

Graphs with at most one crossing

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Abstract

The crossing number of a graph G is the least number of crossings over all possible drawings of G . We present a structural characterization of graphs with crossing number one.

1 Introduction

For a graph G , we let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. Our graphs may have multiple edges but, for simplicity, we assume they have no loops.

We assume the reader is familiar with the concept of drawings of graphs in the plane or sphere. We make no distinction between the elements of a graph and their representations in the drawing. Let D be a drawing of a graph G . If H is a subgraph of G , we use the notation $D[H]$ to denote the drawing of H obtained from D by deleting the corresponding vertices and edges that are not in H .

Let $\text{cr}(D)$ denote the number of crossings of D . The *crossing number* $\text{cr}(G)$ of G is the least number of crossings over all possible drawings of G . A drawing D of G is *optimal* if $\text{cr}(D) = \text{cr}(G)$. We note that a graph G is planar if and only if $\text{cr}(G) = 0$. An optimal drawing of a planar graph is an *embedding*.

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In the context of studying crossing numbers, we can interpret Kuratowski's classic characterization of planar graphs as: a graph has crossing number at least one if and only if it contains a subdivision of K_5 or $K_{3,3}$ (we shall refer to the subdivisions of $K_{3,3}$ and K_5 as *Kuratowski graphs*). Following the same spirit, we answer the question of when does a graph have crossing number at least 2?

We answer with Theorem 4, a characterization of the crossing pairs of a graph. A pair $\{e, f\}$ of edges in a graph G is a *crossing pair* of G if there exists a drawing D of G with $\text{cr}(D) = 1$ (we refer to D as a *1-drawing* of G) in which e and f cross. Clearly, for a nonplanar G , $\text{cr}(G) = 1$ if and only if G has a crossing pair and $\text{cr}(G) \geq 2$ otherwise. This characterization is an extension of a result of Arroyo and Richter [2] (Theorem 1, below).

In Section 2, we present some related work. Section 3 expands on some properties of crossing pairs and details our characterization in Theorem 4. In Section 4, we describe the results and notation used for proving Theorem 4; the proof is in Section 5. The last section contains some remarks on Theorem 4.

2 Related work

The problem of characterizing graphs with crossing number at least two was already studied by Arroyo and Richter [2] in the context of *peripherally 4-connected graphs*.

A graph G is *peripherally 4-connected* if G is 3-connected and, for every vertex 3-cut X of G , and, for any partition of the components of $G - X$ into two non-null subgraphs H and K , at least one of H or K has just one vertex. Two edges $e = x_1y_1$ and $f = x_2y_2$ are *linked* if either e, f are incident with a common vertex or there is a 3-cut X in G such that $X \subset \{x_1, y_1, x_2, y_2\}$ and the vertex in $\{x_1, y_1, x_2, y_2\} \setminus X$ induces a trivial component of $G - X$. Otherwise, e and f are *unlinked*. Two edges e and f of G are *separated by cycles* if there exists two (vertex-)disjoint cycles C_e and C_f in G with $e \in E(C_e)$ and $f \in E(C_f)$.

Theorem 1. [2] *A peripherally 4-connected nonplanar graph G has crossing number at least two if and only if every pair of unlinked edges $\{e, f\}$ in G is separated by cycles.*

We note that if we drop the connectivity requirement, then the converse of Theorem 1 is no longer true. Consider $G = K_{3,4}$. It is 3-connected but the cut consisting of the part of size 3 shows that it is not peripherally 4-connected. We know that $\text{cr}(K_{3,4}) = 2$ [19]; however, no pair of disjoint edges is separated by cycles, as we need at least 8 vertices for 2 disjoint cycles.

There is also some work on a characterization of line graphs that have crossing number one. The line graph $L(G)$ of a graph G is a graph with vertex set $E(G)$ and $a, b \in E(G)$ are adjacent in $L(G)$ if and only if a, b share a common vertex in G . Let $\Delta(G)$ denote the maximum degree of a graph G .

Since the edges incident with a vertex of degree d in G induce a complete graph K_d in $L(G)$, a vertex of degree 6 in G implies $\text{cr}(L(G)) \geq \text{cr}(K_6) = 3$.

Similarly, if G has two vertices with degree 5, then $\text{cr}(L(G)) \geq 2$. Therefore, if $\text{cr}(L(G)) \leq 1$, then G has at most one vertex of degree 5 and $\Delta(L(G)) < 8$.

