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ORIGINAL PAPER

Colouring of $(P_3 \cup P_2)$-free graphs

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Abstract The class of $2K_2$-free graphs and its various subclasses have been studied in a variety of contexts. In this paper, we are concerned with the colouring of $(P_3 \cup P_2)$-free graphs, a super class of $2K_2$-free graphs. We derive a $O(\omega^3)$ upper bound for the chromatic number of $(P_3 \cup P_2)$-free graphs, and sharper bounds for $(P_3 \cup P_2$, diamond)-free graphs and for $(2K_2$, diamond)-free graphs, where ω denotes the clique number. The last two classes are perfect if $\omega \geq 5$ and ≥ 4 respectively.

Keywords Colouring · Chromatic number · Clique number · $2K_2$-free graphs · $(P_3 \cup P_2)$-free graphs · Diamond · Perfect graphs

Mathematics Subject Classification 05C15 · 05C17

1 Introduction

A graph G is said to be H-free, if G does not contain an induced copy of H. More generally, a class of graphs $\mathscr{G}$ is said to be $(H_1, H_2, \ldots)$-free if every $G \in \mathscr{G}$ is H_i-free, for $i \geq 1$. The class of $2K_2$-free graphs and its subclasses are subject of research in various contexts: domination (El-Zahar and Erdős [10]), size (Bermond et al. [2], Chung et al. [9]), vertex colouring (Wagon [19], Nagy and Szentmiklóssy, [16], Gyárfás [12]), edge colouring (Erdős and Nesetril [11]) and algorithmic complexity

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(Blázsik et al. [3]). Here we are concerned with the colouring of $(P_3 \cup P_2)$-free graphs, a super class of $2K_2$-free graphs. A comprehensive result of Král' et al. [15] states that the decision problem of COLOURING H-free graphs is P-time solvable if H is an induced subgraph of P_4 or $P_3 \cup P_1$,and it is NP-complete for any other graph H. In particular, COLOURING $2K_2$-free graphs is NP-complete. However, there have been several studies to obtain tight upper and lower bounds for the chromatic number of $2K_2$-free graphs. A problem of Gyárfás [12] asks for the smallest function $f(x)$ such that $\chi(G) \leq f(\omega(G))$, for every G belonging to the class of $2K_2$-free graphs, where $\chi(G)$ and $\omega(G)$ respectively denote the chromatic number and clique number of G. This problem is still open. An unpublished question of Louis Esperet asks whether it is true that every class of graphs that is χ-bounded admits a χ-bounding function that is a polynomial in the maximum clique size. In this respect, an often quoted result is due to Wagon [19]. It states that if a graph G is $2K_2$-free, then $\chi(G) \leq \binom{\omega(G)+1}{2}$. We look more closely at Wagon's proof and obtain a $O(\omega^3)$ upper bound for the chromatic number of $(P_3 \cup P_2)$-free graphs, and sharper bounds for $(P_3 \cup P_2$, diamond)-free graphs and for $(2K_2$, diamond)-free graphs.The last two classes are perfect if the clique number is ≥ 5 and ≥ 4 respectively. The classes of $(H$, diamond)-free graphs and $(H_1, H_2$, diamond)-free graphs, for various graphs H, H_1 and H_2, have been studied in many papers; see Arbib and Mosca [1], Brandstädt [5], Choudum and Karthick [7], Karthick and Maffray [14], Gyárfás [12], and Randerath and Schiermeyer [17]. See also a comprehensive book on problems of graph colourings by Jensen and Toft [13] and an extensive book of Brandstädt et al. [6], for interesting subclasses and superclasses of $2K_2$-free graphs.

