ON AN EXTENSION OF DIRAC'S THEOREM

FELIX JOOS AND JAEHOON KIM

Abstract. For a collection $\mathbf{G} = \{G_1, \dots, G_s\}$ of not necessarily distinct graphs on the same vertex set V, a graph H with vertices in V is a $\mathbf{G}$-transversal if there exists a bijection $\phi : E(H) \to [s]$ such that $e \in E(G_{\phi(e)})$ for all $e \in E(H)$. We prove that for $|V| = s \geq 3$ and $\delta(G_i) \geq s/2$ for each $i \in [s]$, there exists a $\mathbf{G}$-transversal that is a Hamilton cycle. This confirms a conjecture of Aharoni. We also prove an analogous result for perfect matchings.

1. Introduction

Suppose that we are given a collection $\mathbf{F} = \{F_1, \dots, F_s\}$ of not necessarily distinct subsets of some finite set Ω. Then a set $X \subseteq \Omega$ such that $X \cap F_i \neq \emptyset$ for each $i \in [s]$ is often called a 'transversal' of $\mathbf{F}$ or a 'colourful' object of $\mathbf{F}$. In the case where $\mathbf{F}$ is the edge set of a hypergraph, X is known as a hypergraph transversal. If $X = \{x_1, \dots, x_s\}$ and $x_i \in F_i$ for all $i \in [s]$, then X is also called system of distinct representatives. Frequently, we seek transversals with certain additional properties as for example $|X \cap F_i| = 1$ for all $i \in [s]$.

Other results that deal with transversals include results regarding transversals on Latin squares, a colourful version of Carathéodory's theorem by Holmsen, Pach and Tverberg [5], a colourful version for a topological and a matroidal extension of Helly's theorem by Kalai and Meshulam [6] and a colourful version of the Erdős-Ko-Rado theorem by Aharoni and Howard [2].

Surprisingly, the study of 'transversals' over collections of graphs has not received much attention until recently (for results on this topic see for example [1, 7]). Here, we simply take Ω to be the edge set of the complete graph on some vertex set V, the set $\mathbf{F}$ as a collection of (the edge sets of) graphs with vertex set V, and we ask for transversals (which are then collections of edges) with certain graph properties.

To be more precise, we define the following concept of transversals over a graph collection. Let $\mathbf{G} = \{G_1, \dots, G_s\}$ be a collection of not necessarily distinct graphs with common vertex set V. We say that a graph H with vertices in V is a *partial* $\mathbf{G}$*-transversal* if there exists an injection $\phi\colon E(H) \to [s]$ such that $e \in E(G_{\phi(e)})$ for each $e \in E(H)$. If in addition $|E(H)| = s$, then H is a $\mathbf{G}$*-transversal* (and ϕ a bijection). We also say that H is a path/cycle/triangle/matching (partial) $\mathbf{G}$-transversal if H is a path/cycle/triangle/matching and similarly for other graphs.

Let us consider the following question.

> *Let H be a graph with s edges, $\mathcal{G}$ be family of graphs and $\mathbf{G} = \{G_1, \dots, G_s\}$ be a collection of not necessarily distinct graphs on the same vertex set V such that $G_i \in \mathcal{G}$ for all $i \in [s]$. Which properties imposed on $\mathcal{G}$ yield a $\mathbf{G}$-transversal isomorphic to H?*

By considering the case when $G_1 = \cdots = G_s$, we need to study properties for $\mathcal{G}$ such that H is a subgraph of each graph in $\mathcal{G}$. However, this alone is not sufficient. To see that, let $|V| = s \geq 5$ and $\mathcal{G}$ be the collection of cycles with vertex set V. Consider $s-1$ identical cycles $G_1, \dots, G_{s-1}$ and another cycle G_s which is edge-disjoint from the others. Then there do not exist Hamiltonian $\mathbf{G}$-transversals; that is, one that is a Hamilton cycle (on V). Neither it is sufficient to impose the Turán condition on the number of edges. In [1] (see also [7]), it is shown that there is a triple of n-vertex graphs G_1, G_2, G_3 each having

Date: 4th October 2019.

