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Minimum density of identifying codes of king grids [1]

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Abstract

A set $C \subseteq V(G)$ is an *identifying code* in a graph G if for all $v \in V(G)$, $C[v] \neq \emptyset$, and for all distinct $u, v \in V(G)$, $C[u] \neq C[v]$, where $C[v] = N[v] \cap C$ and $N[v]$ denotes the closed neighbourhood of v in G. The minimum density of an identifying code in G is denoted by $d^*(G)$. In this paper, we study the density of king grids which are strong product of two paths. We show that for every king grid G, $d^*(G) \geq 2/9$. In addition, we show this bound is attained only for king grids which are strong products of two infinite paths. Given $k \geq 3$, we denote by $\mathcal{K}_k$ the (infinite) king strip with k rows. We prove that $d^*(\mathcal{K}_3) = 1/3$, $d^*(\mathcal{K}_4) = 5/16$, $d^*(\mathcal{K}_5) = 4/15$ and $d^*(\mathcal{K}_6) = 5/18$. We also prove that $\frac{2}{9} + \frac{8}{81k} \leq d^*(\mathcal{K}_k) \leq \frac{2}{9} + \frac{4}{9k}$ for every $k \geq 7$.

Keywords: Identifying code, King grid, Discharging Method.

1 Introduction

Let G be a graph. The *neighbourhood* of a vertex v of G, denoted by $N(v)$, is the set of vertices adjacent to v in G, and the *closed neighbourhood* of v is the

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set $N[v] = N(v) \cup \{v\}$. Given a set $C \subseteq V(G)$, let $C[v] = N[v] \cap C$. We say that C is an *identifying code* of G if $C[v] \neq \emptyset$ for all $v \in V(G)$, and $C[u] \neq C[v]$ for all distinct $u, v \in V(G)$. Clearly, a graph G has an identifying code if and only if it contains no *twins* (vertices $u, v \in V(G)$ with $N[u] = N[v]$).

Let G be a (finite or infinite) graph with bounded maximum degree. For any non-negative integer r and vertex v, we denote by $B_r(v)$ the ball of radius r in G centered at v, that is $B_r(v) = \{x \mid \text{dist}(v, x) \leq r\}$. For any set of vertices $C \subseteq V(G)$, the *density* of C in G, denoted by $d(C, G)$, is defined by

$$d(C,G) = \limsup_{r \to +\infty} \frac{|C \cap B_r(v_0)|}{|B_r(v_0)|} ,$$

where v_0 is an arbitrary vertex in G. The infimum of the density of an identifying code in G is denoted by $d^*(G)$. Observe that if G is finite, then $d^*(G) = |C^*|/|V(G)|$, where C^* is a minimum-size identifying code of G.

The problem of finding low-density identifying codes was introduced in [13] in relation to fault diagnosis in arrays of processors. Particular interest was dedicated to grids as many processor networks have a grid topology. Many results have been obtained on square grids [4,1,10,2,12], triangular grids [13,11], and hexagonal grids [6,8,9]. In this paper, we study *king grids*, which are strong products of two paths. The *strong product* of two graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ and edge set :

$$\begin{aligned} E(G \boxtimes H) = \quad & \{(a,b)(a,b') \mid a \in V(G) \text{ and } bb' \in E(H)\} \\ & \cup \{(a,b)(a',b) \mid aa' \in E(G) \text{ and } b \in V(H)\} \\ & \cup \{(a,b)(a',b') \mid aa' \in E(G) \text{ and } bb' \in E(H)\}. \end{aligned}$$

The *two-way infinite path*, denoted by $P_{\mathbb{Z}}$, is the graph with vertex set $\mathbb{Z}$ and edge set $\{\{i, i+1\} \mid \in \mathbb{Z}\}$, and the *one-way infinite path*, denoted by $P_{\mathbb{N}}$, is the graph with vertex set $\mathbb{N}$ and edge set $\{\{i, i+1\} \mid i \in \mathbb{N}\}$. A *path* is a connected subgraph of $P_{\mathbb{Z}}$. For every positive integer k, P_k is the subgraph of $P_{\mathbb{Z}}$ induced by $\{1, 2, \ldots, k\}$. A *king grid* is the strong product of two (finite or infinite) paths. The *plane king grid* is $\mathcal{G}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{Z}}$, the *half-plane king grid* is $\mathcal{H}_K = P_{\mathbb{Z}} \boxtimes P_{\mathbb{N}}$, the *quarter-plane king grid* is $\mathcal{Q}_K = P_{\mathbb{N}} \boxtimes P_{\mathbb{N}}$, and the *king strip of height* k is $\mathcal{K}_k = P_{\mathbb{Z}} \boxtimes P_k$.

In 2001, Cohen et al. [7] proved that $d^*(\mathcal{G}_K) \geq 2/9$. In 2002, Charon et al. [3] obtained an optimal identifying code with density 2/9. They provided the tile depicted in Fig. 1, which generates a periodic tiling of the plane with periods $(0, 6)$ and $(6, 0)$, yielding an identifying code C_∞ of the bidimensional infinite king grid with density $\frac{2}{9}$.

