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Acute Sets

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Abstract A set of points in $\mathbb{R}^d$ is *acute* if any three points from this set form an acute triangle. In this note we construct an acute set in $\mathbb{R}^d$ of size at least 1.618^d. We also present a simple example of an acute set of size at least $2^{d/2}$. Obtained bounds improve the previously best bound.

Keywords Acute set · Euclidean space · Danzer-Grünbaum problem

1 Introduction

A set of points in $\mathbb{R}^d$ is *acute*, if any three points of this set form an acute triangle. In 1962 Danzer and Grünbaum [2] posed the following question: what is the maximum size $f(d)$ of an acute set in $\mathbb{R}^d$? They proved a linear lower bound $f(d) \geqslant 2d - 1$ and conjectured that this bound is tight. However, in 1983 Erdős and Füredi [3] disproved this conjecture in large dimensions. They gave an exponential lower bound

$$f(d) \geqslant \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^d > 0.5 \cdot 1.154^d. \tag{1}$$

Their proof is a very elegant application of the probabilistic method. One drawback of their approach is that only the existence of an acute set of such size is proven, with no possibility to turn it into an explicit construction.

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In 2009 Ackerman and Ben-Zwi [1] improved (1) by a factor $\sqrt{d}$:

$$f(d) \geqslant c\sqrt{d}\left(\frac{2}{\sqrt{3}}\right)^d.$$

In 2011 Harangi [4] refined the approach of Erdős and Füredi and improved the previous bound to

$$f(d) \geqslant c\left(\sqrt[10]{\frac{144}{23}}\right)^d > c \cdot 1.2^d.$$

In this note we give a simple proof of the following inequality:

Theorem 1.1 *$f(d+2) \geqslant 2f(d)$, that is, for any d-dimensional acute set there exists a $(d+2)$-dimensional acute set of twice the size.*

Theorem 1.1 implies lower bound for $f(d)$:

$$f(d) \geqslant 2^{d/2}.$$

Let F_d be the d-th Fibonacci number, that is $F_0 = F_1 = 1$ and $F_{d+2} = F_{d+1} + F_d$. Also, we prove the following inequality:

Theorem 1.2 *There exist d-dimensional acute sets of size F_{d+1}, that is, $f(d) \geqslant F_{d+1}$.*

Using the formula for Fibonacci numbers we can write an asymptotic inequality for $f(d)$:

$$f(d) \geqslant \left(\frac{1+\sqrt{5}}{2}\right)^d \geqslant 1.618^d.$$

The proofs of Theorems 1.1 and 1.2 are explicit and allow to construct acute sets effectively.

The best known upper bound on $f(d)$ is $f(d) \leqslant 2^d - 1$, and follows from the main result of [2]. Danzer and Grünbaum proved that if a set S of points in $\mathbb{R}^d$ determines only acute and right angles, then $|S| \leqslant 2^d$. Moreover, if $|S| = 2^d$ then S must be an affine image of a d-dimensional cube.

2 Proof of Theorem 1.1

The proofs of both theorems are based on two simple propositions:

Proposition 2.1 *For any points a, b, c such that the angle (a, b, c) is acute, there is $\varepsilon > 0$ such that for all $\tilde{a}, \tilde{b}, \tilde{c}$, $\|a - \tilde{a}\|, \|b - \tilde{b}\|, \|c - \tilde{c}\| < \varepsilon$, the angle $(\tilde{a}, \tilde{b}, \tilde{c})$ is acute too.*

The proof of Proposition 2.1 is trivial and we omit it.

Proposition 2.2 (The key fact) *Suppose that $X \subset \mathbb{R}^d$ is an acute set and $r > 0$ is a real number such that $\langle x - y, x - z\rangle > 4r^2$ holds for any $x, y, z \in X$, where $x \neq y$, $x \neq z$. For each $x \in X$ we take an arbitary point $\phi(x) \in \mathbb{R}^2$ on the circle of radius r with center in the origin such that all points $\pm\phi(x)$ are different. Then the set $Y = \{(x, \pm\phi(x)) \mid x \in X\} \subset \mathbb{R}^{d+2}$ is acute as well.*

To prove Theorem 1.1, we apply Proposition 2.2 to a maximal acute set X in $\mathbb{R}^d$, $|X| = f(d)$. We get an acute set $Y \subset \mathbb{R}^{d+2}$ of size $|Y| = 2|X|$ which proves the theorem.

