A vertex ordering characterization of simple-triangle graphs

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ABSTRACT

Consider two horizontal lines in the plane. A point on the top line and an interval on the bottom line define a triangle between two lines. The intersection graph of such triangles is called a simple-triangle graph. This paper shows a vertex ordering characterization of simple-triangle graphs as follows: a graph is a simple-triangle graph if and only if there is a linear ordering of the vertices that contains both an alternating orientation of the graph and a transitive orientation of the complement of the graph.

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1. Introduction

Let L_1 and L_2 be two horizontal lines in the plane with L_1 above L_2. A pair of a point on the top line L_1 and an interval on the bottom line L_2 defines a triangle between L_1 and L_2. The point on L_1 is called the *apex* of the triangle, and the interval on L_2 is called the *base* of the triangle. A *simple-triangle graph* is the intersection graph of such triangles, that is, a simple undirected graph G is called a simple-triangle graph if there is such a triangle for each vertex and two vertices are adjacent if and only if the corresponding triangles have a nonempty intersection. The set of triangles is called a *representation* of G. See Figs. 1(a) and 1(b) for example. Simple-triangle graphs are also known as *PI graphs* [2,3,5], where *PI* stands for *Point-Interval*. Simple-triangle graphs were introduced as a generalization of both interval graphs and permutation graphs, and they form a proper subclass of trapezoid graphs [5]. Although a lot of research has been done for interval graphs, for permutation graphs, and for trapezoid graphs (see [2,10,12,17,19,23] for example), there are few results for simple-triangle graphs [1,3,5]. The polynomial-time recognition algorithm has been given recently [18,25], but the complexity of the graph isomorphism problem still remains an open question [24,26], which makes it interesting to study the structural characterizations of this graph class.

A *vertex ordering* of a graph $G = (V, E)$ is a linear ordering $\sigma = v_1, v_2, \ldots, v_n$ of the vertex set V of G. We use $u <_\sigma v$ to denote that u precedes v in σ. A *vertex ordering characterization* of a graph class $\mathcal{G}$ is a characterization of the following type: a graph G is in $\mathcal{G}$ if and only if G has a vertex ordering fulfilling some properties. For example, it is known that a graph G is an interval graph if and only if G has a vertex ordering σ such that for any three vertices $u <_\sigma v <_\sigma w$, if $uw \in E$ then $uv \in E$ [21]. In other words, a graph is an interval graph if and only if it has a vertex ordering that contains no subordering in Figs. 2(a) and 2(c). It is also known that a graph is a permutation graph if and only if it has a vertex ordering that contains no subordering in Figs. 2(b) and 2(c). See [2,6,13] for other examples of vertex ordering characterizations.

This paper shows a vertex ordering characterization of simple-triangle graphs. We call a vertex ordering σ of a simple-triangle graph G an *apex ordering* if there is a representation of G such that σ coincides with the ordering of the apices of the triangles in the representation. See Fig. 1(d) for example. We characterize the apex orderings of simple-triangle graphs

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(a) A simple-triangle graph G.

(b) The representation of G.

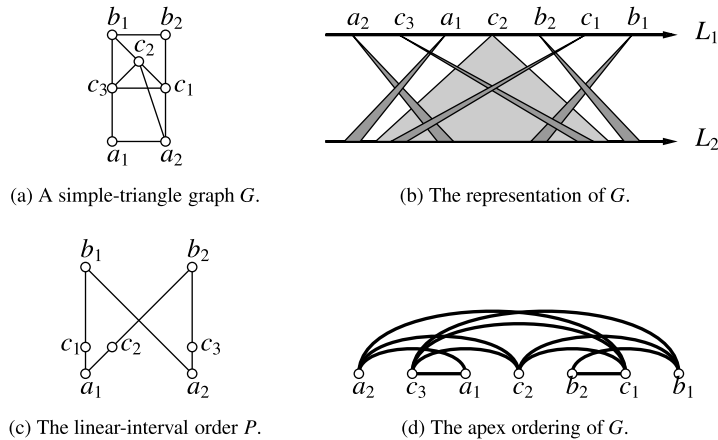(c) The linear-interval order P.

(d) The apex ordering of G.