The research leading to these results was partially supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 339933727 (F. Joos).

more than $n^2/4$ edges with no triangle transversal. In fact, one needs to require (roughly) at least $0.2557n^2$ edges in each G_i to guarantee the existence of a triangle transversal.

On the other hand, Aharoni [1] conjectured that Dirac's theorem [4] can be extended to a colourful version and here we confirm this conjecture.

Theorem 1. *Let $n \in \mathbb{N}$ and $n \geq 3$. Suppose $\mathbf{G} = \{G_1, \dots, G_n\}$ is a collection of not necessarily distinct n-vertex graphs with the same vertex set such that $\delta(G_i) \geq n/2$ for each $i \in [n]$. Then there exists a Hamiltonian $\mathbf{G}$-transversal.*

For the same reason as the bound in Dirac's theorem is sharp, we cannot improve upon the minimum degree bound in Theorem 1. Cheng, Wang and Yi [3] recently proved a weaker version of Theorem 1 with the condition $\delta(G_i) \geq (1/2 + o(1))n$.

We also prove the following theorem concerning perfect matchings.

Theorem 2. *Let $n \in \mathbb{N}$ and $n \geq 2$ even. Suppose $\mathbf{G} = \{G_1, \dots, G_{n/2}\}$ is a collection of not necessarily distinct n-vertex graphs with the same vertex set such that $\delta(G_i) \geq n/2$ for each $i \in [n]$. Then there exists a $\mathbf{G}$-transversal that is a perfect matching.*

2. The proofs

We write $[n] = \{1, \dots, n\}$ and $[m, n] = \{m, m+1, \dots, n\}$. We denote by $\delta(G)$ the minimum degree of a graph G. For a digraph D, we let $A(D)$ be the arc set of D, and $d^-_D(x)$ and $d^+_D(x)$ refer to the indegree and outdegree of a vertex $x \in V(D)$, respectively. We denote by $N^-_D(x)$ the in-neighbourhood of $x \in V(D)$.

It will be also useful to specify a particular injection/bijection for a (partial) $\mathbf{G}$-transversal. To this end, we say that (H, ϕ) is a partial $\mathbf{G}$-transversal if $\phi\colon E(H) \to [s]$ is injective and a $\mathbf{G}$-transversal if ϕ is bijective. If $i \notin \phi(E(H))$ for some $i \in [s]$, we say i is *missed* by ϕ and ϕ *misses* i.

Proof of Theorem 1. Assume for a contradiction that there do not exist Hamiltonian $\mathbf{G}$-transversals. It is routine to check the statement for $n \in \{3, 4\}$, so we may assume that $n \geq 5$. Let V be the common vertex of the graphs in $\mathbf{G}$. For each $e \in \binom{V}{2}$, let

$$c(e) := \{i \in [n] : e \in E(G_i)\}.$$

Claim 1. There exists a partial $\mathbf{G}$-transversal that is a cycle of length $n - 1$.

Proof of claim: Let (C, ϕ) be a partial $\mathbf{G}$-transversal which has the largest number of edges among all paths and cycles. Among cycles and paths with the same number of edges, we prefer cycles.

Suppose $C = (x_1, \dots, x_{\ell+1})$ is an ℓ-edge path with $\ell \in [3, n-1]$ (it is easy to see that $\ell \geq 3$ as $n \geq 5$ by simply picking the edges of C greedily). Consider the $(\ell-1)$-edge path $P = (x_1, \dots, x_\ell)$. The partial $\mathbf{G}$-transversal given by ϕ restricted to $E(P)$ misses at least two integers, say, 1 and 2. Then $1, 2 \notin c(x_1x_\ell)$, as otherwise $(x_1, \dots, x_\ell, x_1)$ forms an ℓ-edge cycle partial $\mathbf{G}$-transversal which contradicts the choice of (C, ϕ). Let

$$I_1 := \{i \in [\ell-2]\colon 1 \in c(x_1x_{i+1})\} \text{ and } I_2 := \{i \in [2, \ell-1]\colon 2 \in c(x_ix_\ell)\}.$$

