NOTE ON THE TURÁN NUMBER OF THE LINEAR 3-GRAPH C_{13}

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ABSTRACT. Let the crown C_{13} be the linear 3-graph on 9 vertices $\{a,b,c,d,e,f,g,h,i\}$ with edges

$$E = \{\{a,b,c\},\{a,d,e\},\{b,f,g\},\{c,h,i\}\}.$$

Proving a conjecture of Gyárfás et. al., we show that for any crown-free linear 3-graph G on n vertices, its number of edges satisfy

$$|E(G)| \leq \frac{3(n-s)}{2}$$

where s is the number of vertices in G with degree at least 6. This result, combined with previous work, essentially completes the determination of linear Turán number for linear 3-graphs with at most 4 edges.

1. Introduction

A **linear 3-graph** $G = (V,E)$ consists of a finite set of vertices $V = V(G)$ and a collection $E = E(G)$ of 3-element subsets of V(edges), such that any two edges in E share at most one vertex. If H and F are linear 3-graphs, then H is F-free if it contains no copy of F. For a linear 3-graph F, and a positive integer n, the **linear Turán number** $\mathrm{ex}(n,F)$ is the maximum number of edges in any F-free linear 3-graph on n vertices.

Let the **crown** C_{13} be the linear 3-graph on 9 vertices $\{a,b,c,d,e,f,g,h,i\}$ with edges

$$E = \{\{a,b,c\},\{a,d,e\},\{b,f,g\},\{c,h,i\}\}.$$

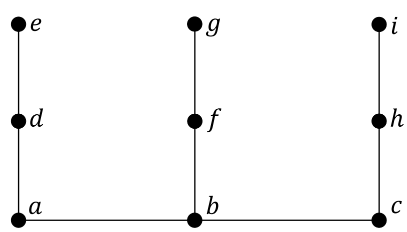

FIGURE 1. The crown C_{13}.

The study of $\mathrm{ex}(n,C_{13})$ was initiated by Gyárfás, Ruszinkó and Sárközy in [3], where they showed the bounds

$$6\lfloor\frac{n-3}{4}\rfloor + \epsilon \leq \mathrm{ex}(n,C_{13}) \leq 2n.$$

where $\epsilon = 0$ if $n-3 \equiv 0,1 \bmod 4$, $\epsilon = 1$ if $n-3 \equiv 2 \bmod 4$, and $\epsilon = 3$ if $n-3 \equiv 3 \bmod 4$. In [1], Gyárfás et. al. showed that every linear 3-graph with minimum degree 4 is not crown-free. They also proposed some

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ideas to obtain the exact bounds. Very recently, Fletcher showed in [2] the improved upper bound

$$\mathrm{ex}(n, C_{13}) < \frac{5}{3}n.$$

In this paper, we show that the lower bound is tight up to a constant, thus resolving a conjecture in [3]. In fact, we show the following stronger result.

Theorem 1.1. *Let G be any crown-free linear 3-graph G on n vertices. Then its number of edges satisfies*

$$|E(G)| \le \frac{3(n-s)}{2}.$$

where s is the number of vertices in G with degree at least 6.

Furthermore, we show that when s is small, the upper bound can be improved.

Theorem 1.2. *Let G be any crown-free linear 3-graph G on n vertices, and let s be the number of vertices in G with degree at least 6. If $s \le 2$, then the number of edges satisfies*

$$|E(G)| \le \frac{10(n-s)}{7}.$$

Combining the two theorems above, we immediately conclude that the lower bound in [3] is exact when $n \equiv 3 \bmod 4$ and $n \ge 63$.

Corollary 1.3. *If $n \ge 63$, then*

$$ex(n, C_{13}) \le \frac{3(n-3)}{2}.$$

The paper is structured as follows. In Section 2 and Section 3 we present the main innovative inequality and prove our main theorems, quotient a technical and familiar lemma that we prove in Section 4.

2. Proof of Theorem 1.1

Let G be any linear 3-graph. For each $v \in V(G)$, let $d(v)$ be the degree of v, which is the number of edges in $E(G)$ that contains v. For each edge $e \in E(G)$ and positive integers $a \ge b \ge c$, we write $D(e) \ge \langle a, b, c\rangle$ if we can write $e = \{x, y, z\}$ such that $d(x) \ge a$, $d(y) \ge b$ and $d(z) \ge c$.

Suppose the contrary. Let G be the smallest linear 3-graph such that G has greater than $3(n-s)/2$ edges. For each $v \in V(G)$, let $\chi(v) = 1$ if $d(v) \le 5$, and $\chi(v) = 0$ otherwise.