In this paper, using the Discharging Method (see Section 3 of [11] for a

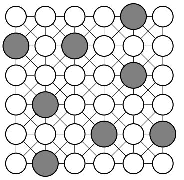

Fig. 1. Tile generating an optimal identifying code of the bidimensional infinite grid. Black vertices are those of the code.

detailed presentation of this technique for identifying codes), we provide the following tight general lower bounds on the minimum density of identifying codes of king grids.

Theorem 1.1 *If G is a (finite or infinite) king grid, then $d^*(G) \geq \frac{2}{9}$.*

Theorem 1.2 *If G is a finite king grid, then $d^*(G) > \frac{2}{9}$.*

Finally, we give some bounds for king strips. Pushing further the proof of Theorem 1.1, we prove the following.

Theorem 1.3 *For every $k \geq 6$, $d^*(\mathcal{K}_k) \geq \frac{2}{9} + \frac{8}{81k}$.*

Changing C_∞, we obtain upper bounds for identifying codes of $\mathcal{K}_k$.

Theorem 1.4 *For every $k \geq 5$,*

$$d^*(\mathcal{K}_k) \leq \begin{cases} \frac{2}{9} + \frac{6}{18k}, & \text{if } k \equiv 0 \mod 3, \\ \frac{2}{9} + \frac{8}{18k}, & \text{if } k \equiv 1 \mod 3, \\ \frac{2}{9} + \frac{7}{18k}, & \text{if } k \equiv 2 \mod 3. \end{cases}$$

Finally, we show some identifying codes of $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5$ and $\mathcal{K}_6$ (see Figs. 2, 3, 4 and 5) and prove that they are optimal. This yields the following.

Theorem 1.5 $d^*(\mathcal{K}_3) = 1/3 = 0.333\ldots$ $\quad d^*(\mathcal{K}_4) = 5/16 = 0.3125$
$d^*(\mathcal{K}_5) = 4/15 = 0.2666\ldots$ $\quad d^*(\mathcal{K}_6) = 5/18 = 0.2777\ldots$

Clearly $d^*(\mathcal{K}_1) = 1/2$ (as $\mathcal{K}_1 = P_\mathbb{Z}$) and $\mathcal{K}_2$ has no identifying code because it has twins. All these results imply that $\mathcal{G}_K$, $\mathcal{H}_K$ and $\mathcal{Q}_K$ are the unique king grids having an identifying code with density 2/9 (one can easily derive from C_∞ identifying codes with density 2/9 of $\mathcal{H}_K$ and $\mathcal{Q}_K$).

2 Sketches of proofs

Sketch of proof of Theorem 1.1. Let C an identifying code of a king grid G. We shall prove that $d(C, G) \geq 2/9$. For this, we use the Discharging Method.

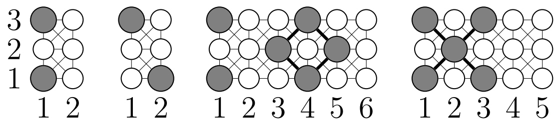

Fig. 2. Four tiles generating optimal identifying codes of $\mathcal{K}_3$ (density 1/3)

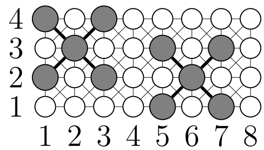

Fig. 3. Tile generating an optimal identifying code of $\mathcal{K}_4$ (density 5/16)

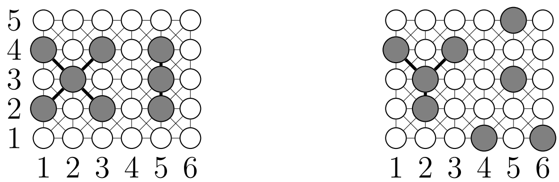

Fig. 4. Two tiles generating optimal identifying codes of $\mathcal{K}_5$ (density 4/15)

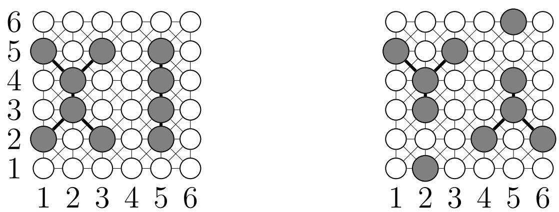

Fig. 5. Two tiles generating optimal identifying codes of $\mathcal{K}_6$ (density 5/18)

The initial charge of a vertex v is 1 if $v \in C$ and 0 otherwise. We apply some local discharging rules. We shall prove that the final charge of every vertex in C is at least 2/9. This would imply the result. Let $U = V(G) \setminus C$. Given $X \subseteq V(G)$ and $i = 1, \ldots, 9$, let X_i (resp. $X_{\geq i}$) the set of vertices $v \in X$ with exactly (resp. at least) i vertices in $N[v] \cap C$. An X*-vertex* is a vertex in X. A vertex is *full* if its 8 neighbours in $\mathcal{G}_K$ are in G; otherwise it is a *side vertex*.

