The geometry and combinatorics of discrete line segment hypergraphs

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ABSTRACT

An r-segment hypergraph H is a hypergraph whose edges consist of r consecutive integer points on line segments in $\mathbb{R}^2$. In this paper, we bound the chromatic number $\chi(H)$ and covering number $\tau(H)$ of hypergraphs in this family, uncovering several interesting geometric properties in the process. We conjecture that for $r \geq 3$, the covering number $\tau(H)$ is at most $(r-1)\nu(H)$, where $\nu(H)$ denotes the matching number of H. We prove our conjecture in the case where $\nu(H) = 1$, and provide improved (in fact, optimal) bounds on $\tau(H)$ for $r \leq 5$. We also provide sharp bounds on the chromatic number $\chi(H)$ in terms of r, and use them to prove two fractional versions of our conjecture.

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1. Introduction

The combinatorial and geometrical properties of families of sets in Euclidean space has been a thriving area of research in discrete geometry and combinatorics for many decades. It was initiated by the classical theorem of Helly [12], asserting that if $\mathcal{F}$ is a family of convex sets in $\mathbb{R}^d$ in which every $d+1$ members have nonempty intersection, then some point lies in every set in $\mathcal{F}$. This landmark result led to the study of *covering numbers* (also sometimes called *piercing numbers* or *hitting numbers*) of families of sets, that is, the minimal number of points needed to "pierce" every set in a family given some local intersection property. Helly's theorem is sharp in general, but improved bounds on covering numbers can be obtained if one restricts the convex sets in question. Indeed, extensive work has been done on the covering numbers of families of disks, boxes, line segments, homothets of centrally symmetric bodies, and other convex sets, as well as in discrete settings, where the vertex set lies in the integer lattice (of some fixed semi-algebraic group). [4,6,7]. Determining covering numbers in discrete arrangements is often harder than in the continuous case, as the covering sets of points are required to lie in the lattice as well.

Hadwiger and Debrunner [10] studied the piercing numbers of families $\mathcal{F}$ satisfying the (p, q) *property* (i.e., among any p sets in H, there are q with a common intersection). When $\mathcal{F}$ consists of convex sets in $\mathbb{R}^d$, a fundamental theorem of Alon and Kleitman [3] asserts that there exists a constant $c = c(d, p, q)$ such that $\mathcal{F}$ can be covered by c points. Optimal upper bounds on the numbers $c(d, p, q)$ in the Alon–Kleitman theorem are not known, despite extensive work (for a survey see [8] and the references therein).

Covering problems have also been studied in the abstract setting of *hypergraphs*, that is, pairs $H = (V, E)$ of a *vertex set* V and an *edge set* $E \subset 2^V$. In this context, one generally wants to obtain an upper bound on the *covering number* $\tau(H)$

in terms of its *matching number* $\nu(H)$ (that is, the maximum number of pairwise disjoint edges in H). If H is an *r-uniform* hypergraph (that is, if every edge has exactly r vertices), then one such bound is $\tau(H) \leq r\nu(H)$, as the union of the edges in any maximal matching is a cover of H of size $r\nu(H)$ (we will refer to this bound as the *trivial bound*).

The trivial bound cannot be improved in general, as is exemplified by the disjoint union of copies of the projective plane of uniformity r. However, for certain families of r-uniform hypergraphs, the trivial bound can be improved. König's theorem [14], for instance, asserts that $\tau(H) = \nu(H)$ in the case where $r = 2$ and H is a bipartite graph. A longstanding conjecture of Ryser (see [13]) generalizes König's theorem and states that any r-partite hypergraph H satisfies $\tau(H) \leq (r-1)\nu(H)$. Ryser's conjecture was proved in the $r = 3$ case [1], and remains open for $r \geq 4$.

The trivial bound can also be improved for the family of *triangle hypergraphs*. Given a graph $G = (V, E)$ let $H = H(G)$ be the 3-uniform hypergraph with vertex set E so that a triple in E is a hyperedge of H if it forms a triangle in G. A famous conjecture of Tuza [19] asserts that for any graph G, $\tau(H) \leq 2\nu(H)$, and the best known upper bound is $\tau(H) \leq \frac{66}{23}\nu(H)$ [11]. For other families of r-uniform hypergraphs in which the trivial bound can be improved, see [2].

In this paper, we introduce a new family of r-uniform hypergraphs whose edges are comprised of consecutive integer points on a line. The covering number of such a hypergraph coincides with the notion of covering number in discrete versions of Helly-type theorems, as each edge is the intersection of a convex set (in this case, a line segment) with the integer lattice, and any covering set consists only of lattice points.

Definition 1.1. Let $r \geq 2$ be an integer. A hypergraph $H = (V, E)$ on vertex set $V \subset \mathbb{Z}^2$ is an *r-segment hypergraph* if

   (i) every edge in E consists of r consecutive integer points on some line in $\mathbb{R}^2$ and
   (ii) every line in $\mathbb{R}^2$ contains at most one edge of H.

This new family of hypergraphs has proven to be a fruitful source for challenging combinatorial questions, the answers to which have yielded surprising geometrical properties. Much of the work in this paper centers around the following conjectured bound on the covering numbers of r-segment hypergraphs.

Conjecture 1.2. *If H is an r-segment hypergraph with $r \geq 3$, then*

$$\tau(H) \leq (r-1)\nu(H).$$

Said another way, if $r \geq 3$ and H is an r-segment hypergraph satisfying the $(p, 2)$ property, then Conjecture 1.2 asserts $\tau(H) \leq (r-1)(p-1)$.

Our main results in this direction are as follows.