Note that we have

$$|N_{G_1}(x_1) \setminus V(P)| + |N_{G_2}(x_\ell) \setminus V(P)| \leq n - \ell, \tag{1}$$

otherwise, by the pigeonhole principle, there exists $y \in V \setminus V(P)$ such that $(x_1, \dots, x_\ell, y, x_1)$ forms an $(\ell+1)$-edge cycle partial $\mathbf{G}$-transversal, again a contradiction to the choice of (C, ϕ). Since $\delta(G_i) \geq n/2$ for all $i \in [n]$ and $1, 2 \notin c(x_1x_\ell)$, equation (1) implies that

$$|I_1| + |I_2| \geq n/2 + n/2 - |N_{G_1}(x_1) \setminus V(P)| - |N_{G_2}(x_\ell) \setminus V(P)| \geq \ell.$$

As $I_1 \cup I_2 \subseteq [\ell-1]$, there exists an integer $j \in I_1 \cap I_2 \subseteq [2, \ell-2]$. Hence deleting x_jx_{j+1} from $E(P)$ and adding x_1x_{j+1}, x_jx_ℓ yields a partial $\mathbf{G}$-transversal that is a cycle of length ℓ, which is a contradiction to the choice of (C, ϕ). Hence we may assume that C is a cycle.

In view of the statement, we may assume that $C = (x_1, \dots, x_\ell, x_1)$ is an ℓ-edge cycle for some $\ell \in [3, n-2]$ and there are two integers, say 1 and 2, that are missed by ϕ. Observe that $\ell \geq n/2 + 1$, since otherwise we have

$$|N_{G_1}(x_1) \setminus V(C)| \geq 1 \text{ and } |N_{G_2}(x_\ell) \setminus V(C)| \geq 1,$$

and we obtain two not necessarily distinct vertices $y, z \in V \setminus V(C)$ with $1 \in c(x_1 y), 2 \in c(x_\ell z)$. Then $(y, x_1, \dots, x_\ell, z)$ is a partial **G**-transversal which is either path or cycle with $\ell+1$ edges and this contradicts the choice of (C, ϕ).

We claim that, for each $v \in V \setminus V(C)$ and $i \in [2]$, we have $N_{G_i}(v) \subseteq V(C)$. Suppose not. Then there exists $i \in [2]$ and $u, v \in V \setminus V(C)$ with $uv \in E(G_i)$. As we have $d_{G_{3-i}}(v) \geq n/2 > |V \setminus V(C)|$, we have, by symmetry, $x_\ell v \in E(G_{3-i})$. Consequently, $(x_1, \dots, x_\ell, v, u)$ contradicts the choice of (C, ϕ). Thus, for each $v \in V \setminus V(C)$ and $i \in [2]$, we have $N_{G_i}(v) \subseteq V(C)$.

Fix some $v \in V \setminus V(C)$. Let

$$I_1 := \{i \in [\ell] \colon 1 \in c(vx_{i+1})\} \text{ and } I_2 := \{i \in [\ell] \colon 2 \in c(vx_i)\},$$

where we identify $x_{\ell+1}$ with x_1. Then

$$|I_1| + |I_2| \geq \delta(G_1) + \delta(G_2) \geq n > \ell,$$

and there exists an integer $j \in I_1 \cap I_2$. Hence deleting $x_j x_{j+1}$ from $E(C)$ and adding vx_j, vx_{j+1} yields partial **G**-transversal that is a cycle of length $\ell+1$, which is a contradiction to the choice of (C, ϕ). This proves Claim 1. $-$