Fig. 1. A simple-triangle graph G, the representation of G, the Hasse diagram of linear-interval order P, and the apex ordering of G.
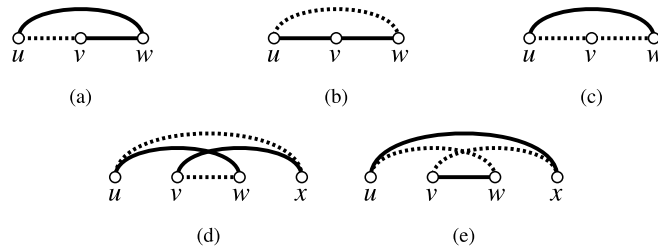

(a)

(b)

(c)

(d)

(e)

Fig. 2. Forbidden patterns. Lines and dashed lines denote edges and non-edges, respectively. Edges that may or may not be present is not drawn.

as follows: a vertex ordering σ of a graph G is an apex ordering if and only if σ contains no subordering in Figs. 2(c)–2(e). Equivalently, we show that a vertex ordering σ of G is an apex ordering if and only if σ contains both an alternating orientation of G and a transitive orientation of the complement $\bar{G}$ of G.

The organization of this paper is as follows. Before describing the vertex ordering characterization, we show in Section 2 a characterization of the linear-interval orders, the partial orders associated with simple-triangle graphs. The vertex ordering characterization of simple-triangle graphs is shown in Section 3. We remark some open questions and related topics in Section 4.

2. Linear-interval orders

A *partial order* is a pair $P = (V, \prec_P)$, where V is a finite set and $\prec_P$ is a binary relation on V that is irreflexive and transitive. The finite set V is called the *ground set* of P. A partial order $P = (V, \prec_P)$ is called a *linear order* if for any two elements $u, v \in V$, $u \prec_P v$ or $u \succ_P v$. A partial order $P = (V, \prec_P)$ is called an *interval order* if for each element $v \in V$, there is a (closed) interval $I(v) = [l(v), r(v)]$ on the real line such that for any two elements $u, v \in V$, $u \prec_P v \iff r(u) < l(v)$, that is, $I(u)$ lies completely to the left of $I(v)$. The set of intervals $\{I(v) \mid v \in V\}$ is called an *interval representation* of P.

Let $P_1 = (V, \prec_1)$ and $P_2 = (V, \prec_2)$ be two partial orders with the same ground set. The *intersection* of P_1 and P_2 is the partial order $P = (V, \prec_P)$ such that $u \prec_P v \iff u \prec_1 v$ and $u \prec_2 v$; it is denoted by $P = P_1 \cap P_2$. A partial order P is called a *linear-interval order* (also known as a *PI order* [3]) if there exist a linear order L and an interval order P_I such that $P = L \cap P_I$. The linear-interval order can also be defined as follows. Recall that L_1 and L_2 are two horizontal lines with L_1 above L_2, and a pair of a point on L_1 and an interval on L_2 defines a triangle between L_1 and L_2. A partial order $P = (V, \prec_P)$ is a linear-interval order if there is such a triangle $T(v)$ for each element $v \in V$, and $u \prec_P v$ if and only if $T(u)$ lies completely to the left of $T(v)$. See Figs. 1(b) and 1(c) for example. Notice that the ordering of the apices of the triangles gives the linear order L, and the bases of the triangles give an interval representation of the interval order P_I.

A linear order $L = (V, \prec_L)$ is called a *linear extension* of a partial order $P = (V, \prec_P)$ if $u \prec_L v$ whenever $u \prec_P v$. Hence, the linear extension L of P has all the relations of P with the additional relations that make L linear. We define two properties of linear extensions.

- Let $\mathbf{2 + 2}$ denote the partial order consisting of four elements a_0, a_1, b_0, b_1 whose only relations are $a_0 \prec_P b_0$ and $a_1 \prec_P b_1$. A linear extension $L = (V, \prec_L)$ of $P = (V, \prec_P)$ is said to fulfill the $\mathbf{2 + 2}$ *rule* if for every suborder $\mathbf{2 + 2}$ in P, either $b_0 \prec_L a_1$ or $b_1 \prec_L a_0$.

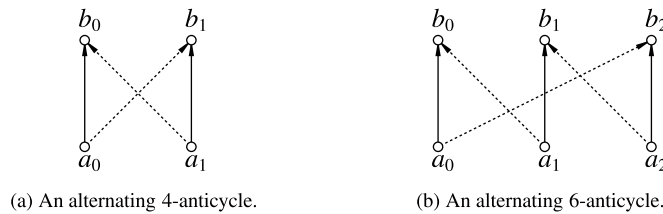(a) An alternating 4-anticycle. (b) An alternating 6-anticycle.

