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A bound on the inducibility of cycles

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ABSTRACT

In 1975, Pippenger and Golumbic conjectured that every n-vertex graph has at most $n^k/(k^k - k)$ induced cycles of length $k \geq 5$. We prove that every n-vertex graph has at most $2n^k/k^k$ induced cycles of length k.

1. Introduction

The study of the number of induced copies of a given graph is a classical topic in extremal combinatorics, which can be traced back to the work of Pippenger and Golumbic [9] from 1975. The *induced density* of a graph H in a graph G, which is denoted by $i(H,G)$, is the number of induced copies of H in G divided by $\binom{|V(G)|}{|V(H)|}$.

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A standard averaging argument shows that for all graphs H and G and all integers $|V(H)| \leq n < |V(G)|$, there exists an n-vertex graph G' such that $i(H,G') \geq i(H,G)$. It follows that the sequence $i(H,n)$ is monotone non-increasing in n, and hence it converges for every H. The *inducibility* of a graph H, which is denoted by $\mathrm{ind}(H)$, is the limit of the sequence $i(H,n)$ where $i(H,n)$ is the maximum induced density of H in an n-vertex graph.

Pippenger and Golumbic [9] showed that the inducibility of every k-vertex graph H is at least $k!/(k^k - k)$ and conjectured that this bound is tight for a cycle of length $k \geq 5$.

Conjecture 1 *(Pippenger and Golumbic [9]). The inducibility of a cycle C_k of length $k \geq 5$ is equal to $\frac{k!}{k^k-k}$.*

In the recent years, the flag algebra method of Razborov [10] led to new bounds on the inducibility of small graphs [1,7], which included the proof of Conjecture 1 for $k = 5$ by Balogh et al. [1]. Other classes of graphs for which the inducibility has been determined include sufficiently balanced complete multipartite graphs [2–4,9] and sufficiently large balanced blow-ups of arbitrary graphs [5].

Motivated by Conjecture 1, we study the inducibility of cycles and provide a new upper bound. In their original paper, Pippenger and Golumbic [9] proved Conjecture 1 within a multiplicative factor of $2e$, i.e., they proved that

$$\mathrm{ind}(C_k) \leq \frac{2k!}{k(k-1)^{k-1}} = (2e + o(1))\,\frac{k!}{k^k}.$$

The multiplicative factor $2e$ has recently been improved to $128e/81$ by Hefetz and Tyomkyn [6] and to e by Pfender and Phillips [8]. Our main result reads as follows.

Theorem 1. *Every n-vertex graph G contains at most $2n^k/k^k$ induced copies of a cycle C_k of length $k \geq 5$.*

This attains the bound of Conjecture 1 up to a multiplicative factor of 2, i.e., we show that

$$\mathrm{ind}(C_k) \leq (2 + o(1))\,\frac{k!}{k^k}. \tag{1}$$

We remark that we convinced ourselves that more detailed arguments could be used to improve the multiplicative factor 2 in (1) to $2 - \varepsilon$ for some tiny $\varepsilon > 0$ but we do not include further details to keep this note short and easily accessible.

2. Proof of Theorem 1

The rest of the paper is devoted to the proof of Theorem 1. Fix an n-vertex graph G and an integer $k \geq 5$. Instead of counting the number of induced copies of C_k, we will count the number of k-tuples of vertices $(z_1, z_2, z_3, z_4, \ldots, z_k)$ such that $z_2 z_1 z_3 z_4 \cdots z_k$ is an induced cycle of length k in G; we call such a k-tuple *good*. We define a weight $w(D)$ of a good k-tuple $D = (z_1, \ldots, z_k)$ as

$$w(D) = \prod_{i=1}^{k} \frac{1}{n_i} \,,$$

where

- n_1 is n,
- n_2 is the number of neighbors of z_1,
- n_3 is the number of neighbors of z_1 that are not neighbors of z_2,
- n_i for $i = 4, \ldots, k-1$ is the number of vertices x such that $z_2 z_1 z_3 z_4 \cdots z_{i-1} x$ is an induced path of length i, and
- n_k is the number of vertices x such that $z_2 z_1 z_3 z_4 \cdots z_{k-1} x$ is an induced cycle of length k.

In other words, n_i is the number of ways that we can extend the $(i-1)$-tuple $(z_1, \ldots, z_{i-1})$ by adding a vertex x in a way that can eventually result in a good k-tuple.

The backward induction on m yields that the total weight of good k-tuples starting with the vertices $z_1, \ldots, z_m$ is at most $(n_1 \cdots n_m)^{-1}$. So, we get the following lemma for $m = 0$. We remark that the lemma can also be proven by considering a carefully chosen probability distribution on some ℓ-tuples, for $\ell < k$, and good k-tuples of vertices of G such that the probability of choosing a good k-tuple D is $w(D)$.