   (a) We prove Conjecture 1.2 in the case where H is *intersecting*, i.e. when $\nu(H) = 1$ (Theorem 2.3). Our proof is highly geometric, using areas of bounded regions to argue that any such hypergraph must have a vertex that lies in only one edge.

   (b) In Section 3, we provide tight upper bounds on $\tau(H)$ in the special case where $r \leq 5$ and H is intersecting. In particular, we show that Conjecture 1.2 is tight when $r = 3$ or $r = 4$, and if $r = 5$ then $\tau(H) \leq 3$. It follows from our proof that any intersecting 5-segment hypergraph that contains a triangle has at most 6 edges.

   (c) In Theorem 4.2 we prove sharp bounds on the *chromatic numbers* $\chi(H)$ of r-segment hypergraphs H (that is, the minimal number of colors needed to color the vertices so that no edge is monochromatic). In particular, we show that if $r = 2$, then $\chi(H) \leq 4$, if $r = 3$ then $\chi(H) \leq 3$ and if $r \geq 4$ then $\chi(H) = 2$.

   (d) Finally, we use our bounds on the chromatic numbers to prove in Theorem 5.2 two fractional relaxations of Conjecture 1.2. Specifically, we show that for any r-segment hypergraph H the ratios $\tau(H)/\nu^*(H)$ and $\tau^*(H)/\nu(H)$ do not exceed $r - 1$, where $\nu^*(H)$ and $\tau^*(H)$ are the fractional matching and covering numbers, respectively (see the beginning of Section 5 for formal definitions).

2. Covering intersecting *r*-segment hypergraphs

The main result of this section is Theorem 2.3, in which we prove Conjecture 1.2 for intersecting r-segment hypergraphs. First, we give two examples demonstrating that no hypothesis in Conjecture 1.2 can be dropped.

Example 2.1. Conjecture 1.2 does not hold for $r = 2$, as exemplified by the complete 2-segment hypergraph R on the vertex set $V(R) = \{(0, 0), (1, 0), (0, 1)\}$, which has $\nu = 1$ and $\tau = 2$.

Example 2.2. Without the "consecutive" assumption in Definition 1.1, Conjecture 1.2 would fail to hold. Consider, for instance, the hypergraph S with vertex set

$$V(S) = \{(-2, 0), (0, 0), (2, 0), (0, 2), (-1, 3), (1, 3), (0, 6)\}$$

depicted in Fig. 2a. This configuration resembles the classical non-Fano plane and has $\nu(H) = 1$ but $\tau(H) = 3$.

In what follows, an *isolated point* in a hypergraph is a vertex lying in only one edge.
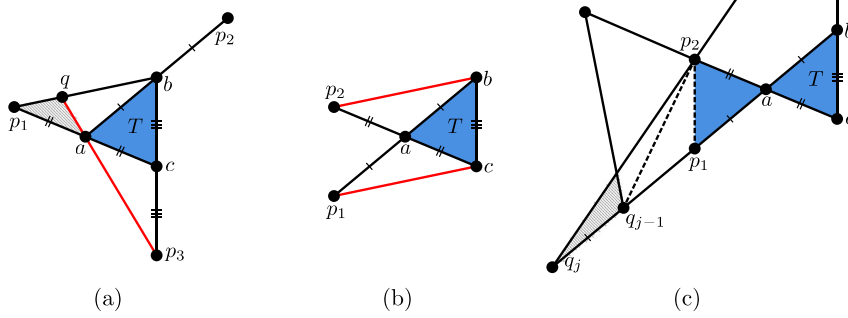
Fig. 1. Configurations for the proof of Theorem 2.3.

Theorem 2.3. *Any intersecting r-segment hypergraph H with $r \geq 3$ contains an isolated vertex. In particular, $\tau(H) \leq r - 1$.*

Proof. Assume for contradiction that H contains no isolated vertices. If all of the edges in H intersect at a vertex, then the theorem is trivial. Otherwise, there are three edges $e_1, e_2, e_3 \in E(H)$ in H whose corresponding line segments form a triangle $T = \{a, b, c\}$ with $a = e_1 \cap e_2$, $b = e_2 \cap e_3$ and $c = e_3 \cap e_1$. Suppose further that T has minimal area among all triangles formed by edges in H. The minimality of the area of T implies that T contains no other integer points in its boundary, and since $r \geq 3$, each edge e_i bounding T contains at least one more vertex.

First, suppose that each of the vertices a, b, and c is an endpoint of one of the edges e_1, e_2, and e_3, and let p_i denote the next sequential vertex along each edge e_i, as depicted in Fig. 1a. Since H contains no isolated vertices, some other edge e_{p_1} must contain p_1, and since any two edges intersect, e_{p_1} must intersect both e_2 and e_3 at b. Similarly, some other edge e_{p_3} contains both p_3 and a, and must also intersect e_{p_1} at some integer point q between p_1 and b. Now, observe that T and the triangle $\{a, p_1, b\}$ have the same area, meaning that the triangle $\{a, p_1, q\}$ has smaller area than T, which is a contradiction.

In all the remaining cases, one of the vertices a, b, and c is not an endpoint of any of the edges e_1, e_2, and e_3. Without loss of generality, assume this vertex is a. Let p_1 and p_2 denote the vertices adjacent to a on e_1 and e_2, respectively, as in Fig. 1b. As before, since H has no isolated vertices, some other edges e_{p_1} and e_{p_2} must contain p_1 and p_2, respectively. Since H is intersecting and the line $\overline{p_1 p_2}$ is parallel to e_3, the edges e_{p_1} and e_{p_2} must be distinct. Moreover, e_{p_1} and e_{p_2} cannot intersect b and c, respectively, as this would make e_{p_1} and e_{p_2} parallel (see Fig. 1b). Thus we may assume that e_{p_2} intersects e_3 at a point past vertex b, as depicted in Fig. 1c.