2 Terminology and notation

We follow standard terminology of Bondy and Murty [4], and West [20]. All our graphs are simple and undirected. If u, v are two vertices of a graph $G(V, E)$, then their adjacency is denoted by $u \leftrightarrow v$, and non-adjacency by $u \not\leftrightarrow v$. P_n, C_n and K_n respectively denote the path, cycle and complete graph on n vertices. A chordless cycle of length ≥ 5 is called a *hole*. If $S \subseteq V(G)$, then $[S]$ denotes the subgraph induced by S. If S and T are two disjoint subsets of $V(G)$, then $[S, T]$ denotes the set of edges $\{st \in E(G) : s \in S \text{ and } t \in T\}$. A subset Q of $V(G)$ is called a *clique* if any two vertices in Q are adjacent. The *clique number* of G is defined to be $\max\{|Q| : Q \text{ is a clique in } G\}$; it is denoted by $\omega(G)$. A clique Q is called a *maximum clique* if $|Q| = \omega(G)$. A subset I of $V(G)$ is called an *independent set* if no two vertices in I are adjacent. A *k-vertex colouring* or a *k-colouring* or a *colouring* is a function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $f(u) \neq f(v)$, for any two adjacent vertices u, v in G. It is also referred to as a proper colouring of G for emphasis. The *chromatic number* $\chi(G)$ of G is defined to be $\min\{k : G \text{ admits a } k\text{-colouring}\}$. If $G_1, G_2, \ldots, G_k$ are vertex disjoint graphs, then $G_1 \cup G_2 \cup \cdots \cup G_k$ denotes the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$. If $G_1 \simeq G_2 \simeq \cdots \simeq G_k \simeq H$, for some H, then $G_1 \cup G_2 \cup \cdots \cup G_k$ is denoted by kH. The three graphs which appear frequently in this paper are shown in Fig. 1.

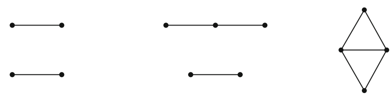


Fig. 1 $2K_2$, $P_3 \cup P_2$, diamond

3 A partition of the vertex set of a graph

Throughout this paper we use a particular partition of the vertex set of a graph $G(V, E)$ and use its properties. Some of these properties are due to Wagon [19], but are restated for ready reference. In what follows, ω denotes the clique number of a graph under consideration.

Let A be a maximum clique in G with vertices $1, 2, \ldots, \omega$. We define the sets C_{ij} for pairs of vertices i, j of A as follows: let $<_L$ denote the lexicographic ordering on the set $\{(i, j) \mid 1 \le i < j \le \omega\}$, and for all (i, j) in that set let

$$C_{ij} = \{v \in V(G) - A \mid v \not\leftrightarrow i \text{ and } v \not\leftrightarrow j\} - \bigcup_{(i',j')<_L(i,j)} C_{i'j'}.$$

By definition, there are $\binom{\omega}{2}$ number of C_{ij} sets and these are pairwise disjoint. Also, every vertex in C_{ij} is adjacent to every vertex k of A, where $1 \le k < j, k \ne i$. Moreover, every vertex in $V(G) - A$ which is non-adjacent to two or more vertices of A is in some C_{ij}. So, every vertex $v \in V(G) - (A \cup C)$ is adjacent to all the vertices of A or $|A| - 1$ vertices of A. The former case is impossible, since A is a maximum clique. Hence we define the following sets. For $a \in A$, let

$$I_a = \{v \in V(G) - (A \cup C) \mid v \leftrightarrow x, \forall x \in A - \{a\} \text{ and } v \not\leftrightarrow a\}.$$

By the above remarks, we conclude that $(A, \bigcup_{i,j} C_{ij}, \bigcup_{a \in A} I_a)$ is a partition of $V(G)$.

4 Colouring of $(P_3 \cup P_2)$-free graphs

We first observe a few properties of the sets C_{ij} and I_a, and then obtain an $O(\omega^3)$ upper bound for the chromatic number of a $(P_3 \cup P_2)$-free graph.

Theorem 1 *If a graph G is $(P_3 \cup P_2)$-free, then $\chi(G) \le \frac{\omega(\omega+1)(\omega+2)}{6}$.*

Proof Let A be a maximum clique in G. Let $(1, 2, 3, \ldots, \omega)$ be a vertex ordering of A. Since G is $(P_3 \cup P_2)$-free, the sets C_{ij} and I_a possess a few more properties, in addition to the ones stated in Sect. 3. □

Claim 1 Each induced subgraph $[C_{ij}]$ of G is P_3-free and hence it is a disjoint union of cliques.

If some C_{ij} contains an induced $P_3 = (x, y, z)$, then $[\{x, y, z\} \cup \{i, j\}] \simeq P_3 \cup P_2$, a contradiction.

Claim 2 *Each I_a is an independent set.*

If some I_a contains an edge vw, then $A \cup \{v, w\} - \{a\}$ is a clique of size $\omega + 1$, a contradiction to the maximality of $|A|$.

Claim 3 $\omega\left([C_{ij}]\right) \leq \omega - (j-2)$, *where* $j \geq 2$.