Kulli, Akka and Beineke [8] characterized planar graphs whose line graph has crossing number one, while Jendrol' and Klevusč [6] completed the characterization by including nonplanar graphs:

Theorem 2. [8] *For every planar graph G , we have $\text{cr}(L(G)) = 1$ if and only if either:*

- (1) $\Delta(G) = 4$ and there is a unique non-cut-vertex of degree 4, or
- (2) $\Delta(G) = 5$, every vertex of degree 4 is a cut vertex, and there is a unique vertex of degree 5 with at most 3 edges in any block.

Theorem 3. [6] *For a nonplanar graph G , we have $\text{cr}(L(G)) = 1$ if and only if the following conditions hold:*

- (1) $\text{cr}(G) = 1$,
- (2) $\Delta(G) \leq 4$, and every vertex of degree 4 is a cut vertex of G , and
- (3) there exists a drawing of G in the plane with exactly one crossing in which each crossed edge is incident with a vertex of degree 2.

Akka, Jendrol', Klešč, and Panshetty [1] obtained a characterization of planar graphs whose line graph has crossing number two.

A graph G is *k-crossing-critical* if $\text{cr}(G) \geq k$ and every proper subgraph H of G has $\text{cr}(H) < k$. The 1-crossing-critical graphs are exactly the Kuratowski graphs. We note that a graph with crossing number at least 2 contains a 2-crossing-critical graph as a subgraph.

A great deal of attention has been given to 2-crossing-critical graphs [3, 4, 5, 7, 9, 10, 16]. For a positive integer $n \geq 3$, the *Möbius Ladder* V_{2n} on $2n$ vertices is the graph obtained from a $2n$ -cycle by joining vertices with distance n in the cycle. Bokal, Oporowski, Richter and Salazar [4] characterized all 2-crossing-critical graphs that: are not 3-connected; are 3-connected and have V_{10} as a minor; or are 3-connected and do not have V_8 as a minor. They also showed that there exists only finitely many 3-connected 2-crossing-critical graphs with no V_{10} minor.

It remains to characterize or enumerate all the 3-connected 2-crossing-critical graphs that have V_8 but not V_{10} as a minor. We hope this work can help determine these remaining 2-crossing-critical graphs.

3 Crossing pairs and statement of the main result

The main point of this section is to introduce our main result Theorem 4. The lead-up to its statement is an analysis of crossing pairs.

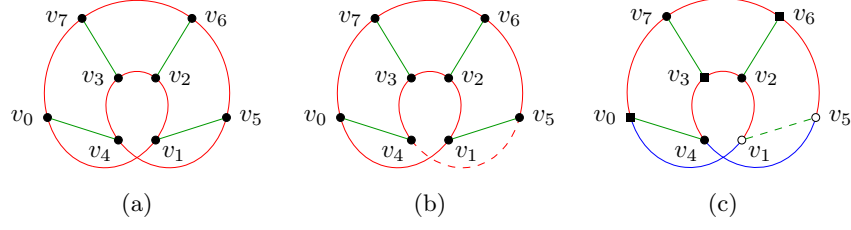


Figure 1: Removing v_4v_5 will result in a planar graph, however removing the edge v_1v_5 will result in a subdivision of $K_{3,3}$. The squares and disks represents the parts of the subdivision of $K_{3,3}$.

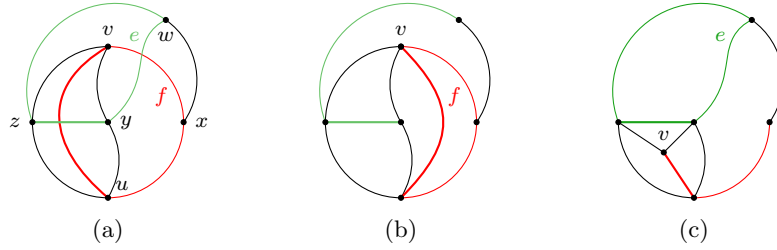


Figure 2: The pair of edges e, f is separated by the highlighted cycles, however removing either will result in a planar graph.

The graph in Figure 1a is V_8 . The figure shows a 1-drawing of the V_8 where v_0v_1 and v_4v_5 are crossing. Removing either v_0v_1 or v_4v_5 results in a planar graph (Figure 1b). In contrast, as shown in Figure 1c, removing v_1v_5 results in a subdivision of $K_{3,3}$. Since any drawing of $V_8 - v_1v_5$ contains a crossing, v_1v_5 is not in a crossing pair.