If e_{p_2} intersects e_3 at the vertex adjacent to b, then e_{p_2} is parallel to e_1. Therefore, e_{p_2} must intersect e_1 at a vertex closer to p_1 than to a. Labeling the vertices on e_1 as $q_0 = p_1, q_1, q_2, \ldots$ according to their distance from a, assume that e_{p_2} is the edge containing p_2 that contains q_j for $j \geq 1$ minimal. Since q_{j-1} is not isolated, it must be contained in some additional edge $e_{q_{j-1}}$ that, by the minimality of j, does not contain p_2. However, the triangle $\{p_2, q_{j-1}, q_j\}$ has the same area as T, and $e_{q_{j-1}}$ must divide this triangle in order to intersect e_{p_2}, contradicting the minimality of T. □

Example 2.4. Theorem 2.3 need not hold for non-intersecting r-segment hypergraphs in general. For example, the 3-hypergraph in Fig. 7b has no isolated vertices.

3. A refined bound for intersecting *r*-segment hypergraphs

Our main goal in this section is to investigate the following refinement of Conjecture 1.2 for intersecting r-segment hypergraphs, which closely resembles the bound in the generalized Tuza conjecture for certain linear hypergraphs (Remark 3.2).

Conjecture 3.1. *If H is an intersecting r-segment hypergraph and $r \geq 5$, then*

$$\tau(H) \leq \left\lceil \frac{r}{2} \right\rceil.$$

We prove two results in support of Conjecture 3.1: (i) we show that the bound in Conjecture 3.1, if true, is sharp (Theorem 3.4), and (ii) we prove that Conjecture 3.1 holds in the case $r = 5$ (Theorem 3.7), confirming a strict departure from the bound in Conjecture 1.2. Note that Conjecture 3.1 fails to hold for $r < 5$, as Conjecture 1.2 is tight in this case. Indeed, adding an isolated vertex to every edge in the 2-segment hypergraph R in Example 2.1 yields a 3-segment hypergraph with $\tau = 2$, and Example 3.3 demonstrates sharpness for $r = 4$.

Remark 3.2. The upper bound in Conjecture 3.1 almost matches the bound conjectured by Aharoni and the third author [2] for a family of hypergraphs $H^{(r-1)}$ (see [2] for formal definitions) generalizing Tuza's conjecture [19]. We believe
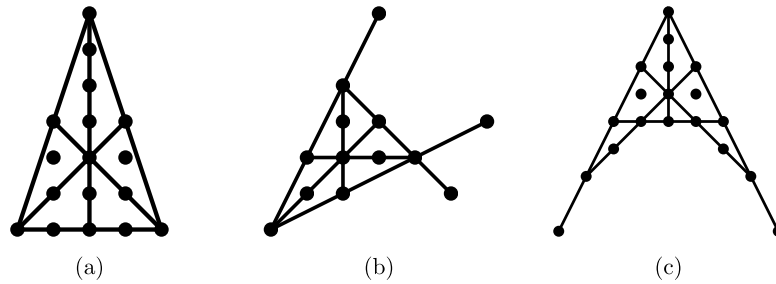
Fig. 2. The configuration lacking the "sequential" property from Example 2.2 (left), the intersecting 4-segment hypergraph with $\tau = 3$ from Example 3.3 (middle), and an intersecting 5-segment hypergraph with 6 edges (right).

this is no coincidence, since both hypergraphs share a similar "strong-linearity" property: given any two vertices u, v in $H^{(r-1)}$, there is at most one possible r-tuple of elements of $V(H^{(r-1)})$ containing both u and v that can serve as an edge; similarly, in an r-segment hypergraph H, any edge e is fully determined by the pair of vertices u, v at the two ends of the segment corresponding to e, in that there is at most one possible r-tuple in $\mathbb{Z}^2$ containing both u and v that can serve as an edge in any r-segment hypergraph.

Example 3.3. The hypergraph H depicted in Fig. 2b is an intersecting 4-segment hypergraph with covering number 3. To see this, note that there are 6 edges in H, and any two vertices of degree 3 share an edge. As such, any two vertices in H cover at most 5 edges.

Note that this hypergraph is the 3-uniform projective plane minus one edge, with an added isolated vertex in every edge. This construction appears to be special to $r = 4$, in part because of the increased difficulty of obtaining straight-line embeddings of the r-uniform projective plane for large r. The authors were unable to obtain a proof, however, as the proof method used in Theorem 3.7 includes too many cases to reasonably check by hand when $r > 5$.

Theorem 3.4. *For every integer $r \geq 5$, there exists an intersecting r-segment hypergraph with covering number $\lceil \frac{r}{2} \rceil$.*

Proof. Consider the vectors

$$\vec{v} = (1, r - 1) \text{ and } \vec{u} = (-1, r - 1).$$

Construct an r-segment hypergraph H by starting with a segment e from $(0, 0)$ to $(r - 1)\vec{v}$ and a segment f from $(0, 0)$ to $(r - 1)\vec{u}$, and for each $i = 1, \ldots, r - 2$, include a segment e_i from $i\vec{v}$ to $(r - 1 - i)\vec{u}$ (connecting e to f). We must show the following.

(i) Each segment has exactly r integer points. Indeed, e and f each have exactly r integer points (one at each integral x-coordinate), and each edge e_i has slope

$$m_i = \frac{i(r - 1) - (r - 1 - i)(r - 1)}{i - (r - 1 - i)(-1)} = 2i - r + 1 \in \mathbb{Z},$$

meaning each of the r points on e_i with integral x-coordinate is an integer point.