By Claim 1, there exists a cycle partial **G**-transversal (C, ϕ) with $C := (x_1, \dots, x_{n-1}, x_1)$. By relabelling colours, we may assume that $\phi(x_i x_{i+1}) = i$ for each $i \in [n-1]$ where we identify x_n with x_1. Hence ϕ misses n. Let $\{y\} = V \setminus V(C)$. We consider the following auxiliary digraph D on vertex set $[n]$ such that

$$A(D) = \bigcup_{i \in [n-1]} \{x_i z : z \neq x_{i+1}, i \in c(x_i z)\}.$$

As $\delta(G_i) \geq n/2$ for all $i \in [n-1]$ and thus $d^+_D(x) \geq n/2 - 1$ for all $x \in V(C)$, we obtain that $|A(D)| \geq (n-1)(n/2 - 1)$. Let

$$I := \{i \in [n-1] \colon x_i y \in A(D)\} \text{ and } I' := \{i \in [n-1] \colon x_{i+1} y \in E(G_n)\}.$$

We claim that $d^-_D(y) \leq \frac{n}{2} - 1$. Otherwise, we have $|I| + |I'| \geq d^-_D(y) + \delta(G_n) > n-1 = |V(C)|$. So, there exists $j \in I \cap I'$ and thus $(E(C) \setminus \{x_j x_{j+1}\}) \cup \{x_j y, y x_{j+1}\}$ is the edge set of a Hamiltonian **G**-transversal, which is a contradiction.

Hence, we assume from now on that $d^-_D(y) \leq \frac{n}{2} - 1$. By our definition of D, we have $d^+_D(y) = 0$ and thus

$$|A(D - y)| \geq (n-1)\left(\frac{n}{2} - 1\right) - \frac{n}{2} + 1 > (n-1)\left(\frac{n}{2} - \frac{3}{2}\right). \tag{2}$$

Let us assume for now that there exists a vertex, say x_1, such that $d^-_{D-y}(x_1) > n/2 - 1$. Consequently, we conclude that

$$|\{i \in [2, n-2] \colon i \in c(x_1 x_i)\}| = d^-_{D-y}(x_1) \geq \frac{n}{2} - \frac{1}{2}. \tag{3}$$

Let

$$I_1 := \{i \in [n-1] \colon x_i y \in E(G_1)\} \text{ and } I_n := \{i \in [n-1] \colon x_{i+1} y \in E(G_n)\}.$$

Clearly, $|I_1| + |I_n| \geq n$, so there exists a $j \in I_1 \cap I_n$. We may assume that $j \neq 1$ as otherwise $(E(C) \setminus \{x_1 x_2\}) \cup \{x_1 y, x_2 y\}$ is the edge set of a Hamiltonian **G**-transversal, which is a contradiction.

Let (P, ϕ') with $P = (x_2, \dots, x_j, y, x_{j+1}, \dots, x_{n-1}, x_1)$ be a path partial **G**-transversal that arises from ϕ by deleting $\{x_1 x_2, x_j x_{j+1}\}$ from its domain and by setting $\phi'(x_j y) := 1$

and $\phi'(x_{j+1}y) := n$. Observe that ϕ' misses (only) j. We write $P = (x^1, \dots, x^n)$ such that $x^1 = x_2$. Let

$$J_1 := \{i \in [n-2] \colon j \in c(x^1 x^{i+1})\} \text{ and } J_n := \{i \in [n-2] \colon x^i \in N^-_{D-y}(x_1)\}.$$

If $j \in c(x^1 x^n)$, then there is a Hamiltonian $\mathbf{G}$-transversal, which is a contradiction; so $|J_1| \geq \delta(G_j)$. Also, as $x^{n-1} \in \{x_{n-1}, y\}$, the definition of D ensures that $x^{n-1} \notin N^-_{D-y}(x_1)$. Hence (3) implies that $|J_n| \geq n/2 - 1/2$ and thus $|J_1| + |J_2| \geq n$. Since $J_1 \cup J_2 \subseteq [n-2]$, there exist at least two integers in $J_1 \cap J_n$ and at least one of them, say k, satisfies $x^{k+1} \neq y$. Moreover, $x^k \neq y$ as $y \notin N^-_{D-y}(x_1)$. Hence, $\phi'(x^k x^{k+1}) \in c(x^k x^n)$ and $(E(P) \setminus \{x^k x^{k+1}\}) \cup \{x^1 x^{k+1}, x^k x^n\}$ forms a Hamiltonian $\mathbf{G}$-transversal, which is a contradiction.