Let B be a maximum clique in $[C_{ij}]$. Every vertex in B is adjacent to every vertex in $K = \{1, 2, \ldots, j-1\} - \{i\} \subseteq A$, by the definition of C_{ij}. So, $B \cup K$ is a clique of G. Hence, $\omega(G) \geq |B \cup K| = \omega([C_{ij}]) + |K| = \omega([C_{ij}]) + j - 2$. Hence the claim.

We now properly colour G as follows:

(1) Colour the vertices $1, 2, \ldots, \omega$ of A with colours $1, 2, \ldots, \omega$ respectively.
(2) Colour the vertices of C_{ij} with $\omega([C_{ij}])$ new colours, $1 \leq i < j \leq \omega$. By Claim 1, $[C_{ij}]$ is a disjoint union of cliques and hence one can properly colour $[C_{ij}]$ with $\omega([C_{ij}])$ colours. Note also that one requires at most $\omega - (j-2)$ colours, by Claim 3.
(3) Each vertex in I_a is given the colour of $a \in A$.

It is a proper colouring of G by Claims 1, 2 and 3. We first estimate the number of colours used in step (2) to colour the vertices of C and then estimate the total number of colours used to colour G entirely.

$$\begin{aligned}
\chi([C]) &\leq 1(\omega) + 2(\omega-1) + 3(\omega-2) + \cdots + (\omega-1)2, \\
&= \sum_{k=1}^{\omega-1} k(\omega+1-k) \\
&= \sum_{k=1}^{\omega-1} k(\omega+1) - \sum_{k=1}^{\omega-1} k^2 \\
&= (\omega+1)\frac{(\omega-1)(\omega)}{2} - \frac{(\omega-1)(\omega)(2\omega-2+1)}{6} \\
&= \frac{\omega(\omega-1)(\omega+4)}{6}
\end{aligned}$$

Hence,

$$\begin{aligned}
\chi(G) &\leq |A| + \chi([C]) \\
&\leq \omega + \frac{\omega(\omega-1)(\omega+4)}{6} \\
&= \frac{\omega(\omega+1)(\omega+2)}{6}
\end{aligned}$$

□

Theorem 2 *If a graph G is $(P_4 \cup P_2)$-free, then $\chi(G) \leq \frac{\omega(\omega+1)(\omega+2)}{6}$.*

Proof The bound for the chromatic number of $(P_3 \cup P_2)$-free graphs holds for $(P_4 \cup P_2)$-free graphs too. In this case, each $[C_{ij}]$ is P_4-free and hence perfect, by a result of

Seinsche [18]. So, we can properly colour each $[C_{ij}]$ with at most $\omega(C_{ij}) \leq \omega-(j-2)$ colours, and the entire G with at most $\frac{\omega(\omega+1)(\omega+2)}{6}$ colours , as in the proof of Theorem 1. □

We next consider ($P_3 \cup P_2$, diamond)-free graphs and obtain sharper bounds for the chromatic number. If $\omega = 1$, then obviously the chromatic number is 1. So in the following, all graphs have $\omega \geq 2$.

Theorem 3 *If a graph G is* ($P_3 \cup P_2$, *diamond*)*-free, then*

$$\chi(G) \leq \begin{cases} 4 & \textit{if } w = 2 \\ 6 & \textit{if } \omega = 3 \\ 5 & \textit{if } \omega = 4 \end{cases}$$

and G is perfect if $\omega \geq 5$.

Proof We continue to use the terminology and notation of Sects. 2 and 3. In particular, we use the sets A, C_{ij}, I_a, and Claims 1, 2 and 3.

Claim 4 If G is C_5*-free, then it is a perfect graph.*

Clearly, every hole $C_{2k+1}, k \geq 3$ contains an induced $P_3 \cup P_2$, and the complement $\overline{C}_{2k+1}, k \geq 3$ of the hole contains an induced diamond. So G is $(C_{2k+1}, \overline{C}_{2k+1})$-free for all $k \geq 3$. Hence if G is C_5-free, then G is perfect, by the Strong Perfect Graph Theorem [8].

Claim 5 $C_{ij} = \emptyset$*, for every* $j \geq 4$.

On the contrary, let $x \in C_{ij}$, for some $j \geq 4$. Then by the definition of C_{ij}, there exist two distinct vertices $p, q \in \{1, 2, 3\} \subseteq A$ such that $x \leftrightarrow p$ and $x \leftrightarrow q$. But then $[\{x, j, p, q\}] \simeq$ diamond, a contradiction.