(ii) Any two segments intersect at an integer point. Indeed, segments e and f intersect at the origin, and the endpoints of each edge e_i intersect e and f, respectively. Moreover, two distinct edges e_i and e_j intersect at

$$x = i + j + 1 - r,$$

since the corresponding y-coordinate is

$$
\begin{aligned}
m_i(x - i) + i(r - 1) &= (2i - r + 1)(i + j + 1 - r - i) + i(r - 1) \\
&= 2ij - (r - 1)(i + j) + (r - 1)^2 \\
&= (2j - r + 1)(i + j + 1 - r - j) + j(r - 1) \\
&= m_j(x - j) + j(r - 1).
\end{aligned}
$$

(iii) No 3 segments intersect at the same integer point. This holds since, on a given edge, each intersection point identified in part (ii) has a distinct x-coordinate.

From these facts, we conclude $\tau(H) = \lceil \frac{1}{2}|E(H)| \rceil = \lceil \frac{r}{2} \rceil$. □

For the proof of Theorem 3.7, we first prove Lemma 3.5. This geometric result was also used by the authors to systematically verify Conjecture 3.1 for fixed values of $r \geq 6$ in only finitely many cases.

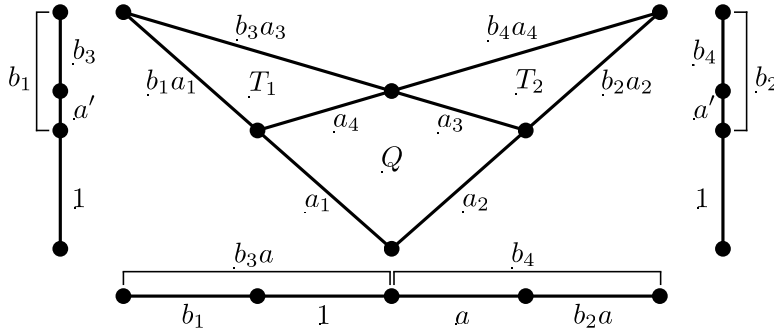**Fig. 3.** The configuration from Lemma 3.5, together with each projection used in the proof.

Lemma 3.5. *Fix a configuration of four pairwise intersecting line segments in $\mathbb{R}^2$ in which no three intersect at the same point. These segments necessarily bound a quadrilateral Q and two triangles T_1 and T_2, as in Fig. 3. Denote the lengths of the sides of the quadrilateral Q by $a_1, \ldots, a_4$, and the lengths of the remaining sides of T_1 and T_2 by $b_1a_1, \ldots, b_4a_4$, as shown in Fig. 3 (here, we assume only that each labeled length is positive). The ratios $b_1, \ldots, b_4$ must satisfy the relations*

$$b_2b_3 = b_1(b_2 + 1) \quad \text{and} \quad b_1b_4 = (b_1 + 1)b_2.$$

In particular, specifying rational values for any two of the b_i's uniquely determines rational values for the other two.

Proof. Let $A_1, \ldots, A_4, B_1, \ldots, B_4$ denote the given segments in the lemma of lengths $a_1, \ldots, a_4, a_1b_1, \ldots, a_4b_4$, respectively. Note the ratios $b_1, \ldots, b_4$ are preserved under rotation and scaling of the entire configuration, as well as under projection onto a line.

First, consider the projection P along the diagonal of Q connecting the points $A_1 \cap A_2$ and $A_3 \cap A_4$ (the "vertical" diagonal in Fig. 3). Scale the original configuration so that $P(A_1)$ (which coincides with $P(A_4)$) has unit length, and let a be the length of $P(A_2) = P(A_3)$. Comparing projected lengths yields

$$b_3a = b_1 + 1 \quad \text{and} \quad (b_2 + 1)a = b_4$$

which can be solved to obtain

$$(b_1 + 1)(b_2 + 1) = b_3b_4. \tag{3.1}$$

Next, consider the projection P' along the diagonal of Q connecting the points $A_1 \cap A_4$ and $A_2 \cap A_3$ (the "horizontal" diagonal in Fig. 3). Scaling the original configuration so that $P'(A_1) = P'(A_2)$ has unit length and writing a' for the length of $P'(A_3) = P'(A_4)$, we obtain

$$(b_3 + 1)a' = b_1 \quad \text{and} \quad (b_4 + 1)a' = b_2,$$

yielding

$$b_1(b_4 + 1) = b_2(b_3 + 1). \tag{3.2}$$

Solving (3.1) for b_4 and substituting into (3.2) yields

$$b_2b_3^2 + (b_2 - b_1)b_3 + b_1(b_1 + 1)(b_2 + 1) = 0,$$

implying

$$b_3 = \frac{(b_1 - b_2) \pm (b_1 + b_2 + 2b_1b_2)}{2b_2}.$$

Disregarding the negative solution yields $b_3 = (b_2 + 1)b_1/b_2$. Substituting back into (3.1) and (3.2) yields the desired equations. □

Example 3.6. Suppose $r = 5$, and consider the configuration in Fig. 4a. Using the labeling in Lemma 3.5, $b_1 = b_2 = 2$, which forces $b_3 = b_4 = 3$. As such, the additional integer points depicted in Fig. 4b must also appear in the configuration. On the other hand, the configuration in Fig. 4c has $b_1 = 1$ and $b_2 = 2$, so by Lemma 3.5, we have $b_4 = 4$, which is impossible if $r < 6$. As such, this configuration cannot occur in any 5-segment hypergraph.

We are now ready to prove Theorem 3.7.