Therefore, we may assume that $d^-_{D-y}(x_i) \leq n/2 - 1$ for all $i \in [n-1]$. We define

$$\mathcal{J} := \left\{ i \in [n-1] : d^-_{D-y}(x_i) = \left\lfloor \frac{n}{2} - 1 \right\rfloor \right\}.$$

Then (2) implies that

$$\left\lfloor \frac{n}{2} - 1 \right\rfloor |\mathcal{J}| + \left\lfloor \frac{n}{2} - 2 \right\rfloor (n - 1 - |\mathcal{J}|) \geq |A(D-y)| > (n-1)\left(\frac{n}{2} - \frac{3}{2}\right).$$

Hence, we have

$$|\mathcal{J}| > (n-1)\left(\frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{2}\right) \geq \frac{n-1}{2}.$$

Let $\mathcal{J}' := \{i \in [n-1] \colon x_{i+1}y \in E(G_n)\}$. Clearly, $|\mathcal{J}| + |\mathcal{J}'| \geq n$ and so there exists a $j \in \mathcal{J} \cap \mathcal{J}'$. Let (Q, ϕ') with $Q = (y, x_{j+1}, x_{j+2}, \dots, x_{n-1}, x_1, \dots, x_j)$ be a path partial $\mathbf{G}$-transversal that arises from ϕ by deleting $\{x_j x_{j+1}\}$ from its domain and by setting $\phi'(x_{j+1}y) := n$. Observe that ϕ' misses j. We write $Q = (x^1, \dots, x^n)$ such that $x^1 = y$. Let

$$J_1 := \{i \in [n-2] \colon j \in c(x^1 x^{i+1})\} \text{ and } J_n := \{i \in [2, n-2] \colon x^i \in N^-_{D-y}(x^n)\}.$$

If $j \in c(x^1 x^n)$, then there is a Hamiltonian $\mathbf{G}$-transversal, which is a contradiction; so $|J_1| \geq \delta(G_j) \geq n/2$. Note that $x^1 = y \notin N^-_{D-y}(x^n)$ and $x^{n-1} = x_{j-1} \notin N^-_{D-y}(x^n)$ by the definition of D. As $x^n = x_j \in \mathcal{J}$, we infer that $|J_n| = \lfloor n/2 - 1 \rfloor$. We obtain $|J_1| + |J_n| \geq n - 1$. As $J_1 \cup J_n \subseteq [n-2]$, there exists an integer $k \in J_1 \cap J_n \subseteq [2, n-2]$. Since $x^k \neq y = x^1$, we conclude that $\phi'(x^k x^{k+1}) \in c(x^k x^n)$ and $(E(P) \setminus \{x^k x^{k+1}\}) \cup \{x^1 x^{k+1}, x^k x^n\}$ contains a Hamiltonian $\mathbf{G}$-transversal. This is the final contradiction. $\square$

Proof of Theorem 2. We use similar notation as in the proof of Theorem 1; in particular, let V be the common vertex set of the graphs in $\mathbf{G}$ and for each $e \in \binom{V}{2}$, let

$$c(e) := \{i \in [n/2] : e \in E(G_i)\}.$$

For a partial $\mathbf{G}$-transversal (M, ϕ), we refer to $|E(M)|$ as the *size* of (M, ϕ). We assume for a contradiction that there does not exist a matching $\mathbf{G}$-transversal.