So, we conclude that $C = C_{12} \cup C_{13} \cup C_{23}$.

Claim 6 If $a \in A$*, then* I_a *is an empty set if* $\omega \geq 3$*, and it is an independent set if* $\omega = 2$.

If $\omega \geq 3$, and $x \in I_a$, for some $a \in A - \{1, 2\}$, then $[\{x, a, 1, 2\}] \simeq$ diamond, a contradiction; if a = 1 or 2, then [{x,1,2,3}] is a diamond. If $\omega = 2$, then the assertion follows by Claim 2.

Therefore, $V(G) = A \cup C_{12} \cup C_{13} \cup C_{23}$, if $\omega \geq 3$.

Recall that by Claim 3, $\omega([C_{13}]) \leq \omega - 1$, and $\omega([C_{23}]) \leq \omega - 1$. But $[C_{12}]$ may contain an ω-clique. However, we have the following claim.

Claim 7 If $\omega(G) \geq 3$ *and* $C_{23} \cup C_{13} \neq \emptyset$*, then* $\omega([C_{12}]) \leq \omega - 1$.

On the contrary suppose $[C_{12}]$ contains an ω-clique Q, and for definiteness suppose $C_{23} \neq \emptyset$ (if $C_{13} \neq \emptyset$, proof is similar). Let $x \in C_{23}$. If x is adjacent to all the vertices of Q or $|Q| - 1$ vertices of Q, then we have an $(\omega + 1)$-clique or a diamond in G, both impossible. Else, there exist two vertices u and v in Q such that $x \not\leftrightarrow u$ and $x \not\leftrightarrow v$. Then $[\{x, 1, 2\} \cup \{u, v\}] \simeq P_3 \cup P_2$, a contradiction. Hence the claim.

Claim 8 $[C_{13},\ A-\{2\}]=\emptyset$, *and* $[C_{23}, A-\{1\}]=\emptyset$.

If there exists an edge $xi\ \in\ [C_{13},\ A-\{2\}]$, then $[\{x,i,1,2\}]\simeq$ diamond, a contradiction. Similarly, $[C_{23},\ A-\{1\}]=\emptyset$.

Claim 9 A vertex of C_{12} *is adjacent to at most one vertex of* A.

The claim is obvious for $\omega=2,3$. Next, assume that $\omega\geq 4$. If some vertex $x\in C_{12}$ is adjacent to two distinct vertices say, i and j of $A-\{1,2\}$, then $[\{1,x,i,j\}]\simeq$ diamond, a contradiction.

We now prove the theorem for different values of ω, by making the cases as stated in the theorem.

- $\omega=2$; so $A=\{1,2\}$.

Colouring G with four colours is easy in this case, since $V(G)=A\cup C_{12}\cup I_1\cup I_2$, $\omega([C_{12}])\leq\omega=2$, and I_1, I_2 are independent sets, by Claim 6. Moreover, $\omega[C_{12}]\leq\omega(G)=2$. The following is a proper 4-colouring of G:

(1) Colour the vertices 1 and 2 of A with colours 1 and 2 respectively.
(2) Colour $[I_1]$ with colour 1.
(3) Colour $[I_2]$ with colour 2.
(4) Colour $[C_{12}]$ with two new colours.

An extremal ($P_3\cup P_2$, diamond)-free graph G with $\omega(G)=2$, and $\chi(G)=4$ is the Mycielski–Grötzsch graph; see Fig. 2. It is well known that this graph has clique number 2 and chromatic number 4. The graph is clearly diamond free since it is triangle free. It can be observed that this graph is $(P_3\cup P_2)$-free by selecting every edge P_2 and then verifying that the second neighborhood of P_2, is P_3-free. There are not too many cases for such a verification because of the symmetry of edges; we need to choose only three kinds of edges: v_1v_2, v_1u_2 and u_1w.

- $\omega=3$; so $A=\{1,2,3\}$.