Our key innovation is the following observation

$$\sum_{e\in E(G)} \sum_{v\in V(G), v\in e} \frac{\chi(v)}{d(v)} = \sum_{v\in V(G)} \sum_{e\in E(G), v\in e} \frac{\chi(v)}{d(v)} = \sum_{v\in V(G)} \chi(v) = n - s.$$

As $|E(G)| > 3(n-s)/2$, we conclude that there exists an edge $e = \{x, y, z\}$ such that

$$\frac{\chi(x)}{d(x)} + \frac{\chi(y)}{d(y)} + \frac{\chi(z)}{d(z)} < \frac{2}{3}. \tag{1}$$

Without loss of generality, assume $d(x) \le d(y) \le d(z)$. First we note that $d(x) \ge 2$ and $d(y) \ge 4$, as otherwise (1) would be violated. If $d(z) \ge 6$, then we can easily find a C_{13} by choosing an edge $e_1 \ne e$ adjacent to x, choosing an edge e_2 adjacent to y that does not share a vertex with e_1, and finally choosing an edge e_3 adjacent to z that does not share a vertex with e_1 and e_2, contradiction. Therefore, we have $d(z) \le 5$, and (1) implies that $D(e) \ge \langle 5, 5, 4\rangle$.

We use the following lemma to handle the case $D(e) \geq \langle 5,5,4 \rangle$. As the lemma is quite straightforward using the techniques in [1], [2] and [3], we delay the lengthy proof to Section 4.

Lemma 2.1. *Let G be a crown-free graph and $e = \{x, y, z\} \in E(G)$ satisfy $D(e) \geq \langle 5,5,4 \rangle$. Then, the vertex set of all vertices sharing an edge with $\{x, y, z\}$,*

$$S = \bigcup_{f \in E(G), f \cap \{x,y,z\} \neq \emptyset} f,$$

contains exactly 11 vertices and all vertices in S have degree at most 5. The set of edges that contains at least one vertex in S,

$$E_S = \{f : f \in E(G), f \cap S \neq \emptyset\},$$

contains at most 13 edges, and all elements of E_S are subsets of S. In other words, the subgraph $G[S]$ is a connected component of G.

Let $G - S$ be the graph obtained by deleting the vertices S and the edges in E_S. By the lemma, the graph $G - S$ has $n' = n - 11$ vertices and at least $|E(G)| - 13$ edges. Furthermore, the number of vertices in $G - S$ of degree at least 6 is exactly s. Therefore, we conclude that

$$|E(G - S)| \geq |E(G)| - 13 > \frac{3(n - s)}{2} - 13 > \frac{3(n' - s)}{2}$$

contradicting the assumption that G is the smallest counterexample to Theorem 1.1. So we have shown Theorem 1.1.

3. Proof of Theorem 1.2

We use the same notations as Section 2.

Suppose the contrary. Let G be the smallest linear 3-graph such that G has at most 2 vertices with degree at least 6 and has greater than $10(n - s)/7$ edges.

For each $e \in E(G)$ and $v \in e$, we define a weight $\chi(v, e)$ as follows: let $\chi(v, e) = 1$ if $d(v) = 1, 2, 4, 5$, and $\chi(v, e) = 0$ if $d(v) \geq 6$. If $d(v) = 3$, let $\chi(v, e) = 1.05$ if there exists at least one vertex in e with degree at least 6, and $\chi(v, e) = 0.9$ otherwise.

Since $s \leq 2$, we have

$$\sum_{e \in E(G)} \sum_{v \in V(G), v \in e} \frac{\chi(v, e)}{d(v)} = \sum_{v \in V(G)} \sum_{e \in E(G), v \in e} \frac{\chi(v, e)}{d(v)} \leq n - s.$$

As $|E(G)| > 10(n - s)/7$, we conclude that there exists an edge $e = \{x, y, z\}$ such that

$$\frac{\chi(x, e)}{d(x)} + \frac{\chi(y, e)}{d(y)} + \frac{\chi(z, e)}{d(z)} < \frac{7}{10}. \tag{2}$$

Without loss of generality, assume $d(x) \leq d(y) \leq d(z)$. First we note that $d(x) \geq 2$, as otherwise (2) would be violated. Then note that if $d(y) \leq 3$, no matter $d(z)$ is greater than 6 or not (2) would also be violated, thus $d(y) \geq 4$.

The rest of the proof proceeds exactly the same as Section 2, other than the following inequality which leads to contradiction. Theorem 1.2 then follows.

$$|E(G - S)| \geq |E(G)| - 13 > \frac{10(n - s)}{7} - 13 > \frac{10(n' - s)}{7}.$$

4. Proof of Lemma 2.1

In this section we show our lemma on the case $D(e) \geq \langle 5,5,4\rangle$. Our proof follows similar techniques as in [1], [2] and [3]. In particular, [1] analyzed the case $D(e) \geq \langle 4,4,4\rangle$, [2] analyzed the case $D(e) \geq \langle 5,5,5\rangle$, and [3] analyzed the case $D(e) \geq \langle 5,5,3\rangle$. We use a slight variation of their methods to prove our lemma.