We first prove the following properties of C: (i) two C_2-vertices are not adjacent, (ii) every C-vertex has at most one neighbour in U_1, (iii) every full C_2-vertex has at least three neighbours in $U_{\geq 3}$, (iv) every full C_3-vertex has a neighbour in $U_{\geq 3}$, and (v) every C_1-vertex (a, b) has no neighbour in U_1 and at most six neighbours in U_2; furthermore, if it has six neighbours in U_2, then either $\{(a-1,b-2),(a-2,b-1),(a+2,b+1),(a+1,b+2)\} \subseteq C$ or $\{(a+1,b-2),(a+2,b-1),(a-2,b+1),(a-1,b+2)\} \subseteq C$.

A *defective vertex* is a vertex in C_1 with six neighbours in U_2. Let $v=(a,b)$ be a defective vertex. The *team* of v is a set among $\{(a-1,b-2),(a-2,b-1),(a+2,b+1),(a+1,b+2)\}$ and $\{(a+1,b-2),(a+2,b-1),(a-2,b+1),(a-1,b+2)\}$ which is included in C. By Property (v), the team exists. Moreover, by Property (i), at least two vertices of the team are in $C_{\geq 3}$. Those vertices are the *partners* of v. We apply the following discharging rules.

(R1) Every C-vertex sends $\frac{2}{9i}$ to each of its neighbours in U_i.

(R2) Every defective vertex receives $\frac{1}{54}$ from each of its partners.

Using the above properties, we then prove that the final charge of every vertex v is at least $2/9$. □

Sketch of proof of Theorem 1.2. We only need to prove that, at the end of the proof of Theorem 1.1, one vertex has final charge greater than $2/9$. To do so, we prove that there is a side C-vertex or a $C_{\geq 3}$-vertex and check that such a vertex has final charge at least $\frac{2}{9}+\frac{1}{27}$. □

Sketch of proof of Theorem 1.3. Applying initially the same rules R1 and R2 as in the proof of Theorem 1.1, we prove that in average, for every column, there is an extra charge of at least $\frac{4}{81}$ on the three top vertices and an extra charge of at least $\frac{4}{81}$ on the three bottom vertices. □

Sketch of proof of Theorem 1.4. If $k \equiv 0,2 \mod 3$, let $C'_k = (C_\infty \cap \mathbb{Z} \times [k]) \cup \{(6a+2,3),(6a+5,3),(6a+2,k-2),(6a+5,k-2) | a \in \mathbb{Z}\}$. If $k \equiv 1 \mod 3$, let $C'_k = (C_\infty \cap \mathbb{Z} \times \{2,\dots,k+1\}) \cup \{(6a+2,4),(6a+5,4),(6a+2,k-1),(6a+5,k-1) | a \in \mathbb{Z}\}$. One can check that C'_k is an identifying code of $\mathcal{K}_k$ when $k \equiv 0,2 \mod 3$, and that C'_k is an identifying code of the strip induced by the rows 2 to $k+1$ (which is isomorphic to $\mathcal{K}_k$), with the desired densities. As an example, C'_5 and C'_6 are the 2nd identifying codes of Figs. 4 and 5. □

Sketch of proof of Theorem 1.5. The bth row of $\mathcal{K}_k$ is $R_b = \{(a,b) \mid a \in \mathbb{Z}\}$. We have $d(C,\mathcal{K}_k) = \frac{1}{k}\sum_{i=1}^{k} d(C,R_i)$. We show that if C is an identifying code of $\mathcal{K}_k$ ($k \geq 3$), then $d(C,R_1)+d(C,R_2) \geq 1/2$, $d(C,R_k)+d(C,R_{k-1}) \geq 1/2$, $d(C,R_3) \geq 1/3$ and $d(C,R_{k-2}) \geq 1/3$. As a consequence, we obtain that, if C is an identifying code of $\mathcal{K}_5$ (resp. $\mathcal{K}_6$), then $d(C,\mathcal{K}_5) \geq 4/15$. (resp. $d(C,\mathcal{K}_6) \geq 5/18$.) To prove lower bounds on $d^*(\mathcal{K}_k)$ for $k \in \{3,4\}$, we use the Discharging Method on the columns $Q_a = \{(a,b) \mid 1 \leq b \leq k\}$, $a \in \mathbb{Z}$. Let C be an identifying code of $\mathcal{K}_k$. We set the initial charge of every integer $a \in \mathbb{Z}$ to $\mathrm{chrg}_0(a) = |Q_a \cap C|$. We say that $a \in \mathbb{Z}$ is *satisfied* if its charge is at least q_k and *unsatisfied* otherwise, where $q_3 = 1$ and $q_4 = 5/4$. We apply

five discharging rules, Rule R_i ($i = 1, \ldots, 5$) one after another. We then prove that, once the rules R_1 to R_5 are applied, every $a \in \mathbb{Z}$ is satisfied. This implies $d(C, \mathcal{K}_k) \geq q_k/k$. Let $\text{chrg}_i(a)$ the charge of a after applying Rule R_i.
(R_i) Every unsatisfied $a \in \mathbb{Z}$ receives $\min\{\text{chrg}_{i-1}(a-i) - q_k, q_k - \text{chrg}_{i-1}(a)\}$ from $a - i$, if $a - i$ is satisfied (before Rule R_i). □

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