At the outset, recall that every $I_a=\emptyset$, by Claim 6. So, $V(G)=A\cup C_{12}\cup C_{23}\cup C_{13}$. If $C_{23}\cup C_{13}=\emptyset$, then G is C_5-free since at most two vertices of C_5 can belong to the clique A and C_{12} is P_3-free. Therefore G is perfect by Claim 4. Hence by Claims 7 and 3, $\omega[C_{12}]\leq 2$, $\omega[C_{13}]\leq 2$, $\omega[C_{23}]\leq 2$. We colour G with six colours as follows:

Fig. 2 Mycielski–Grötzsch graph

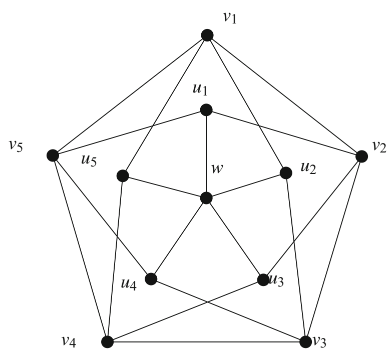
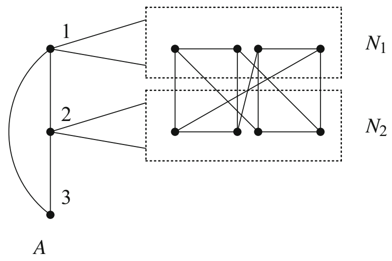


Fig. 3 ($P_3 \cup P_2$, diamond)-free graph with $\omega = 3$ and $\chi = 4$



(1) Colour the vertices 1, 2, 3 of A with colours 1, 2, 3 respectively.
(2) Colour $[C_{12}]$ with colours 1 and 2.
(3) Colour $[C_{23}]$ with colours 3 and 4.
(4) Colour $[C_{13}]$ with colours 5 and 6.

It is a proper colouring by the above observations.

Remarks (i) If some C_{ij} is empty, we may not require all the six colours.
(ii) We do not have extremal graphs with chromatic number 6.
(iii) However, we do have a graph with chromatic number 4 (see Fig. 3). In this figure, A is an ω-clique and $N_i \subseteq V(G)$ is such that every vertex of N_i is adjacent to i and only i of A, $i \in \{1, 2\}$.

- $\omega = 4$; so $A = \{1, 2, 3, 4\}$.

We colour G with five colours by considering two cases.
Case 1 $[C_{23}, C_{13}] \neq \emptyset$; let $ab \in [C_{23}, C_{13}]$.
Clearly, $[\{a, b, 2\}] \simeq P_3$.

Claim 10 *a is an isolated vertex in $[C_{23}]$, and b is an isolated vertex in $[C_{13}]$.*

Suppose, $a \leftrightarrow c$, for some $c \in C_{23}$. If $c \leftrightarrow b$, then $[\{a, b, c, 1\}] \simeq$ diamond, a contradiction. If $c \not\leftrightarrow b$, then $[\{a, b, c\} \cup \{3, 4\}] \simeq P_3 \cup P_2$, since no vertex of $C_{23} \cup C_{13}$ is adjacent to the vertex $4 \in A$, by Claim 8. Hence, we conclude that a is an isolated vertex in C_{23}. Similarly, b is an isolated vertex in C_{13}.

Claim 11 *C_{23} and C_{13} are independent sets.*

Suppose there exists an edge cd in $[C_{23}]$, where $c \neq a$ and $d \neq a$, by Claim 10. If $c \not\leftrightarrow b$ and $d \not\leftrightarrow b$, then $[\{a, b, 2\} \cup \{c, d\}] \simeq P_3 \cup P_2$. Next, without loss of generality, suppose that $c \leftrightarrow b$. Then $[\{a, b, c\} \cup \{3, 4\}] \simeq P_3 \cup P_2$, by Claim 8 and by the definition of C_{ij}'s, a contradiction. Hence, C_{23} is independent. Similarly C_{13} is independent.

We now colour G with five colours as follows:

(1) Colour the vertices 1, 2, 3, 4 of A with colours 1, 2, 3, 4 respectively.
(2) Colour $[C_{12}]$ with colours 1, 2 and a new colour 5.
(3) Colour $[C_{13}]$ with colour 3.
(4) Colour $[C_{23}]$ with colour 4.

It is a proper colouring by Claims 8, 7 and 11.
Case 2 $[C_{23}, C_{13}] = \emptyset$.

If both C_{23} and C_{13} are empty sets, then G is C_5-free, since $[C_{12}]$ is P_3-free and any induced 5-cycle contains at most two vertices of A. So, G is perfect, by Claim 4. If one of the sets C_{23} or C_{13} is nonempty, then we have the following assertion.