Fig. 3. Alternating anticycles. An arrow $a \to b$ denotes the relation $a \prec_P b$, and a dashed arrow $a \dashrightarrow b$ denotes the relation $a \prec_L b$ but $a \nprec_P b$.

- An *alternating 2k-anticycle* of a linear extension $L = (V, \prec_L)$ of $P = (V, \prec_P)$ is an induced suborder consisting of distinct $2k$ elements $a_0, b_0, a_1, b_1, \dots, a_{k-1}, b_{k-1}$ with $a_i \prec_P b_i$ and $a_{i+1} \prec_L b_i$ but $a_{i+1} \nprec_P b_i$ for any $i = 0, 1, \dots, k-1$ (indices are modulo k). See Fig. 3 for example.

Notice that a linear extension L of P fulfills the $\mathbf{2 + 2}$ rule if and only if L contains no alternating 4-anticycle. These properties characterize the linear-interval orders as follows.

Theorem 1. *For a partial order P, the following conditions are equivalent:*

  (i) *P is a linear-interval order;*
 (ii) *P has a linear extension fulfilling the $\mathbf{2 + 2}$ rule;*
(iii) *P has a linear extension that contains no alternating 4-anticycle.*

Proof. It is obvious that **(ii)** $\iff$ **(iii)**. The implications **(i)** $\implies$ **(ii)** and **(iii)** $\implies$ **(i)** are proved by Lemmas 2 and 3, respectively. $\square$

Lemma 2. *If a partial order $P = (V, \prec_P)$ has a linear order $L = (V, \prec_L)$ and an interval order $P_I = (V, \prec_I)$ with $P = L \cap P_I$, then for any suborder $\mathbf{2 + 2}$ in P, the equivalences $a_0 \prec_L a_1 \iff b_0 \prec_L a_1 \iff b_0 \prec_L b_1 \iff a_0 \prec_L b_1$ hold, where the suborder $\mathbf{2 + 2}$ consists of four elements a_0, a_1, b_0, b_1 whose only relations are $a_0 \prec_P b_0$ and $a_1 \prec_P b_1$.*

Proof. For an element v of P_I, let $I(v)$ denote the interval of v in the interval representation of P_I. We first show that $a_0 \prec_L a_1 \implies b_0 \prec_L a_1$. Suppose for a contradiction that $a_0 \prec_L a_1 \prec_L b_0$. Since $a_0 \prec_P b_0$, the interval $I(a_0)$ lies completely to the left of $I(b_0)$. Then since $a_0 \prec_L a_1 \prec_L b_0$, the interval $I(a_1)$ must intersect both $I(a_0)$ and $I(b_0)$. Since $a_1 \prec_P b_1$, the interval $I(a_1)$ lies completely to the left of $I(b_1)$, and consequently, $I(a_0)$ lies completely to the left of $I(b_1)$, that is, $a_0 \prec_I b_1$. From $a_1 \prec_P b_1$, we also have $a_0 \prec_L a_1 \prec_L b_1$, which implies $a_0 \prec_P b_1$, contradicting the definition of $\mathbf{2 + 2}$. Thus, $a_0 \prec_L a_1 \implies b_0 \prec_L a_1$. By similar arguments, we have the other implications $b_0 \prec_L a_1 \implies b_0 \prec_L b_1 \implies a_0 \prec_L b_1 \implies a_0 \prec_L a_1$. We note that this proof is implicit in [5]. $\square$

Lemma 3. *If a partial order $P = (V, \prec_P)$ has a linear extension $L = (V, \prec_L)$ that contains no alternating 4-anticycle, then there is an interval order $P_I = (V, \prec_I)$ with $P = L \cap P_I$. The interval order P_I and its interval representation can be obtained in $O(n^2)$ time from P and L, where n is the number of elements of P.*

Proof. We prove the lemma by showing an algorithm to construct an interval representation of P_I from P and L. We note that this algorithm is inspired by the algorithms that solve the sandwich problems for chain graphs and for threshold graphs [7,11,16,22,25]. As in Fig. 3, we use in this proof an arrow $a \to b$ to denote the relation $a \prec_P b$, and a dashed arrow $a \dashrightarrow b$ to denote the relation $a \prec_L b$ but $a \nprec_P b$. Notice that for a partial order Q, the intersection $L \cap Q = P$ if and only if Q has all the relations of $\to$ but has no relations of $\dashrightarrow$. The following facts are central to the proof of the correctness of the algorithm.