There are three obvious facts about a crossing pair $\{e, f\}$ in a graph G . First, as illustrated above, if D is a 1-drawing of G in which e and f cross, then $D[G - e]$ and $D[G - f]$ are planar embeddings of $G - e$ and $G - f$, respectively.

Second, if H is a Kuratowski subgraph of G , then e and f are both in H and make the unique crossing of $D[H]$. That is, $\{e, f\}$ is a crossing pair in H . This is readily seen to be equivalent to the assertion that e and f are not in either the same branch or adjacent branches of H . (A *branch* in H is a path P joining two vertices with degrees different from 2 such that all internal vertices of P have degree 2 in H .)

Third, since vertex-disjoint cycles cross an even number of times in a drawing, e and f are not separated by cycles in G (separated by cycles is defined just before Theorem 1). Figure 2 shows edges e and f that are separated by cycles such that $G - e$ and $G - f$ are both planar.

The second fact is easily seen to imply the first: If, for every Kuratowski subgraph H of G , $\{e, f\}$ is a crossing pair of H , then $G - e$ and $G - f$ are

both planar. However, the first fact does not imply the second: if e and f are in branches of a Kuratowski graph H sharing a vertex, both $H - e$ and $H - f$ are planar, but $\{e, f\}$ is not a crossing pair of H . In Theorem 4 we describe conditions under which the first and second facts are equivalent.

We are now ready to state our main result, which characterizes crossing pairs in nonplanar graphs in terms of these three facts.

Theorem 4. *Let G be a nonplanar graph and let $e, f \in E(G)$. The following are equivalent:*

- (i) $\{e, f\}$ is a crossing pair of G ;
- (ii) $\{e, f\}$ is a crossing pair of every Kuratowski subgraph of G and e, f are not separated by cycles in G ; and
- (iii) e, f are not separated by cycles in G , both $G - e$ and $G - f$ are planar and there exists a Kuratowski subgraph H of G such that $\{e, f\}$ is a crossing pair in H .

Theorem 4 gives an answer to our original question: what makes a graph have crossing number at least 2? Let G be a non-planar graph and let H be a Kuratowski subgraph of G . Then $\text{cr}(G) \geq 2$ if and only if, for every crossing pair $\{e, f\}$ of H , e, f are separated by cycles in G or at least one of $G - e$ and $G - f$ is not planar.

The following example from Širáň [15] shows that the two conditions in Theorem 4 (ii) are independent. Let $(\{u, v, w\}, \{x, y, z\})$ be a bipartition of $K_{3,3}$ and let G be the graph $K_{3,3} + \{uv, yz\}$ (see Figure 2a). As Širáň notes, there is only one Kuratowski graph contained in G . The edges ux and wz are separated by cycles in G . However, $\{ux, wz\}$ is a crossing pair of every Kuratowski subgraph of G . On the other hand, $\{uy, wz\}$ is also a crossing pair of every Kuratowski subgraph of G , but uy and wz are not separated by cycles in G .

The proof of Theorem 4 is in Section 5. This is preceded by some preparatory work in the next section.

4 Preliminaries

In this section, we present some definitions and results used in the proof of Theorem 4.

Given a subgraph H of G , a path P in G is H -avoiding or avoids H if no edge or internal vertex of P is in H . For $u, v \in V(G)$, a uv -path in G is a path whose ends are u and v .

Let G be a graph and let H be a subgraph of G . An H -bridge B of G is a subgraph of G consisting of either a single edge of $E(G) \setminus E(H)$ with both ends in H , or a component F of $G - V(H)$ together with the edges of G with one end in F and another in H . The vertices of $\text{Att}(B) = V(H) \cap V(B)$ are the *attachments* of B , and the *nucleus* $\text{Nuc}(B)$ is $B - \text{Att}(B)$. Although, the

definitions of attachment and nucleus depend on G and on the subgraph H , we omit them, since these are always clear from the context.

Let C be a cycle of G . Two distinct C -bridges B_1 and B_2 *overlap* if they have exactly three attachments in common or if there exist distinct vertices a, x, b, y occurring in this cyclic order in C such that $a, b \in \text{Att}(B_1)$ and $x, y \in \text{Att}(B_2)$. The following simple observations are useful to us.