It is easy to see that G contains a matching partial $\mathbf{G}$-transversal of size $n/2 - 1$. Indeed, consider a matching partial $\mathbf{G}$-transversal (M, ϕ) of maximum size ℓ. Assume for a contradiction that $\ell < n/2 - 1$ and ϕ misses 1 and 2, say. Clearly, $\{1, 2\} \cap c(xx') = \emptyset$ for all $xx' \in \binom{V \setminus V(M)}{2}$. Fix two vertices $x, x' \in V \setminus V(M)$. Let the weight of an edge $e = uv$ be $\mathbb{1}_{1 \in c(xu)} + \mathbb{1}_{1 \in c(xv)} + \mathbb{1}_{2 \in c(x'u)} + \mathbb{1}_{2 \in c(x'v)}$. Since $\delta(G_i) \geq n/2$ for $i \in [2]$, we deduce that the sum of the weights of the edges in M is at least n. Hence there is an edge $e = yy'$ in M with weight at least 3. Replacing e by $\{xy, x'y'\}$ or $\{x'y, xy'\}$ yields a contradiction to our assumption that the size of (M, ϕ) is maximum.

For a contradiction, we assume that there is no matching $\mathbf{G}$-transversal. Let $\ell := n/2 - 1$. For a matching partial $\mathbf{G}$-transversal (N, ϕ), we let D^ϕ_N be a digraph with vertex set V and

$$A(D^\phi_N) := \{xy : \phi(xz) \in c(xy), y \neq z, xz \in E(N)\}.$$

Claim 1. $d^-_{D^\phi_M}(x) \leq \ell - 1$ for all matching partial **G**-transversals (M, ϕ) of size ℓ and $x \in V \setminus V(M)$.

Proof of claim: We define $D := D^\phi_M$. We assume for a contradiction that $d^-_D(x) \geq \ell$. Let $\{x'\} = V \setminus (V(M) \cup \{x\})$. Say ϕ misses 1. Clearly, $1 \notin c(xx')$. As $\delta(G_1) \geq n/2$ and $d^-_D(x) \geq \ell$, there exists an edge $yy' \in V(M)$ such that $yx \in A(D)$ and $1 \in c(x'y')$. However, then removing yy' from M and adding xy and $x'y'$ yields a matching **G**-transversal, which is a contradiction. —

Claim 2. $d^-_{D^\phi_M}(x) \leq \ell$ for all matching partial **G**-transversals (M, ϕ) of size ℓ and $x \in V$.

Proof of claim: We define $D := D^\phi_M$. We assume for a contradiction that $d^-_D(x) \geq \ell + 1$ and ϕ misses 1, say. By Claim 1, we conclude that $x \in V(M)$. Let y be the neighbour of x in M and $\phi(xy) = 2$, say. Let $\{z, z'\} = V \setminus V(M)$. Suppose $i \in c(y\tilde{z})$ for some $i \in [2], \tilde{z} \in \{z, z'\}$. Then let (M', ϕ') be the matching partial **G**-transversal where (M', ϕ') arises (M, ϕ) by deleting xy from M, adding $y\tilde{z}$, and assigning i on $y\tilde{z}$. Hence, for $D' := D^{\phi'}_{M'}$, we obtain $d^-_{D'}(x) \geq \ell + 1$, which is a contradiction to Claim 1. So we may assume that $\{1, 2\} \cap (c(yz) \cup c(yz')) = \emptyset$.

Let $V' := V \setminus \{x, y, z\}$. Then $|N_{G_2}(y) \cap V'| \geq n/2 - 1$ and $|N_M(N_{G_1}(z)) \cap V'| \geq n/2 - 1$. Consequently, there exists a vertex $u \in V' \cap N_{G_2}(y) \cap N_M(N_{G_1}(z))$. Observe that $u \notin \{x, y, z, z'\}$. Let u' be the neighbour of u in M. Let (M'', ϕ'') be the matching partial **G**-transversal where M'' arises M by deleting xy, uu' and adding $uy, u'z$ and ϕ'' arises from ϕ by assigning $u'z$ to 1 and uy to 2. We write D'' for $D^{\phi''}_{M''}$ and observe that $d^-_{D''}(x) \geq d^-_D(x) - 1$ as $y \in N^-_{D''}(x) \setminus N^-_D(x)$ and $N^-_D(x) \setminus N^-_{D''}(x) \subseteq \{u, u'\}$. However, exploiting Claim 1, (M'', ϕ'') yields a contradiction. —

Claim 3. For all matching partial **G**-transversals (M, ϕ) of size ℓ, there are at least $n/2$ vertices $x \in V(M)$ with $d^-_{D^\phi_M}(x) \geq \ell - 1$.