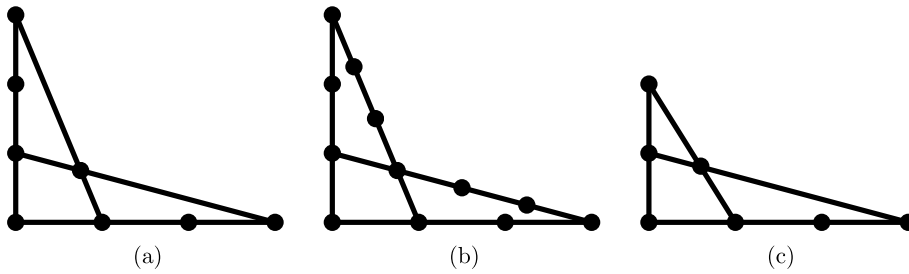**Theorem 3.7.** *If H is an intersecting 5-segment hypergraph, then $\tau(H) \leq 3$.*

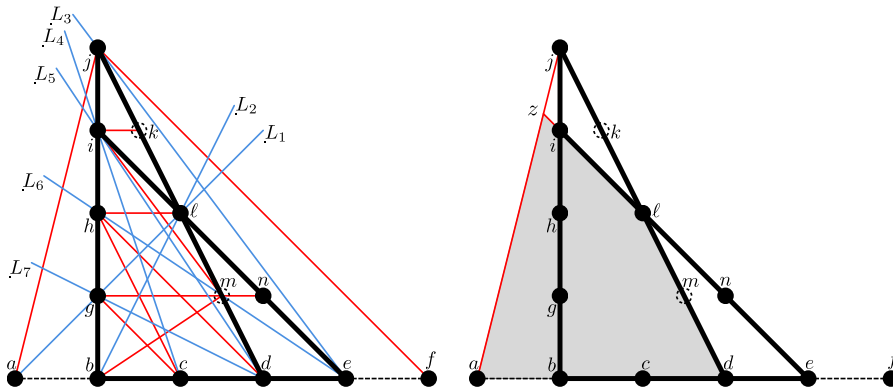**Fig. 4.** Configurations in Example 3.6.

Fig. 5. The thick lines (colored black) are the edges in Q, with ratios $(b_1, b_2, b_3, b_4) = (\frac{1}{3}, \frac{1}{2}, 1, 2)$. The thin lines in the left hand image labeled $L_1, \ldots, L_7$ (colored blue) may contain additional edges of H, while the remaining thin lines (colored red) cannot contain any edges of H. Applying Lemma 3.5 to the shaded quadrilateral in the right hand image implies that $\overline{aj}$ cannot contain any edges in H. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Proof. Let $H = (V, E)$ be an intersecting 5-segment hypergraph. We may assume that $|E| \geq 7$, for otherwise the claim $\tau(H) \leq 3$ is trivial. If all of the edges in H intersect in a single vertex, then $\tau(H) = 1$. Otherwise, choose two edges $e_1, e_2 \in E$, and let e_3 be an edge not intersecting $e_1 \cap e_2$. If the three vertices of the triangle formed by e_1, e_2, e_3 form a cover of H, then again $\tau(H) \leq 3$. Otherwise, there is an edge e_4 not containing any of these vertices, so $Q = \{e_1, e_2, e_3, e_4\}$ is a set of four edges satisfying the conditions of Lemma 3.5.

Let $b_1, \ldots, b_4$ be the ratios defined in Lemma 3.5 for Q. Since $r = 5$, each b_i must lie in the set $\{\frac{1}{3}, \frac{1}{2}, 1, 2, 3\}$. Since the values of b_1 and b_2 uniquely determine the values of b_3 and b_4 by Lemma 3.5, trying each possible pair (b_1, b_2) yields

$$(1, 1, 2, 2), \quad (2, 2, 3, 3), \quad (\tfrac{1}{2}, 1, 1, 3), \quad \text{and} \quad (\tfrac{1}{3}, \tfrac{1}{2}, 1, 2)$$

as the only possible tuples (b_1, b_2, b_3, b_4) up to symmetry. We can now divide the proof into four cases, one for each possibility. We include here only the case where $(b_1, b_2, b_3, b_4) = (\frac{1}{3}, \frac{1}{2}, 1, 2)$, as similar arguments yield the remaining three cases.

The thick black edges depicted in Fig. 5 correspond to the edges in Q, with ratios $(b_1, b_2, b_3, b_4) = (\frac{1}{3}, \frac{1}{2}, 1, 2)$. We first demonstrate that any additional edges in H must lie in one of the lines labeled $L_1, \ldots, L_7$ in Fig. 5 (i.e. those colored blue). The lines $\overline{gn}$, $\overline{hl}$ and $\overline{ik}$ are parallel to $\overline{be}$ and therefore, since H is intersecting, they cannot contain an edge. Similarly, the lines $\overline{hd}$, $\overline{gc}$ and $\overline{jf}$ are parallel to $\overline{ie}$, and $\overline{hc}$ is parallel to $\overline{jd}$, and thus cannot contain an edge. Applying Lemma 3.5 to the shaded quadrilateral in Fig. 5 implies the length ratio of $\overline{jz}$ to $\overline{az}$ is $\frac{1}{4}$, which is impossible if $\overline{aj}$ has only 5 consecutive integer points. As such, we conclude the line $\overline{aj}$ cannot contain an edge. By similar arguments, the lines $\overline{im}$ and $\overline{bm}$ cannot contain an edge since they do not intersect $\overline{be}$ and $\overline{ie}$, respectively, at integer points.