Observation 5. *Let C be a cycle in a graph G with overlapping C -bridges B_1 and B_2 .*

- (i) *If D is a planar embedding of G , then B_1 and B_2 are embedded in distinct faces of $D[C]$.*
- (ii) *If u_1 and u_2 are in the nuclei of B_1 and B_2 , respectively, then $(C \cup B_1 \cup B_2) + u_1u_2$ is not planar. \square*

For vertices x and y of a graph G , a cycle $C \subseteq G$ *detaches* x from y if there exists two overlapping C -bridges, each containing exactly one of x and y in its nucleus.

The vertices x and y are *cofacial* in an embedding D if x and y are incident with a common face of D . The following theorem by Tutte (and its slight modification in Corollary 7) is an important tool in the proof of Theorem 4.

Theorem 6. *(Tutte [18]) Let G be a planar graph and let $x, y \in V(G)$. Then G has an embedding such that x and y are cofacial unless G contains a cycle C that detaches x from y . \square*

We need a slight modification of Tutte's result that holds for a vertex x and an edge f . A cycle C *detaches* x from f if there exist overlapping C -bridges with one containing x and its nucleus and the other containing f . A vertex x and an edge f are *cofacial* in an embedding D if x and f are incident with a common face of D .

Corollary 7. *Let G be a planar graph, let $x \in V(G)$ and let $f \in E(G)$. Then G has an embedding such that x and f are incident with a common face unless G contains a cycle C that detaches x from f . \square*

5 Proof of Theorem 4

In this section we give the proof of Theorem 4. The proof that (i) implies (ii) follows from the second and third facts on the discussion preceding the statement of Theorem 4. To see that (ii) implies (iii), by (ii), $G - e$ and $G - f$ both have no Kuratowski subgraphs, so both are planar. Moreover, G is not planar, so G has a Kuratowski subgraph H . By (ii), $\{e, f\}$ is a crossing pair in H and also e and f are not separated by cycles in G . The rest of this section is devoted to (iii) implies (i).

For a drawing D of a graph K , a *side* is the closure of a face of D .

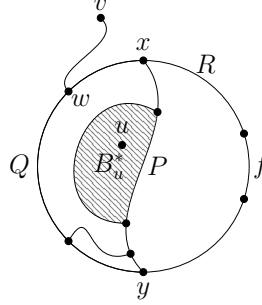


Figure 3: An abstract representation of the elements introduced for the proof of Theorem 4.

Proof. The hypotheses of (iii) are that e and f are not separated by cycles in G , $G - e$ and $G - f$ are planar, and there is a Kuratowski subgraph H in which $\{e, f\}$ is a crossing pair. Let u and v be the ends of e .

Let D_H be a 1-drawing of H in which e and f cross. The embedding $D_H[H - e]$ shows that $H - e$ is planar. Since H is not planar, no embedding of $H - e$ can have both u and v incident with the same face. By Theorem 6, there exists a cycle C of $H - e$, together with distinct overlapping C -bridges B_u^H and B_v^H containing u and v in their nuclei, respectively.

Since B_u^H and B_v^H overlap, they must be embedded in distinct sides of $D_H[C]$ in $D_H[H - e]$. Therefore e crosses at least one edge of C in D_H , and, since f is the only edge that crosses e in D_H , f must be in C , while neither u nor v is in C .

Let D be a planar embedding of $G - e$ and let B_u and B_v be the C -bridges in $G - e$ containing u and v , respectively. If $B_u = B_v$, then there is a C -avoiding uv -path P in B_u . Then $P + e$ and C are disjoint cycles that show e and f are separated by cycles in G , a contradiction.

Thus, $B_u \neq B_v$. Since $B_u^H \subseteq B_u$ and $B_v^H \subseteq B_v$, B_u and B_v are overlapping C -bridges in G . In particular, they are in different sides of C in D .

Let G_u and G_v be the subgraphs of G embedded in the side of C containing u and v in D , respectively. Our goal is to prove there exist planar embeddings of G_u and G_v such that, in both cases, C bounds a face and, in G_u , u and f are cofacial, while in G_v , v and f are cofacial. This obviously implies that G has a 1-drawing in which e and f are the crossing pair. We prove the result for G_u and the result for G_v follows by symmetry.