Proof of claim: We define again $D := D^\phi_M$. Observe that the number of arcs in D is at least $2\ell^2$, as $d^+_D(x) \geq \ell$ for all $x \in V(M)$. Assuming that there are at most $n/2 - 1 = \ell$ vertices $x \in V(M)$ with $d^-_D(x) \geq \ell - 1$, implies in view of Claims 1 and 2 that $|A(D)| \leq \ell^2 + \ell(\ell - 2) + 2(\ell - 1) < 2\ell^2$, which is a contradiction. —

Let (M, ϕ) be some matching partial **G**-transversal of maximum size. In view of the above, the size of M equals ℓ and so ϕ misses 1, say. Let $\{z, z'\} = V \setminus V(M)$ and $D := D^\phi_M$. By Claim 3 and as $\delta(G_1) \geq n/2$, there exists $xy \in V(M)$ with $d^-_D(x) \geq \ell - 1$ and $1 \in c(yz)$. Say, $\phi(xy) = 2$. Let (M', ϕ') arise from (M, ϕ) by deleting xy from M, adding yz and assigning yz to 1. Let $D' := D^{\phi'}_{M'}$.

Claim 4. The following hold:

(a) $|N^-_{D'}(x) \cap (V \setminus \{x, z, z'\})| \geq \ell - 1$;
(b) $|N_{G_2}(z') \cap (V \setminus \{x, y, z'\})| \geq n/2$.

Proof of claim: Statement (a) is obvious. To see (b), we first observe that if $2 \in c(xz')$, then we can delete xy from M and add xz' and yz and obtain a matching **G**-transversal. Moreover, if $2 \in c(yz')$, then the matching that arises from M by deleting xy, adding yz', and assigning 2 to yz' contradicts Claim 1. This proves (b). —

Observe that $N_{G_2}(z') \cap (V \setminus \{x, y, z'\}) \subseteq V(M')$. Let A be the set of vertices that are joined by an edge in M' to a vertex in $N_{G_2}(z') \cap (V \setminus \{x, y, z'\})$. Consequently, $A \subseteq V \setminus \{x, z, z'\}$ and $|A| \geq n/2$ by Claim 4(b). As $|V \setminus \{x, z, z'\}| = n - 3 < n/2 + \ell - 1 \leq |A| + |N^-_{D'}(x) \cap (V \setminus \{x, z, z'\})|$, there is a vertex $u \in A \cap N^-_{D'}(x) \cap (V \setminus \{x, z, z'\})$. Let v be the neighbour of u in M'. Deleting uv and adding ux and vz' to M' gives rise to a matching **G**-transversal. This is the final contradiction and completes the proof. □

References

1. R. Aharoni, M. DeVos, S. González Hermosillo de la Maza, A. Montejano, and R. Šámal, *A rainbow version of Mantel's Theorem*, arXiv:1812.11872 (2018).
2. R. Aharoni and D. Howard, *A rainbow r-partite version of the Erdős-Ko-Rado theorem*, Combin. Probab. Comput. **26** (2017), 321–337.
3. Y. Cheng, G. Wang, and Y. Zhao, *Rainbow pancyclicity in graph systems*, arXiv:1909.11273 (2019).
4. G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. **2** (1952), 69–81.
5. A. F. Holmsen, J. Pach, and H. Tverberg, *Points surrounding the origin*, Combinatorica **28** (2008), no. 6, 633–644.
6. G. Kalai and R. Meshulam, *A topological colorful Helly theorem*, Adv. Math. **191** (2005), 305–311.
7. C. Magnant, *Density of Gallai multigraphs*, Electron. J. Combin. **22** (2015), Paper 1.28, 6pp.

Felix Joos
<f.joos@bham.ac.uk>
School of Mathematics
University of Birmingham
United Kingdom

Jaehoon Kim
<jaehoon.kim@kaist.ac.kr>
Department of Mathematical Sciences
KAIST
Republic of Korea