Lemma 2. *The sum of the weights $w(D)$ of all good k-tuples D is at most* 1*.*

We continue the proof of Theorem 1. Consider an induced cycle $v_1 v_2 v_3 \cdots v_k$ of length k in G, and define D_j to be the good k-tuple $(v_j, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_{j+k-2})$ for $j = 1, \ldots, k$ (indices are modulo k). We will show that

$$\frac{k^k}{4n^k} \leq w(D_1) + \cdots + w(D_k) \,. \tag{2}$$

The inequality (2) implies that the sum of the $2k$ good k-tuples corresponding to a single induced cycle of length k is at least $\frac{k^k}{2n^k}$. Since the sum of all such k-tuples is at most 1 by Lemma 2, the number of induced cycles of length k in G is at most $\frac{2n^k}{k^k}$. Hence, the proof of Theorem 1 will be completed when we establish (2).

We now focus on proving (2) and start with applying the AM-GM inequality.

$$\left(\prod_{j=1}^{k} w(D_j)\right)^{\frac{1}{k}} \leq \frac{w(D_1) + \cdots + w(D_k)}{k} \tag{3}$$

Let $n_{j,i}$ be the quantity n_i appearing in the definition of the weight $w(D_j)$. We obtain the following estimate using the definition of the weight $w(D_j)$, the identity $n_{j,1} = n$ and the AM-GM inequality.

$$\begin{aligned}\left(\prod_{j=1}^{k}\frac{1}{w(D_j)}\right)^{\frac{1}{k(k-1)}} &= \left(\prod_{j=1}^{k} 4n_{j,1}\frac{n_{j,2}}{2}\frac{n_{j,3}}{2}n_{j,4}\cdots n_{j,k}\right)^{\frac{1}{k(k-1)}}\\ &= \left((4n)^k\prod_{j=1}^{k}\frac{n_{j,2}}{2}\frac{n_{j,3}}{2}n_{j,4}\cdots n_{j,k}\right)^{\frac{1}{k(k-1)}}\\ &\le \frac{(4n)^{\frac{1}{k-1}}}{k(k-1)}\sum_{j=1}^{k}\frac{n_{j,2}}{2}+\frac{n_{j,3}}{2}+n_{j,4}+\cdots+n_{j,k}\ . \end{aligned} \tag{4}$$

We next establish that each vertex x contributes at most $k-1$ to the sum in (4).

We start with showing that each vertex x contributes at most 1 to the sum $\frac{n_{j,2}}{2} + \frac{n_{j,3}}{2} + n_{j,4} + \cdots + n_{j,k}$ for every $j = 1, \ldots, k$. By symmetry, it is enough to analyze the case $j = 1$. Let i be the smallest index such that x is adjacent to v_i. If $i = 1$, then x can contribute only to $n_{1,2}$ and $n_{1,3}$, and if $i = 2$, then only to $n_{1,k}$. If $i = 3, \ldots, k-2$, then x can contribute only to $n_{1,i+1}$. Finally, if $i > k-2$ or x is not adjacent to any vertex v_i, then x does not contribute to any of the summands. Since the contribution of a vertex x to the sum in (4) is at most 1 for every j, the total contribution of x to the sum in (4) is at most k; we improve this bound by 1 in the next paragraph.

Fix a vertex x. If the vertex x is adjacent to all the vertices $v_1, \ldots, v_k$, then x contributes $1/2$ to the sum $\frac{n_{j,2}}{2} + \frac{n_{j,3}}{2} + n_{j,4} + \cdots + n_{j,k}$ for every j, and its total contribution to the whole sum in (4) is at most $k/2 < k-1$. Otherwise, let i be the smallest index such that x is adjacent to v_{i-1} but not to v_i (all indices in this paragraph are modulo k). If x is adjacent to any of the vertices $v_{i+1}, \ldots, v_{i+k-4}$ or it is not adjacent to the vertex $v_{i+k-3} = v_{i-3}$, then the contribution of x to the sum for $j = i$ is 0. Hence, it remains to analyze the following two cases:

- x is adjacent to the vertices v_{i-3} and v_{i-1} only, and
- x is adjacent to the vertices v_{i-3}, v_{i-2} and v_{i-1} only.

In the former case, the contribution of x to the sum for $j = i-2$ is 0, and in the latter case, the contribution of x to the sum for $j = i-2$ and for $j = i-1$ is $1/2$. We conclude that the contribution of each vertex x to the sum in (4) is at most $k-1$.

Since the contribution of each vertex x to the sum in (4) is at most $k-1$, the whole sum is at most $n(k-1)$ and we derive the following from (4).

$$\left(\prod_{j=1}^{k}\frac{1}{w(D_j)}\right)^{\frac{1}{k(k-1)}} \le \frac{(4n)^{\frac{1}{k-1}}}{k(k-1)}\cdot n(k-1) = \frac{(4n)^{\frac{1}{k-1}}n}{k}$$

It follows that

$$\left(\prod_{j=1}^{k}\frac{1}{w(D_j)}\right)^{\frac{1}{k}} \le \frac{4n^k}{k^{k-1}}\,,$$

which is equivalent to

$$\frac{k^{k-1}}{4n^k} \le \left(\prod_{j=1}^{k} w(D_j)\right)^{\frac{1}{k}}\,. \qquad (5)$$

The desired estimate (2) now follows from (3) and (5).

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