Let $E' = E \setminus Q$. By the above argument, $|E'| \leq 7$ and each edge in E' must lie in a distinct labeled line $L_1, \ldots, L_7$ in Fig. 5. We now consider which pairs of lines can simultaneously contain edges. For instance, if L_1 and L_3 both contained edges, then applying Lemma 3.5 to the quadrilateral formed with $\overline{de}$ and $\overline{e\ell}$ implies the edge in L_1 contains more than 5 integer points. Similarly, if L_1 and L_5 both contained edges, then applying Lemma 3.5 to the quadrilateral formed with $\overline{de}$ and $\overline{e\ell}$ implies L_1 contains more than 5 integer points. Continuing in this way, the only pairs of lines that can contain two distinct edges is E' are the following:

$$\{L_1, L_2\}, \quad \{L_1, L_4\}, \quad \{L_1, L_7\}, \quad \{L_2, L_6\}, \quad \{L_3, L_5\},$$
$$\{L_3, L_6\}, \quad \{L_3, L_7\}, \quad \{L_4, L_5\}, \quad \{L_5, L_7\}, \quad \{L_6, L_7\}.$$

From this, we must have $|E'| \leq 3$, as the maximal sets of lines that could simultaneously contain edges in E' are $\{L_3, L_5, L_7\}$ or $\{L_3, L_6, L_7\}$. However, both triples require L_3 to contain more than 5 integer points by Lemma 3.5. We conclude that H contains at most 6 edges, which completes the proof of this case. $\square$

Remark 3.8. One consequence of the proof of Theorem 3.7 is that any intersecting 5-segment hypergraph has at most 6 edges if it contains a triangle. Indeed, in any such configuration, the degree of each vertex v is at most r since every edge through v must intersect an edge not containing v at a different vertex. As such, any configuration with more than $r + 1$ edges contains a configuration of the form in Lemma 3.5, and the proof of Theorem 3.7 verifies that no more than 2 edges can be added in each case. One configuration achieving the maximum 6 edges is illustrated in Fig. 2c.

Question 3.9. What is the maximum number of edges that can occur in an intersecting r-segment hypergraph containing a triangle?

4. Coloring r-segment hypergraphs

A hypergraph H is k-*colorable* for a positive integer k if one can assign one of k colors to each vertex in such a way that no edge in H is monochromatic, and the *chromatic number* of H, denoted $\chi(H)$, is the least number k such that H is k-colorable.

Coloring problems in geometric settings have a long and vibrant history. Particularly noteworthy examples include the 4-color theorem for planar graphs and the question of how to color every point in $\mathbb{R}^2$ so that no two points that are unit distance apart are colored by the same color [5]. Chromatic numbers of certain families of r-uniform hypergraphs are also of interest, such as the family of Kneser hypergraphs $KG^r(n, k)$, whose vertices are the k-element subsets of $\{1, \ldots, n\}$ and whose edges are collections of r pairwise disjoint subsets [17].

In this section, we give an optimal upper bound on the chromatic numbers of r-segment hypergraphs in terms of r (Theorem 4.2). Our proof uses geometric projections to explicitly construct proper colorings (Proposition 4.1). We also demonstrate that the bounds in Theorem 4.2 are sharp (Examples 4.3 and 4.4).

Like many other hypergraph families in the literature, r-segment hypergraphs are *linear*, meaning any two edges intersect in at most one vertex. It is worth noting that in general, the problem of deciding whether an r-uniform hypergraph H is k-colorable is NP-hard, even in the case where H is linear.

For each integer $k \geq 2$, let $\mathbb{Z}_k$ denote the additive group $\mathbb{Z}/k\mathbb{Z} = \{\overline{0}, \overline{1}, \ldots, \overline{k-1}\}$. In what follows, denote by Z_k the k-uniform hypergraph with vertex set $V(Z_k) = \mathbb{Z}_k^2$ and edge set

$$E(Z_k) = \{\overline{u} + \mathbb{Z}_k \overline{v} : u, v \in \mathbb{Z}_k^2, \ |\overline{u} + \mathbb{Z}_k \overline{v}| = k\},$$

where $\overline{u} = (\overline{u}_1, \overline{u}_2) \in \mathbb{Z}_k^2$.

Proposition 4.1. *Fix integers $r \geq k \geq 2$ and an r-segment hypergraph H. The image of any edge of H under the projection*

$$\begin{aligned} P_k : \mathbb{Z}^2 &\longrightarrow \mathbb{Z}_k^2 \\ (v_1, v_2) &\longmapsto (\overline{v}_1, \overline{v}_2) \end{aligned}$$

is an edge in Z_k. In particular, $\chi(H) \leq \chi(Z_k)$.

Proof. Fix an edge $e \in E(H)$ with endpoint $u \in \mathbb{Z}^2$ and vector $v = (v_1, v_2) \in \mathbb{Z}^2$ pointing from u to the next integer point in e. Since $e = u + \{0, 1, \ldots, r-1\}v$, the image of e under P_k is $\overline{u} + \mathbb{Z}_k \overline{v}$, so to complete the proof, we claim the images of any k sequential vertices in e are distinct vertices of Z_k, or, equivalently, that no nonzero element of $\mathbb{Z}_k$ annihilates $\overline{v}$. The fact that u and $u + v$ are two consecutive integer points on a line in $\mathbb{R}^2$ implies $\gcd(v_1, v_2) = 1$, so the claim follows from the fact that $\operatorname{ann}(\overline{v}) = \operatorname{ann}(\overline{v}_1) \cap \operatorname{ann}(\overline{v}_2)$ has nonzero elements only if $\gcd(v_1, v_2) > 1$. $\square$

We are now ready to prove our main theorem on colorations.

Theorem 4.2. *Let H be an r-segment hypergraph.*

(a) *If $r = 2$ then $\chi(H) \leq 4$.*
(b) *If $r = 3$ then $\chi(H) \leq 3$.*
(c) *If $r \geq 4$ then $\chi(H) = 2$.*

Additionally, the inequalities in parts (a) and (b) are sharp.

