-Sum random variable. Then, for any $0 < \epsilon < 1$ and $q' \leq q$*

$$\mathbb{P}\left[X < (1 - \epsilon) q'\right] \leq e^{-\frac{\epsilon^2 q'}{2}}.$$

3 Analysis of the Algorithm

In this section we prove Theorem 1.2. We first observe that the cuts of G are precisely the cuts of G^0 that have u_0 and v_0 on the same side of the cut, and for any such cut the set of edges crossing the cut in G and in G^0 is the same (once u_0 and v_0 are contracted). We begin by showing that the output of Algorithm 1 is k-edge connected (in G) with probability 1.

Lemma 3.1 (k-Connectivity of the Output)**.** *For any $\alpha \geq 0$, the output of Algorithm 1, $F \uplus B \uplus T^*$ is a k-edge connected subgraph of G.*

Proof. Fix spanning trees $T_1, \cdots, T_{k/2}$ in G^0 and let $C = \delta(S)$ for some S, where $(u_0, v_0) \notin C$. We show that $C_{T^* \uplus F \uplus B} \geq k$. If $C_{T^*} \geq k - \alpha\sqrt{k/2-1}$, then since B has $\alpha\sqrt{k/2-1}$ copies of the minimum spanning tree, $C_{T^* \uplus B} \geq k$ and we are done. Otherwise $C_{T^*} < k - \alpha\sqrt{k/2-1}$. Then, we know that for any tree T_i, either $C_{T_i} \geq 2$ or $C_{T_i} = 1$. If $C_{T_i} = 1$, since $(u_0, v_0) \notin C_{T_i}$, F has one extra copy of the unique edge of T_i in C. Therefore, including those cases where an extra copy of the edge e is added, each T_i has at least two edges in C, so $C_{T^* \uplus F} \geq k$ as desired. ☐

Lemma 3.2. *For any $0 \leq \alpha \leq \sqrt{k/2-1}$, $1 \leq i \leq k/2$, and any $e \in E$,*

$$\mathbb{P}\left[C_{T_i}(e)_{T^*} \leq k - \alpha\sqrt{k/2-1} | e \in T_i \wedge (u_0, v_0) \notin C_{T_i}(e)\right] \leq e^{-\alpha^2/2}.$$

where the randomness is over spanning trees $T_1, \cdots, T_{i-1}, T_{i+1}, \cdots, T_{k/2}$ independently sampled from μ_λ.

Proof. Condition on tree T_i such that $e \in T_i$ and $(u_0, v_0) \notin C_T(e_i)$.

By Lemma 2.5, for any $j \leq k/2$ such that $j \neq i$, $C_{T_i}(e)_{T_j}$ is a $BS(\mathbb{E}\left[C_{T_i}(e)_{T_j}\right])$ random variable, with $\mathbb{E}\left[C_{T_i}(e)_{T_j}\right] = \frac{2}{k}x(C_{T_i}(e)) \geq 2$. Also, by definition, $C_{T_i}(e)_{T_i} = 1$ (with probability 1). Since $T_1, \cdots, T_{k/2}$ are independently chosen, by Fact 2.4 the random variable $C_{T_i}(e)_{T^*}$ is distributed as $BS(q)$ for $q \geq k-1$. Since each T_j has at least one edge in $C_{T_i}(e)$, $C_{T_i}(e)_{T^*} \geq k/2$ with probability 1. So, by Theorem 2.6, with $q' = k - 1 - k/2$, when $0 \leq \alpha \leq \sqrt{k/2-1}$,

$$\begin{aligned}
&\mathbb{P}\left[C_{T_i}(e)_{T^*} < k - \alpha\sqrt{k/2-1} | e \in T_i \wedge (u_0, v_0) \notin C_{T_i}(e)\right] \\
&= \mathbb{P}\left[C_{T_i}(e)_{T^*} - k/2 < k/2 - \alpha\sqrt{k/2-1} | e \in T_i \wedge (u_0, v_0) \notin C_{T_i}(e)\right] \leq e^{-\frac{(\alpha/\sqrt{k/2-1})^2(k/2-1)}{2}} = e^{-\alpha^2/2}.
\end{aligned}$$

Averaging over all realizations of T_i satisfying the required conditions proves the lemma. ☐

Proof of Theorem 1.2. Let x be an optimum solution of LP (1). Since the output of the algorithm is always k-edge connected we just need to show $\mathbb{E}\left[c(F \cup T^* \cup B)\right] \leq \left(1 + \sqrt{\frac{8\ln k}{k}}\right)c(x)$. By linearity of expectation,

$$\mathbb{E}\left[c(T^*)\right] = \sum_{i \in [\frac{k}{2}]} \mathbb{E}\left[c(T_i)\right] = \frac{k}{2}\sum_{e \in E} c(e)\mathbb{P}_{\mu_\lambda}[e] = \frac{k}{2}\sum_{e \in E} c(e) \cdot \frac{2}{k} \cdot x_e = c(x),$$

where for simplicity we ignored the $1 + 2^{-n}$ loss in the marginals. On the other hand, since by Fact 2.1, $\frac{2x}{k}$ is in the spanning tree polytope of G^0, $c(B) \leq \frac{2c(x)}{k} \cdot \alpha\sqrt{k/2-1} \leq \frac{\alpha c(x)}{\sqrt{k/2}}$. It remains to bound the expected cost of F. By Lemma 3.2, we have,

$$\begin{aligned}
\mathbb{E}\left[c(F)\right] &= \sum_{e \in E} c(e) \sum_{i=1}^{k/2} \mathbb{P}\left[e \in T_i \wedge (u_0, v_0) \notin C_{T_i}(e)\right] \mathbb{P}\left[C_{T_i}(e)_{T^*} < k - \alpha\sqrt{k/2-1} | e \in T_i \wedge (u_0, v_0) \notin C_{T_i}(e)\right] \\
&\leq \sum_{e \in E} c(e)x_e e^{-\alpha^2/2} \leq e^{-\alpha^2/2}c(x).
\end{aligned}$$

Putting these together we get, $\mathbb{E}\left[c(T^* \cup B \cup F)\right] \leq (1 + \alpha/\sqrt{k/2} + e^{-\alpha^2/2})c(x)$. Setting $\alpha = \sqrt{\ln\left(\frac{k}{2}\right)}$ finishes the proof. ☐

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