We refer the reader to Figure 3 for a visual aid in what follows. Let F be the face of $D[C \cup B_u]$ incident with f but not bounded by C . There is a unique cycle C_f in $C \cup B_u$ that contains f and is contained in the boundary of F . To see that C_f exists, the cycle C is one cycle that has F on one side and contains f . Choose C_f to be (the) one that has a minimal subgraph of G on the side containing F . This choice implies that every C_f -bridge embedded on the side containing F has at most one attachment.

Let R be the component of $C \cap C_f$ including f . As B_v overlaps B_u on C , B_u has at least two attachments on C . Thus, R is the subpath of C that contains f and joins two consecutive attachments of B_u . Let x and y be the two ends of R , let P be the other xy -path in C_f , and let Q be the other xy -subpath of C .

We note that every $(C \cup P)$ -bridge in $C \cup B_u$ is a $(P \cup Q)$ -bridge and, if a $(C \cup P)$ -bridge B in $C \cup B_u$ is contained in the face of $C \cup P$ bounded by $P \cup R$, then $\text{Att}(B)$ is a single vertex in P .

To show that G_u has an embedding in which u and f are cofacial and C bounds a face, we shall show that $C \cup B_u$ has such an embedding. If u is in P , then D is already such an embedding, so we assume u is not in P . Recall that, since C detaches u and v in $G - e$, u is not in C .

Therefore, there is a $(C \cup P)$ -bridge B_u^* in $C \cup B_u$ containing u . Because x and y are consecutive attachments of B_u on C and B_v overlaps B_u on C , B_v has an attachment w in $Q - \{x, y\}$. Thus,

(*) there is a C -avoiding vw -path P_v in B_v .

Suppose B_u^* has an attachment z in $Q - \{x, y\}$; let P^* be a $(C \cup P)$ -avoiding uz -path in B_u^* . Letting $Q[w, z]$ denote the wz -subpath of Q , $(P_v \cup Q[w, z] \cup P^*) + e$ and $P \cup R$ are disjoint cycles in G containing e and f , respectively. This contradicts the assumption that e and f are not separated by cycles. Thus,

(**) $\text{Att}(B_u^*)$ is contained in P .

To show that $C \cup B_u$ has an embedding in which u and f are cofacial and C bounds a face, we first show that $C \cup P \cup B_u^*$ has such an embedding. For the sake of contradiction, suppose that there is no such embedding of $C \cup P \cup B_u^*$.

By Corollary 7, there is a cycle C' in $C \cup P \cup B_u^*$ that detaches u and f . Let B'_u and B'_f be the C' -bridges containing u and f , respectively. Note that $B'_u \subseteq B_u^*$ and B'_u and B'_f are overlapping C' -bridges.

To see that $Q \subseteq C'$, suppose $C' \subseteq B_u^* \cup P$. Let z be any attachment of B'_f . Then there is a C' -avoiding path P'_z in $B'_f - f$ joining an end of f to z . Choose the labelling of x and y so that x is in P'_z .

As we saw in (*) above, there is a C -avoiding vw -path P_v in B_v . Extend this path to x using the wx -subpath of Q , and finally extend to z using the xz -subpath of P'_z . The result is a C' -avoiding vz -path in $G - \{e, f\}$.

Consider the graph K consisting of $(C - f) \cup B_u^* \cup B_v$. We have shown that the C' -bridge B'_v in K that contains B_v has all the attachments of B'_f as attachments. Since B'_f overlaps B'_u , it follows that B'_v overlaps B'_u .

Therefore, Observation 5 shows that $K + e$ is not planar. However, $K + e$ is contained in $G - f$, contradicting the hypothesis that $G - f$ is planar. Thus, $Q \subseteq C'$, so C' consists of Q and an xy -path contained in $B_u^* \cup P$.

It follows that $B'_f = R$ and, therefore, B'_u has an attachment in each component of $C' - \{x, y\}$; in particular, we have the contradiction that B'_u , and therefore B_u^* , has an attachment in $Q - \{x, y\}$, contradicting (**).

Consequently, u and f are cofacial in an embedding D_1 of $C \cup P \cup B_u^*$ in which C bounds a face. We first extend this to an embedding D_2 of $C \cup B_u$ in which u and f are cofacial and C bounds a face.