Proof of Theorem 4.2. By Proposition 4.1, it suffices to show $\chi(Z_2) \leq 4$, $\chi(Z_3) \leq 3$ and $\chi(Z_4) \leq 2$. The first inequality is clear since Z_2 is a complete graph on 4 vertices, and the other two follow from the colorings given in Figs. 6a and 6b, respectively (one can manually verify that no monochromatic edges are present). The remaining claims about the sharpness of parts (a) and (b) of the theorem follow from Examples 4.3 and 4.4, respectively. $\square$
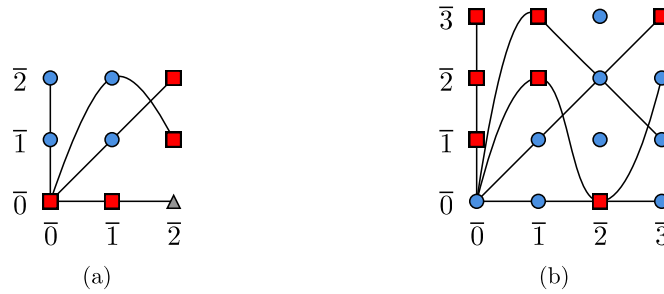
Fig. 6. Colorings of Z_3 (left) and Z_4 (right) for the proof of Theorem 4.2. In each, every edge through the point $(\bar{0}, \bar{0})$ is also depicted.
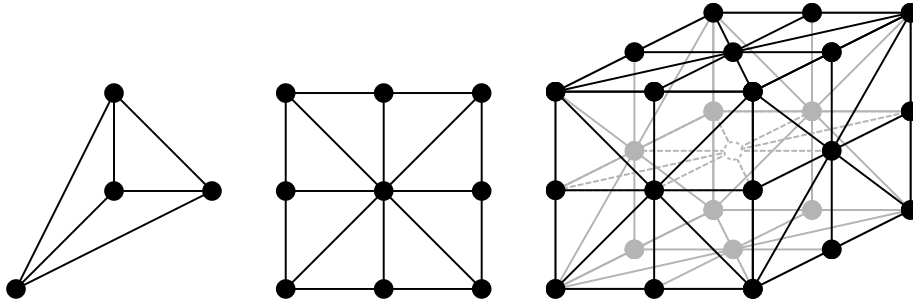

Fig. 7. Hypergraphs from Examples 4.3 and 4.4.

Example 4.3. The 2-segment hypergraph H with vertex set

$$V(H) = \{(0, 0), (1, 1), (1, 2), (2, 1)\}$$

depicted in Fig. 7a is the complete graph on 4 vertices, and thus has $\chi(H) = 4$.

Example 4.4. The following 3-segment hypergraph H has $\chi(H) = 3$. We begin by constructing the hypergraph C depicted in Fig. 7c with vertex set $\{0, 1, 2\}^3$. Start with the 1-skeleton of the cube with vertex set $\{0, 2\}^3$. Next, add edges so that each side of the cube forms a square with 3 horizontal edges, 3 vertical edges, and 2 diagonal edges (see Fig. 7b). Finally, include edges so that the center layer (i.e. $z = 1$) matches Fig. 7b as well. This yields 40 edges: 8 within each layer parallel to the xy-plane, 8 "vertical" edges parallel to the z-axis, and 2 diagonal edges within each vertical side. One can check via computation that C is not 2-colorable (a brute-force implementation in Python takes approximately 1 min).

Next, let H denote the image of C under the projection map

$$
\begin{array}{ccc}
\mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 \\
(x, y, z) & \longmapsto & (x + 3z, y + 5z).
\end{array}
$$

The lower left corners of the squares comprising the bottom ($z = 0$), middle ($z = 1$), and top ($z = 2$) layers of C are $(0, 0)$, $(3, 5)$, and $(6, 10)$, respectively. It suffices to check that the image of each "vertical" edge in H' still only contains 3 integer points, but the vector v in the direction of any such edge from the bottom layer ($z = 0$) to the middle layer ($z = 1$) either has x-coordinate 3 or y-coordinate 5, ensuring $\gcd(v_1, v_2) = 1$. As such, H is a 3-segment hypergraph with chromatic number 3.

One immediate consequence of Theorem 4.2 is a bound on $\tau(H)$ in terms of $V(H)$ for any r-segment hypergraph H.

Corollary 4.5. *Let* $H = (V, E)$ *be an r-segment hypergraph.*

(a) *If $r = 2$, then $\tau(H) \le \frac{3}{4}|V|$.*

(b) *If $r = 3$, then $\tau(H) \le \frac{2}{3}|V|$.*

(c) *For every $r \ge 4$, $\tau(H) \le \frac{1}{2}|V|$.*

Proof. Suppose H is k-colorable. For any color c, the set of all vertices not colored by c is a cover. By the pigeon hole principle, some color must label at least $1/k$ of the vertices of H, so $\tau(H) \leq \frac{k-1}{k}|V|$. The result now follows from Theorem 4.2. □

Remark 4.6. One surprising consequence of Theorem 4.2 is that some graphs are not realizable as 2-segment hypergraphs. For example, K_5 (the complete graph on 5 vertices) is the smallest complete graph that cannot be realized as a 2-segment hypergraph, and is also the smallest non-planar complete graph. On the other hand, $K_{3,3}$ (the complete bipartite graph with 3 vertices of each color) is realizable as a 2-segment hypergraph (in fact, so is any bipartite graph).

Schnyder proved that every planar graph can be embedded in the plane so that every vertex is an integer point and every edge is a straight line segment [18]. This, together with Remark 4.6, prompts the following.