Claim 12 If C_{23} or C_{13} is non empty, then the other is independent.

Suppose $C_{23} \neq \emptyset$ and $x \in C_{23}$. If uv is an edge in $[C_{13}]$, then $[\{x, 1, 3\} \cup \{u, v\}] \simeq P_3 \cup P_2$, a contradiction. Hence C_{13} is independent. Similarly, C_{23} is independent if $C_{13} \neq \emptyset$.

Without loss of generality, we henceforth assume that $C_{23} \neq \emptyset$. Since C_{13} is nonempty or empty, we consider two subcases.
Subcase 2.1 C_{13} is nonempty.

This implies that both C_{23} and C_{13} are independent sets, by Claim 12.

(1) Colour the vertices 1, 2, 3, 4 of A with colours 1, 2, 3, 4 respectively.
(2) Colour $[C_{12}]$ with colours 1, 2 and a new colour 5.
(3) Colour $[C_{13}]$ with colour 3.
(4) Colour $[C_{23}]$ with colour 3.

It is a proper 5-colouring by Claims 7, 12 and the fact that $[C_{23}, C_{13}] = \emptyset$.
Subcase 2.2 C_{13} is empty.

We now examine this subcase based on whether C_{23} is an independent set or not.
Case 2.2.a C_{23} is an independent set. We colour G with five colours as follows:

(1) Colour the vertices 1, 2, 3, 4 of A with colours 1, 2, 3, 4 respectively.
(2) Colour $[C_{23}]$ with colour 3.
(3) Colour $[C_{12}]$ with colours 1, 2 and a new colour 5.

It is a proper 5-colouring by Claim 7 and by our assumptions.
Case 2.2.b C_{23} contains an edge. Let cd be an edge in $[C_{23}]$ (see Fig. 4). We claim that C_{12} is independent. Else, there is an edge ab in $[C_{12}]$. If c is neither adjacent to a nor adjacent to b, then $[\{c, 1, 2\} \cup \{a, b\}] \simeq P_3 \cup P_2$, a contradiction. Without loss of generality, assume that $a \leftrightarrow c$. But then $a \not\leftrightarrow d$; else, $[\{a, c, d, 1\}] \simeq$ diamond. By definition of C_{12} and C_{23}, no vertex in $\{a, c, d\}$ is adjacent to vertex 2 of A. By Claim 9, a is adjacent to at most one vertex of $A - \{1, 2\}$, namely 3 or 4. So $[\{a, c, d\} \cup \{2, 3\}] \simeq P_3 \cup P_2$ or $[\{a, c, d\} \cup \{2, 4\}] \simeq P_3 \cup P_2$, a contradiction. Thus, C_{12} is independent. Recall that $\omega([C_{23}]) \leq 3$ by Claim 3.

We colour G with four colours:

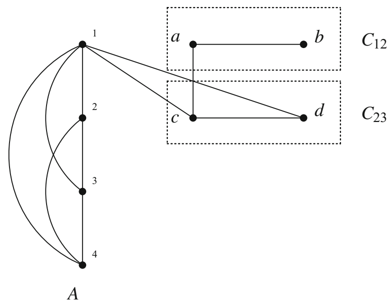
(1) Colour the vertices 1, 2, 3, 4 of A with colours 1, 2, 3, 4 respectively.
(2) Colour C_{23} with colours 2, 3 and 4.
(3) Colour C_{12} with colour 1.

It is a proper 4-colouring by Claims 3 and 8.

- $\omega \geq 5$.

It is enough to show that G is C_5-free, in view of Claim 4. On the contrary, suppose that G contains an induced C_5. As before, $V(G) - A = C = C_{12} \cup C_{13} \cup C_{23}$. Since

Fig. 4 $[C_{23}]$ contains an edge



at most two vertices of C_5 can belong to the clique A, a $P_3 = (a, b, c)$ is an induced subgraph of $[C]$. Since each C_{ij} is P_3-free, either (i) two vertices are in one C_{ij}, and the third vertex is in one of the other two C_{ij}'s, or (ii) each C_{ij} contains a vertex.