Claim 4. *L contains no alternating anticycle.*

Proof. Suppose for a contradiction that L contains an alternating anticycle. Let C be an alternating $2k$-anticycle of L with the least number of elements, and let $a_0, b_0, a_1, b_1, \dots, a_{k-1}, b_{k-1}$ be the consecutive elements of C with $a_i \to b_i$ and $a_{i+1} \dashrightarrow b_i$ for any $i = 0, 1, \dots, k-1$ (indices are modulo k). Since L contains no alternating 4-anticycle, we have $k \geq 3$. We consider the relation between a_0 and b_1. If $a_0 \to b_1$ then the elements $a_0, b_1, a_2, b_2, a_3, b_3, \dots, a_{k-1}, b_{k-1}$ induce an alternating $(2k - 2)$-anticycle, contradicting the minimality of C. If $b_1 \to a_0$ then $a_1 \to b_0$ by the transitivity of $\prec_P$, a contradiction. If $a_0 \dashrightarrow b_1$ then the elements a_0, b_0, a_1, b_1 induce an alternating 4-anticycle, a contradiction. Therefore, we have $b_1 \dashrightarrow a_0$. Similarly, we have $b_{i+1} \dashrightarrow a_i$ for any $i = 0, 1, \dots, k-1$. However, it follows from $a_i \to b_i$ that L is not a linear order, a contradiction. $\square$

An element a of P is said to be *minimal* if there is no element b of P with $b \prec_P a$. Let S be the set of all minimal elements of P.

Claim 5. *There is a minimal element $a \in S$ such that for any element $b \in V \setminus S$, if $a \prec_L b$ then $a \prec_P b$. In other words, there is an element $a \in S$ that has no element $b \in V \setminus S$ with $a \dashrightarrow b$.*

Proof. Suppose for a contradiction that for any minimal element $a \in S$, there is an element $b \in V \setminus S$ with $a \dashrightarrow b$. Notice that for any element $b \in V \setminus S$, there is a minimal element $a \in S$ with $a \to b$. Thus, starting at some vertex in V, we can grow a walk alternating between S and $V \setminus S$ without stopping. The first time the walk revisits a previously visited vertex, we obtain an alternating anticycle, contradicting that L contains no alternating anticycle. □

Algorithm 1: Constructing of the interval representations

    Data: The partial order $P = (V, \prec_P)$ and the linear extension $L = (V, \prec_L)$ of P
    Result: An interval representation $\{I(v) = [l(v), r(v)] \mid v \in V\}$ of $P_I = (V, \prec_I)$ with $P = L \cap P_I$
1 $S \leftarrow \emptyset, i \leftarrow 0$;
2 repeat
3 $i \leftarrow i + 1$;
4   **foreach** *element $a \in V \setminus S$* **do**
5   **if** *a has no element $b \in V$ with $b \prec_P a$* **then**
6 $S \leftarrow S \cup \{a\}$;
7 $l(a) \leftarrow i$;
8   **end**
9   **end**
10 $i \leftarrow i + 1$;
11   **foreach** *element $a \in S$* **do**
12   **if** *a has no element $b \in V \setminus S$ with $a \prec_L b$ but $a \nprec_P b$* **then**
13 $V \leftarrow V \setminus \{a\}, S \leftarrow S \setminus \{a\}$;
14 $r(a) \leftarrow i$;
15   **end**
        /* Claim 5 ensures that at least one element of S fulfills the **if** condition.   */
16   **end**
17 until $V = \emptyset$;

The algorithm to construct an interval representation of P_I is given as Algorithm 1. At the end of the loop on lines 4–9, S has all the minimal elements of the suborder of P induced by V (recall that elements may be removed from V at line 13). Hence, Claim 5 ensures that S has at least one element fulfilling the **if** condition at line 12. Since such an element is removed from V and S at line 13, we can see by induction that Algorithm 1 eventually terminates. For any two elements $a, b \in V$, the **if** condition at line 5 ensures that $r(a) < l(b)$ whenever $a \to b$, and the **if** condition at line 12 ensures that $a \not\dashrightarrow b$ whenever $r(a) < l(b)$; the interval order P_I has all the relations of $\to$ but has no relations of $\dashrightarrow$. Hence, Algorithm 1 gives an interval representation of P_I with $P = L \cap P_I$. The tests in lines 5 and 12 take $O(n)$ time for each element, and hence the loops on lines 4–9 and 11–16 take $O(n^2)$ time. Therefore the running time of this algorithm is $O(n^2)$, and we have Lemma 3. □