Problem 4.7. Which graphs are realizable as 2-segment hypergraphs?

5. Fractional covers and matchings in r-segment hypergraphs

Let $H = (V, E)$ be a hypergraph. A function $f : E \to \mathbb{R}_{\geq 0}$ is a *fractional matching* if $\sum_{e \ni v} f(e) \leq 1$ for every $v \in V$. We denote by

$$\nu^*(H) = \max\left\{\sum_{e \in E} f(e) : f \text{ is a fractional matching of } H\right\}$$

the *fractional matching number* of H. Similarly, a *fractional cover* is a function $g : V \to \mathbb{R}_{\geq 0}$ such that $\sum_{v \in e} g(v) \geq 1$ for every $e \in E$. We denote by

$$\tau^*(H) = \min\left\{\sum_{v \in V} g(v) : g \text{ is a fractional cover of } H\right\}$$

the *fractional covering number* of H. In every hypergraph H, we have

$$\nu(H) \leq \nu^*(H) = \tau^*(H) \leq \tau(H),$$

where the equality $\nu^*(H) = \tau^*(H)$ follows from linear programming duality. Moreover, the *complementary slackness condition* asserts that if some minimum fractional cover $g : V \to \mathbb{R}_{\geq 0}$ satisfies $g(u) > 0$ for some $u \in V$, then every maximum fractional matching $f : E \to \mathbb{R}_{\geq 0}$ satisfies $\sum_{e \ni u} f(e) = 1$.

Bounding the fractional ratios τ/ν^* and τ^*/ν is often easier than the integral one. In the context of Ryser's conjecture, Lovász proved [16] that $\tau(H) \leq \frac{r}{2}\nu^*(H)$, and the bound $\tau^*(H) \leq (r-1)\nu(H)$ follows from a theorem of Füredi:

Theorem 5.1 ([9])**.** *If every edge of a hypergraph H has at most r vertices with $r \geq 3$, and H does not contain a copy of the r-uniform projective plane, then $\nu(H) \geq \frac{\nu^*(H)}{r-1}$.*

For Tuza's conjecture, Krivelevich [15] showed both $\tau(H(G)) \leq 2\nu^*(H(G))$ and $\tau^*(H(G)) \leq 2\nu(H(G))$ for every G. The latter bounds follow from Theorem 5.1 as well.

Here we prove that both fractional bounds are true when H is an r-segment hypergraph with $r \geq 3$. As in the case of r-partite hypergraphs and triangle hypergraphs of graphs, the proof of the bound on τ^*/ν follows from Theorem 5.1.

Theorem 5.2. *If H is an r-segment hypergraph with $r \geq 3$, then $\tau^*(H) \leq (r-1)\nu(H)$ and $\tau(H) \leq (r-1)\nu^*(H)$.*

Proof. By Theorem 5.1, for the first statement of the theorem it is enough to show that H does not contain a copy of the r-uniform projective plane. Suppose to the contrary that H contains a subhypergraph H' isomorphic to the r-uniform projective plane. Since H' is an intersecting r-segment hypergraph, Theorem 2.3 implies H' contains an isolated vertex, contradicting the fact that each vertex in the r-uniform projective plane belongs to r edges.

Suppose $H = (V, E)$ has the minimal number of edges such that

$$\tau(H) > (r-1)\nu^*(H) = (r-1)\tau^*(H).$$

Let $f : E \to \mathbb{R}_{\geq 0}$ and $g : V \to \mathbb{R}_{\geq 0}$ be a maximal fractional matching and a minimal fractional cover of H, respectively. By removing vertices from V if necessary, we may assume that every vertex in V belongs to some edge.

Suppose first that some vertex $u \in V$ has $g(u) = 0$. Given an edge e containing u, we have $\sum_{e \ni v} g(v) \geq 1$ since g is the fractional cover, but $g(u) = 0$, so there must be a vertex $v \in e$ with $g(v) \geq \frac{1}{r-1}$. Consider the hypergraph H' obtained from H by removing v and all the edges containing it.

Clearly $\tau(H') \geq \tau(H) - 1$, since otherwise a minimal cover in H' together with v is a cover of H of size smaller than $\tau(H)$. Moreover, by the minimality of H, we have $\tau(H') \leq (r-1)\tau^*(H')$, and since the restriction of g to $V(H')$ is a fractional cover of H', we have $\tau^*(H') \leq \tau^*(H) - g(v)$. Thus,

$$\tau^*(H') \leq \tau^*(H) - g(v) < \frac{\tau(H)}{r-1} - \frac{1}{r-1} = \frac{\tau(H)-1}{r-1} \leq \frac{\tau(H')}{r-1} \leq \tau^*(H'),$$

which is a contradiction.

We may now assume $g(u) > 0$ for all $u \in V$ (if the second inequality does not hold for some $u \in V$ then we can repeat the argument above by removing u from H). By the complementary slackness conditions, each vertex u satisfies $\sum_{e \ni u} f(e) = 1$, so

$$|V| = \sum_{u \in V} 1 = \sum_{u \in V} \sum_{e \ni u} f(e) = \sum_{e \in E} rf(e) = r\nu^*(H) = r\tau^*(H).$$

We conclude $\tau^*(H) = \frac{1}{r}|V|$, but since $\tau(H) \leq \frac{2}{3}|V|$ by Corollary 4.5, we have

$$\frac{\tau(H)}{\tau^*(H)} \leq \frac{\frac{2}{3}|V|}{\frac{1}{r}|V|} = \frac{2r}{3} \leq r - 1,$$

for every $r \geq 3$. This concludes the proof of the theorem. $\square$

Declaration of competing interest

Acknowledgments

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