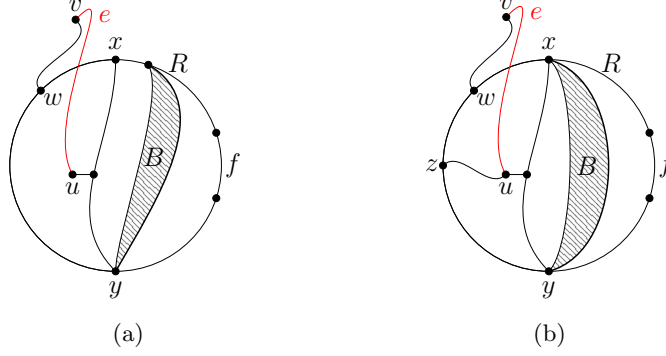


Figure 4

Let B be a $(C \cup P \cup B_u^*)$ -bridge in $C \cup B_u$. If B is embedded by D in the face of $C \cup P$ bounded by $P \cup R$, then B has only one attachment and this is in P . Thus, it may be added to D_1 while retaining the facts that u and f are cofacial and C bounds a face.

The remaining $(C \cup P \cup B_u^*)$ -bridges are embedded by D in the face of $C \cup P$ bounded by $P \cup Q$. These may all be added in that same face of D_1 , completing the required embedding D_2 .

Finally, we add all the remaining $(C \cup P)$ -bridges in G_u to maintain the facts that u and f are cofacial and C bounds a face. If B is embedded by D in the face of $C \cup P$ bounded by $P \cup Q$, then we use that embedding in D_2 .

In the last case, suppose B is embedded by D in the face of $C \cup P$ bounded by $P \cup R$ (see Figures 4a and 4b). If B has an attachment in $P - \{x, y\}$, then $B \subseteq B_u$ is already embedded in D_2 . Therefore, $Att(B)$ is contained in R .

If all the attachments of B are in one of the components of $R - f$, then we can use the embedding of B in D to add B to D_2 . Thus, we may assume that B has an attachment in each component of $R - f$.

If either B_u has an attachment z in $Q - \{x, y\}$ or B has an attachment not in $\{x, y\}$, then there are vertex-disjoint paths $P_u \subseteq B_u$ from u to Q , and P' in B joining attachments in each component of $R - f$ such that $V(P_u) \cap V(P') = \emptyset$. Thus, the cycle in G containing e , P_u , P_v , and the zw -subpath of Q is disjoint from the cycle containing f , P' , and a subpath of R . That is, e and f are separated by cycles, a contradiction.

Therefore $Att(B) = Att(B_u) = \{x, y\}$. This shows that we can add B in the face of $D_2[C \cup P]$ bounded by $P \cup Q$. This completes the embedding of G_u for which u and f are cofacial and C bounds a face. □

6 Final remarks

Let G be a nonplanar graph. We say that a pair of edges $\{e, f\}$ of G is a *potential crossing pair* if, for every Kuratowski subgraph H of G , e and f is a crossing pair of H . Clearly, a crossing pair is also a potential crossing pair and, as Theorem 4 (ii) shows, a potential crossing pair not separated by cycles is also a crossing pair. This raises the question on the existence of potential crossing pairs on graphs with crossing number at least 2. Given Theorem 4, such a pair would necessarily be separated by cycles.

Conjecture 8. *A non-planar graph G has crossing number at least 2 if and only if it does not have a potential crossing pair.*

Širáň's analysis [15] is instructive here. Deleting any two edges of K_6 results in a graph with crossing number 1. Thus, there is necessarily a potential crossing pair. Deleting any edge e of K_6 results in a graph with crossing number 2, but a simple case check shows there is no pair of edges that is in every $K_{3,3}$. That is, $K_6 - e$ has no potential crossing pair.

Regarding the proof of Theorem 4, we are aware of a shorter proof that shows that (ii) and (i) are equivalent (see Silva's thesis [14, Theorem 3.12]). This proof uses the famous Two Disjoint Paths theorem, proved independently by many authors [11, 12, 13, 17]. The proof relies on the fact that a pair of edges $\{e, f\}$ of a graph G is a crossing pair if and only if $G - e - f$ has a planar embedding such that the ends of e and f alternate in a face of the embedding. The proof of the equivalence of (i) and (ii) using the Two Disjoint Paths Theorem is shorter and somewhat straightforward. However, proving that (iii) implies (ii) seems significantly more complicated and we were unable to make it work using only the Two Disjoint Paths Theorem.

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