By Claim 9, for any two vertices $x, y \in C_{12}$, there is a vertex in $A - \{1, 2\}$, say 5, which is neither adjacent to x nor y. Also, by Claim 8, $[C_{13} \cup C_{23}, \{3, 4, 5\}] = \emptyset$. So, whether (i) or (ii) holds, there exists an edge ij in $[A]$ such that $[\{a, b, c\} \cup \{i, j\}] \simeq P_3 \cup P_2$, a contradiction. For the choice of an appropriate edge ij, it is enough if we consider the following four cases:

(a) If P_3 is an induced subgraph of $[\{C_{12} \cup C_{13}\}]$, then $[\{a, b, c, 1, 5\}] \simeq P_3 \cup P_2$.
(b) If P_3 is an induced subgraph of $[\{C_{12} \cup C_{23}\}]$, then $[\{a, b, c, 2, 5\}] \simeq P_3 \cup P_2$.
(c) If P_3 is an induced subgraph of $[\{C_{13} \cup C_{23}\}]$, then $[\{a, b, c, 4, 5\}] \simeq P_3 \cup P_2$.
(d) If (ii) holds, then $[\{a, b, c, 4, 5\}] \simeq P_3 \cup P_2$, where without loss of generality we assume that the vertex of (a, b, c) that is in C_{12} is non-adjacent to the vertices $4, 5 \in A$.

□

5 ($2K_2$, Diamond)-free graphs

The Claims of Sect. 4 are valid for ($2K_2$, diamond)-free graphs too. So we continue to use the Claims made in Sects. 3 and 4. In what follows, we assume that graphs have clique number at least 2, as before.

Theorem 4 *If a graph G is ($2K_2$, diamond)-free, then $\chi(G) \leq 3$ for $\omega = 2, 3$ and G is perfect if $\omega \geq 4$.*

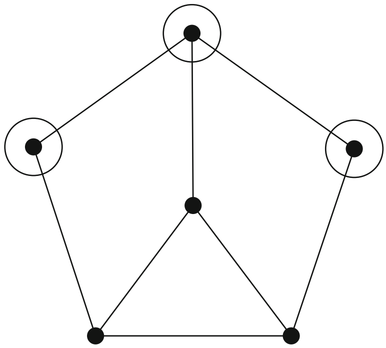
Proof Since the proof is similar to the proof of Theorem 3, we give an outline. As before, consider the partition $(A, \bigcup C_{ij}, \bigcup I_a)$ of $V(G)$. In this case, every C_{ij} is K_2-free, and so it is an independent set.

If $\omega = 2$, then $V(G) = A \cup C_{12} \cup I_1 \cup I_2$. So one can easily colour G with three colours.

Next suppose $\omega \geq 3$. We already know (by Claims 5 and 6) that $C_{ij} = \emptyset$ when $j \geq 4$ and that $I_j = \emptyset$ for all $j \in A$. Hence $V(G) = A \cup C_{12} \cup C_{13} \cup C_{23}$. An ω-colouring of G is obtained as follows:

(1) Colour the vertices $1, 2, \ldots, \omega$ of A, by colours $1, 2, \ldots, \omega$.

Fig. 5 Graphs that are not perfect and have $\chi(G) = \omega(G)$



(2) Colour every vertex of C_{12} with colour 1, colour every vertex of C_{13} with colour 3, colour every vertex of C_{23} with colour 2.

Remark There exist ($2K_2$, diamond)-free graphs with $\omega = 3$, which are not perfect. See Fig. 5, where each circled vertex is multiplied by an independent set.

Now we prove perfectness for $\omega \geq 4$.

It is similar to the proof of Theorem 3, Case $\omega = 5$. By Claim 4, it is enough if we show that G is C_5-free. On the contrary, if G contains an induced 5-cycle, then $C(= C_{12} \cup C_{13} \cup C_{23})$ contains an edge xy of the 5-cycle. Since C_{ij}'s are independent, no $[C_{ij}]$ contains xy. We use Claims 8 and 9 and arrive at a contradiction:

(a) If $xy \in [C_{12}, C_{13}]$, then $[\{x, y, 1, 3\}] \simeq 2K_2$ or $[\{x, y, 1, 4\}] \simeq 2K_2$.
(b) If $xy \in [C_{12}, C_{23}]$, then $[\{x, y, 2, 3\}] \simeq 2K_2$ or $[\{x, y, 2, 4\}] \simeq 2K_2$.
(c) If $xy \in [C_{13}, C_{23}]$, then $[\{x, y, 3, 4\}] \simeq 2K_2$.

So, G is C_5-free and hence it is perfect. □

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