Without loss of generality, assume $d(y), d(z) \geq 5$ and $d(x) \geq 4$. As we must not have $D(e) \geq \langle 6,4,2\rangle$, we must have $d(y) = d(z) = 5$. For $p \in \{x,y,z\}$, let $G(p)$ be the set of all vertices distinct from x,y,z that lie on the same edge with p. We first note that we must have $G(y) = G(z)$. Suppose the contrary, and some edge $e_1 \neq e$ adjacent to y contain some vertex not in $G(z)$. Then at most one edge adjacent to z other than e contains a vertex in e_1, so at least three edges F adjacent to z are disjoint from e_1. Thus, we can take an edge e_2 containing x that is disjoint from e_1, then take an edge e_3 from F that is disjoint from e_2. So e, e_1, e_2, e_3 forms a C_{13}, contradiction.

Similarly, we must have $G(x) \subset G(y)$. Suppose the contrary, and some edge $e_1 \neq e$ adjacent to x contain some vertex not in $G(y)$. Then, we can take an edge e_3 containing z that is disjoint from e_1. Among the four edges adjacent to y distinct from e, at most two can intersect e_3, and at most one can intersect e_1. Thus, we can choose e_2 containing y that is disjoint from e_1 and e_3. So e, e_1, e_2, e_3 forms a C_{13}, contradiction.

Thus $S\backslash\{x,y,z\} = G(y) = G(z) \supset G(x)$. We define F as the set of all edges in $E(G)$ that contains one of the vertices in S, but is disjoint from $\{x,y,z\}$. It suffices to show that F must be empty.

We denote the vertices in $G(z)$ by a,b,c,d,r,s,p,q, such that $\{z,a,b\}, \{z,c,d\}, \{z,r,s\}, \{z,p,q\}$ are edges in $E(G)$.

StepI. We construct an auxiliary bipartite graph $H = (X_H, Y_H, E_H)$, where $X_H = \{e_i | y \in e_i\}, Y_H = \{e_j | z \in e_j\}, E_H = \{\{e_i, e_j\} | e_i \cap e_j \neq \emptyset\}$. H is a 2-regular bipartite graph with order 8. Thus, $H = C_8$ or $H = C_4 \biguplus C_4$.

We claim that if G contains no crown, H contains a $K_{2,2}$. Arbitrarily choose $e \in G(x)$. Define $V_1 = e \cap S \subset E_H, W_1 = \{e_i | e_i \cap V_1 \neq \emptyset\} \subset X_H \biguplus Y_H$, we have $|V_1| \leq 2, |W_1| \leq 4$, $|H - W_1| \geq 4$. To find a crown, we only need to choose $e_i \in X_G$, $e_j \in Y_G$ s.t. $\{e_i, e_j\} \notin E_{G-W_1}$. Therefore, if there is no crown in H, $H - W_1$ has to be a completed bipartite graph. Since $|G - W_1| \geq 4$ and two parts have the same order, there is definitely a $K_{2,2}$ in $H - W_1$. So H contains a $K_{2,2}$, furthermore, $H = C_4 \biguplus C_4$.

By symmetry we can assume $\{z,a,b\}, \{z,c,d\}$ are in a C_4 and $\{z,r,s\}, \{z,p,q\}$ are in the other one. Without loss of generality we can further assume $\{y,b,d\}$, $\{y,a,c\}$ lie in $E(G)$, and $\{y,s,q\}, \{y,r,p\}$ lie in $E(G)$.

StepII. Now let $V_1 = \{a,b,c,d\}$, $V_2 = \{r,s,p,q\}$, We have symmetry between V_1 and V_2, and symmetry inside $V_i, i = 1,2$ as well. We claim that there exists no edge containing x that contains exactly one vertex in V_1 and another one in V_2. Otherwise we can let it be $\{x,a,r\}$ by symmetry. Then $\{z,a,b\}, \{y,b,d\}, \{z,p,q\}, \{x,a,r\}$ form a C_{13}, contradiction. Thus the edges other than e containing x must be a subset of $\{\{x,a,d\}, \{x,b,c\}, \{s,r,q\}, \{x,s,p\}\}$.

StepIII. Let f be any element of F. By symmetry we can let $a \in f$ without loss of generality. Then we can see $b, c \notin f$. Firstly, we claim that f cannot contain exactly one element a of S. Otherwise $\{z,a,b\}, \{y,b,d\}, \{z,r,s\}, f$ form a C_{13}, contradiction. Secondly, we claim that $d \notin f$. Otherwise $G(x) = \{\{x,b,c\}, \{s,r,q\}, \{x,s,p\}\}$ since $d(x) \geq 4$. Since at most one edge of $\{z,r,s\}$ and $\{z,p,q\}$ intersect f, we can assume $\{z,r,s\} \cap f = \emptyset$. Then$\{z,a,b\}, \{x,b,c\}, \{z,r,s\}, f$ form a C_{13}, contradiction.

Therefore we can assume $r \in f$ by symmetry. Similarly we know that $q \notin f$ since a, d and r, q are symmetric. So f has exactly two elements a, r of S. While $\{z,a,b\}, \{x,b,c\}, \{z,p,q\}, f$ form a C_{13} in this case, contradiction.

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