3. Apex orderings

An *orientation* of a graph G is an assignment of a direction to each edge of G. A *transitive orientation* of G is an orientation such that for any three vertices u, v, w of G, if $u \to v$ and $v \to w$ then $u \to w$. A transitively oriented graph is used to represent a partial order $P = (V, \prec_P)$, where an edge $u \to v$ denotes the relation $u \prec_P v$. A graph is called a *comparability graph* if it has a transitive orientation. For a graph $G = (V, E)$, the *complement* of G is the graph $\bar{G} = (V, \bar{E})$ such that for any two vertices $u, v \in V$, $uv \in \bar{E} \iff uv \notin E$. The complement of a comparability graph is called a *cocomparability graph*. The vertex ordering characterizations of these graph classes are known as follows. Recall that if σ is a vertex ordering of G, we use $u <_\sigma v$ to denote that u precedes v in σ.

- A graph $G = (V, E)$ is a comparability graph if and only if there is a vertex ordering σ of G such that for any three vertices $u <_\sigma v <_\sigma w$, if $uv \in E$ and $vw \in E$ then $uw \in E$. In other words, a graph is a comparability graph if and only if it has a vertex ordering that contains no subordering in Fig. 2(b). We call such an ordering a *comparability ordering*.
- A graph $G = (V, E)$ is a cocomparability graph if and only if there is a vertex ordering σ of G such that for any three vertices $u <_\sigma v <_\sigma w$, if $uw \in E$ then $uv \in E$ or $vw \in E$ [15]. In other words, a graph is a cocomparability graph if and only if it has a vertex ordering that contains no subordering in Fig. 2(c). We call such an ordering a *cocomparability ordering*.

Cocomparability graphs form a superclass of trapezoid graphs, and hence, every simple-triangle graph is a cocomparability graph. In order to characterize simple-triangle graphs, we define the following vertex ordering properties.

- Let $C_4 = (u, v, w, x)$ denote a chordless cycle of length 4. A vertex ordering σ of G is said to fulfill the C_4 *rule* if for every cycle C_4 in G, the equivalences $u <_\sigma v \iff w <_\sigma v \iff w <_\sigma x \iff u <_\sigma x$ hold.
- Let $2K_2$ denote the graph consisting of four vertices u, v, w, x whose only edges are uw and vx. A vertex ordering σ of G is said to fulfill the $2K_2$ *rule* if for every subgraph $2K_2$ in G, the equivalences $u <_\sigma v \iff w <_\sigma v \iff w <_\sigma x \iff u <_\sigma x$ hold. We note that the $2K_2$ rule is also used to characterize co-threshold tolerance graphs [20].

Notice that a vertex ordering of G fulfills the C_4 rule if and only if it is a vertex ordering of the complement $\overline{G}$ of G fulfilling of the $2K_2$ rule. These rules characterize the simple-triangle graphs as follows.

Theorem 6. *For a graph G, the following conditions are equivalent:*

  (i) *G is a simple-triangle graph;*
 (ii) *G has a cocomparability ordering fulfilling the C_4 rule;*
(iii) *G has a vertex ordering that contains no subordering in* Figs. 2(c)–2(e)*;*
(iv) *$\overline{G}$ has a comparability ordering fulfilling the $2K_2$ rule;*
 (v) *$\overline{G}$ has a vertex ordering that contains no subordering in* Figs. 2(b), 2(d) *and* 2(e)*.*

Proof. It is obvious that **(ii)** $\iff$ **(iv)** and **(iii)** $\iff$ **(v)**.

  (i) $\implies$ **(ii):** It suffices to show that the apex ordering σ of a simple-triangle graph $G = (V, E)$ is a cocomparability ordering since an apex ordering is known to fulfill the C_4 rule [5] (see also Lemma 2). Suppose that G has three vertices $u <_\sigma v <_\sigma w$ with $uw \in E$. Let $T(v)$ denote the triangle of a vertex $v \in V$ in the representation of G. Since $u <_\sigma v <_\sigma w$ and $T(u) \cap T(w) \neq \emptyset$, the triangle $T(v)$ must intersect $T(u)$ or $T(w)$. Hence, we have $uv \in E$ or $vw \in E$.

  (iv) $\implies$ **(i):** Let σ be a comparability ordering of $\overline{G} = (V, \overline{E})$ fulfilling the $2K_2$ rule. From $\overline{G}$, we can obtain the partial order P if we orient the edges of $\overline{G}$ transitively so that $u \to v \iff u <_\sigma v$ since σ is a comparability ordering. Since σ fulfills the $2K_2$ rule, σ is also a linear extension of P fulfilling the **2 + 2** rule. By Theorem 1, P is a linear-interval order, and hence, G is a simple-triangle graph.

  (ii) $\implies$ **(iii):** Let σ be a cocomparability ordering of $G = (V, E)$ fulfilling the C_4 rule. The ordering σ contains no subordering in Fig. 2(c) since σ is a cocomparability ordering. Suppose for a contradiction that there are four vertices $u <_\sigma v <_\sigma w <_\sigma x$ on σ that induce a subordering in Fig. 2(d). We have $uv \in E$ for otherwise the vertices $u <_\sigma v <_\sigma w$ would induce a subordering in Fig. 2(c). We also have $wx \in E$ for otherwise the vertices $v <_\sigma w <_\sigma x$ would induce a subordering in Fig. 2(c). Hence, the vertices $u <_\sigma v <_\sigma w <_\sigma x$ induce C_4 that violates the C_4 rule, a contradiction. Similarly, suppose for a contradiction that there are four vertices $u <_\sigma v <_\sigma w <_\sigma x$ on σ that induce a subordering in Fig. 2(e). We have $uv \in E$ for otherwise the vertices $u <_\sigma v <_\sigma x$ would induce a subordering in Fig. 2(c). We also have $wx \in E$ for otherwise the vertices $u <_\sigma w <_\sigma x$ would induce a subordering in Fig. 2(c). Hence, the vertices $u <_\sigma v <_\sigma w <_\sigma x$ induce C_4 that violates the C_4 rule, a contradiction.

  (ii) $\impliedby$ **(iii):** Let σ be a vertex ordering that contains no subordering in Figs. 2(c)–2(e). The ordering σ is a cocomparability ordering since σ contains no subordering in Fig. 2(c). We can verify that any four vertices of C_4 that violate the C_4 rule induce the subordering in either Figs. 2(d) or 2(e). Hence, σ fulfills the C_4 rule.

  We can also prove **(iv)** $\iff$ **(v)** by a similar argument in the proof of **(ii)** $\iff$ **(iii)**. $\square$

We can also describe the characterization in terms of orientations of graphs. An orientation of a graph is called *acyclic* if it has no directed cycle. An orientation of a graph is called *alternating* if it is transitive on every chordless cycle of length greater than or equal to 4, that is, the directions of the oriented edges alternate. A graph is called *alternately orientable* [14] if it has an alternating orientation. Since cocomparability graphs have no chordless cycle of length greater than 4 (see [2,9] for example), we have the following from Theorem 6.

Corollary 7. *A graph G is a simple-triangle graph if and only if there is an alternating orientation of G and a transitive orientation of the complement $\overline{G}$ of G such that the union of the oriented edges of G and $\overline{G}$ forms an acyclic orientation of the complete graph.*

Moreover, we have the following from Theorem 1 since being a linear-interval order is a comparability invariant [3] (that is, any transitive orientation of the complement of a simple-triangle graph gives a linear-interval order).

Corollary 8. *Let G be a simple-triangle graph. For any transitive orientation of the complement $\overline{G}$ of G, there is an alternating orientation of G such that the union of the oriented edges of G and $\overline{G}$ forms an acyclic orientation of the complete graph.*

4. Concluding remarks

We have shown a vertex ordering characterization of simple-triangle graphs based on the ordering of the apices of the triangles in the representation. We conclude this paper with some miscellaneous topics related to this characterization.

Corollary 7 indicates that a simple-triangle graph is a cocomparability graph that has an alternating orientation [8], but we can see the converse is not true. The separating example is the graph $\overline{W}$ in Fig. 4(b). This graph $\overline{W}$ has a unique alternating orientation (up to reversal), and the complement of $\overline{W}$ has the unique transitive orientation (up to reversal) whose Hasse

(a) The order W.   (b) The graph $\overline{W}$.
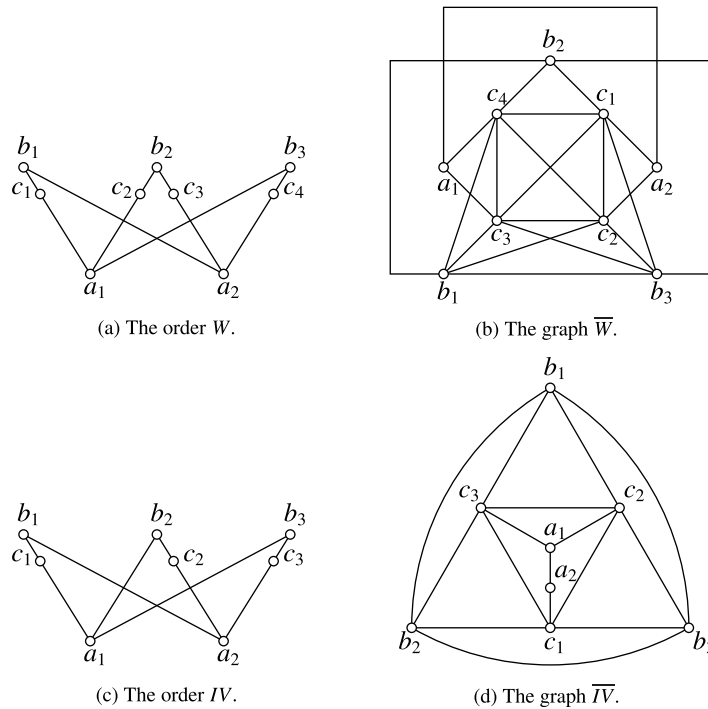
(c) The order IV.   (d) The graph $\overline{IV}$.

Fig. 4. The partial orders and the graphs.

diagram is shown in Fig. 4(a). Suppose that $a_1 \to a_2$. Then the cycle (a_1, a_2, c_2, c_3) requires that $a_1 \to a_2 \iff c_2 \to a_2$, while the cycles (a_1, a_2, c_1, c_3), (c_1, c_3, b_1, b_2), and (c_1, c_2, b_1, b_2) require that $a_1 \to a_2 \iff c_1 \to c_3 \iff b_1 \to b_2 \iff b_1 \to c_2$. Hence, we have a directed cycle (b_1, c_2, a_2) in the union of the oriented edges of G and $\overline{G}$. Suppose on the contrary that $a_1 \leftarrow a_2$. Then the cycle (a_1, a_2, c_2, c_3) requires that $a_1 \leftarrow a_2 \iff a_1 \leftarrow c_3$, while the cycles (a_1, a_2, c_2, c_4), (c_2, c_4, b_2, b_3), and (c_3, c_4, b_2, b_3) require that $a_1 \leftarrow a_2 \iff c_2 \leftarrow c_4 \iff b_2 \leftarrow b_3 \iff c_3 \leftarrow b_3$. Hence, we have a directed cycle (b_3, c_3, a_1) in the union of the oriented edges of G and $\overline{G}$, and Corollary 7 indicates that $\overline{W}$ is not a simple-triangle graph.

A graph is a permutation graph if and only if it is simultaneously a comparability graph and a cocomparability graph (see [10] for example). A permutation graph G is known to have a unique transitive orientation (up to reversal) when the complement $\overline{G}$ of G has a unique transitive orientation. This derives the polynomial-time algorithm for testing isomorphism of permutation graphs [4]. Hence, it is interesting to ask whether a simple-triangle graph G has a unique alternating orientation when the complement $\overline{G}$ of G has a unique transitive orientation (up to reversal). We give the negative answer to this question. The graph $\overline{IV}$ in Fig. 4(d) is alternately orientable but does not have a unique alternating orientation since we can reverse the orientation of edges on the cycle (b_2, b_3, c_2, c_3), while the complement of $\overline{IV}$ has the unique transitive orientation (up to reversal) whose Hasse diagram is shown in Fig. 4(c).

We finally pose two open questions for simple-triangle graphs. The first question is related to the recognition problem. The polynomial-time recognition algorithm is already known [18,25], but the running time of it is $O(n^2\bar{m})$, where n and $\bar{m}$ is the number of vertices and non-edges of the graph, respectively. The algorithm reduces the recognition to a problem of covering an associated bipartite graph by two chain graphs with additional conditions. Our first question is whether we can recognize simple-triangle graphs in polynomial time by using the vertex ordering characterization in Theorem 6?

Problem 1. By using the vertex ordering characterization of simple-triangle graphs, find a recognition algorithm faster than the existing ones [18,25].

The second question is related to the isomorphism problem. A *canonical ordering* of a graph G is a vertex ordering of G such that every graph that is isomorphic to G has the same canonical ordering as G. Hence, the graph isomorphism problem can be solved by computing the canonical orderings of the two given graphs and testing whether these two ordered graphs are identical. Our second question is whether there is any canonical ordering of simple-triangle graphs based on the vertex ordering characterization in Theorem 6?

Problem 2. By using the vertex ordering characterization of simple-triangle graphs, find a canonical ordering computable in polynomial time.

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References

[1] K.P. Bogart, J.D. Laison, S.P. Ryan, Triangle, parallelogram, and trapezoid orders, Order 27 (2) (2010) 163–175.
[2] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph Classes: A Survey, SIAM, Philadelphia, PA, USA, 1999.
[3] M.R. Cerioli, F. de S. Oliveira, J.L. Szwarcfiter, Linear-interval dimension and PI orders, Electron. Notes Discrete Math. 30 (2008) 111–116.
[4] C.J. Colbourn, On testing isomorphism of permutation graphs, Networks 11 (1) (1981) 13–21.
[5] D.G. Corneil, P.A. Kamula, Extensions of permutation and interval graphs, Congr. Numer. 58 (1987) 267–275.
[6] D.G. Corneil, J. Stacho, Vertex ordering characterizations of graphs of bounded asteroidal number, J. Graph Theory 78 (1) (2015) 61–79.
[7] S. Dantas, C.M.H. de Figueiredo, M.C. Golumbic, S. Klein, F. Maffray, The chain graph sandwich problem, Ann. Oper. Res. 188 (1) (2011) 133–139.
[8] S. Felsner, Tolerance graphs, and orders, J. Graph Theory 28 (3) (1998) 129–140.
[9] T. Gallai, Transitiv orientierbare graphen, Acta Math. Acad. Sci. Hungar. 18 (1–2) (1967) 25–66.
[10] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, second ed., in: Ann. Discrete Math., vol. 57, Elsevier B.V., Amsterdam, The Netherlands, 2004.
[11] M.C. Golumbic, H. Kaplan, R. Shamir, Graph sandwich problems, J. Algorithms 19 (3) (1995) 449–473.
[12] M.C. Golumbic, A.N. Trenk, Tolerance Graphs, Cambridge Univ. Press, 2004.
[13] P. Hell, B. Mohar, A. Rafiey, Ordering without forbidden patterns, in: A. Schulz, D. Wagner (Eds.), ESA 2014, in: LNCS, vol. 8737, Springer, Berlin, Heidelberg, 2014, pp. 554–565.
[14] C.T. Hoàng, Alternating orientation and alternating colouration of perfect graphs, J. Combin. Theory Ser. B 42 (3) (1987) 264–273.
[15] D. Kratsch, L. Stewart, Domination on cocomparability graphs, SIAM J. Discrete Math. 6 (3) (1993) 400–417.
[16] N. Mahadev, U. Peled, Threshold Graphs and Related Topics, in: Ann. Discrete Math., vol. 56, Elsevier Science B.V., Amsterdam, The Netherlands, 1995.
[17] T.A. McKee, F.R. McMorris, Topics in Intersection Graph Theory, SIAM, Philadelphia, PA, USA, 1999.
[18] G.B. Mertzios, The recognition of simple-triangle graphs and of linear-interval orders is polynomial, SIAM J. Discrete Math. 29 (3) (2015) 1150–1185.
[19] G.B. Mertzios, D.G. Corneil, Vertex splitting and the recognition of trapezoid graphs, Discrete Appl. Math. 159 (11) (2011) 1131–1147.
[20] C.L. Monma, B.A. Reed, W.T. Trotter, Threshold tolerance graphs, J. Graph Theory 12 (3) (1988) 343–362.
[21] S. Olariu, An optimal greedy heuristic to color interval graphs, Inf. Process. Lett. 37 (1) (1991) 21–25.
[22] T. Raschle, K. Simon, Recognition of graphs with threshold dimension two, in: Proc. STOC 1995, 1995, pp. 650–661.
[23] J.P. Spinrad, Efficient graph representations, in: Fields Institute Monographs, vol. 19, AMS, Providence, RI, USA, 2003.
[24] A. Takaoka, Graph isomorphism completeness for trapezoid graphs, IEICE Trans. Fundamentals E98–A (8) (2015) 1838–1840.
[25] A. Takaoka, Recognizing simple-triangle graphs by restricted 2-chain subgraph cover, in: S.H. Poon, M.S. Rahman, H.C. Yen (Eds.), WALCOM 2017, in: LNCS, vol. 10167, Springer International Publishing, 2017, pp. 177–189.
[26] R. Uehara, The graph isomorphism problem on geometric graphs, Discrete Math. Theor. Comput. Sci. 16 (2) (2014) 87–96.