Proof of Proposition 2.2 The scalar product of two vectors u, v is denoted by $\langle u, v\rangle$. Put

$$s := \min\{\langle y - x, z - x\rangle \mid x, y, z \in X, x \neq y, x \neq z\}.$$

Since the set X is acute, we have $s > 0$, and we can take a positive number r such that $4r^2 < s$.

Our aim is to prove that Y is acute. Take three distinct points $\tilde{x}, \tilde{y}, \tilde{z} \in Y$, where

$$\tilde{x} = (x, a\phi(x)), \quad \tilde{y} = (y, b\phi(y)), \quad \tilde{z} = (z, c\phi(z)), \quad a, b, c \in \{\pm 1\}.$$

Suppose that $x \neq y$ and $x \neq z$. Then

$$\langle \tilde{y} - \tilde{x}, \tilde{z} - \tilde{x}\rangle = \langle y - x, z - x\rangle + \langle b\phi(y) - a\phi(x), c\phi(z) - a\phi(x)\rangle.$$

The first scalar product on the right hand side is at least s by the definition of s, while the second scalar product is at least $-4r^2$. By the choice of r, the sum of these two scalar products is positive, which means that the angle $(\tilde{y}, \tilde{x}, \tilde{z})$ is acute.

Suppose that $x = y$ (the case $x = z$ is treated in the same way). We have $a + b = 0$, so

$$\begin{aligned}\langle \tilde{y} - \tilde{x}, \tilde{z} - \tilde{x}\rangle &= \langle b\phi(y) - a\phi(x), c\phi(z) - a\phi(x)\rangle = \langle 2a\phi(x), a\phi(x) - c\phi(z)\rangle \\ &= 2\big(\|\phi(x)\|^2 \pm \langle \phi(x), \phi(z)\rangle\big) > 0,\end{aligned}$$

because $\phi(x) \neq \pm\phi(z)$. Thus, the angle $(\tilde{y}, \tilde{x}, \tilde{z})$ is acute in this case as well. □

3 Proof of Theorem 1.2

Sketch of the proof. We prove that there exists a d-dimensional acute set of size F_{d+1} with the property that there exists a $(d-1)$-dimensional hyperplane $H \subset \mathbb{R}^d$ such that H contains F_d points and the remaining F_{d-1} points are on the same side of H.

The proof is by induction. The basic idea is the same as in the first construction: we want to replace a point v with two points $v \pm \phi(v)$. However, this time we have only one extra dimension. So we do this only for the points on the hyperplane H. It is not hard to see that if we choose the vectors $\phi(v)$ carefully, then this results in a $(d+1)$-dimensional acute set of size F_{d+2}.

To get the "hyperplane property", one needs to modify this procedure a bit. At first we construct an auxiliary set, which is almost what we need: hyperplane and size properties hold but some angles of this set became right. We correct them by perturbing the set and by invoking Proposition 2.2.

We will derive by induction Theorem 1.2 from the following lemma:

Lemma 3.1 *Suppose that $X \subset \mathbb{R}^d$ is an acute set and h is a hyperplane such that X lies on one side of h. Then there is an acute set $X' \subset \mathbb{R}^{d+1}$ and a hyperplane H in $\mathbb{R}^{d+1}$ such that $|X'| = |X| + |X \cap h|$, $|X' \cap H| = |X|$ and X' lies on one side of H.*

Proof of Theorem 1.2 For $d = 1$ we take $X = \{0, 1\} \subset \mathbb{R}^1$ and a hyperplane $h = \{x \in \mathbb{R}^1 \mid x = 0\}$. Clearly, $|X| = F_2$, $|X \cap h| = F_1$ and the pair (X, h) satisfies conditions of Lemma 3.1.

Suppose that we constructed an acute set $X \subset \mathbb{R}^d$ and a hyperplane h such that $|X| = F_{d+1}$, $|X \cap h| = F_d$ and X lies on one side of h. Then, by Lemma 3.1, there is an acute set $X' \subset \mathbb{R}^{d+1}$ and a hyperplane H such that

$$|X'| = |X| + |X \cap h| = F_{d+1} + F_d = F_{d+2}, \qquad |X' \cap H| = |X| = F_{d+1}$$

and X' lies on one side of H. So the induction step is completed. □

The proof of the lemma is based on Propositions 2.1 and 2.2.

Proof of Lemma 3.1 By embedding X into $\mathbb{R}^{d+1}$ we can assume that

$$\begin{aligned} h &= \{(x_1, \dots, x_{d-1}, 0, 0) \mid x_i \in \mathbb{R}\}, \\ X \subset P &= \{(x_1, \dots, x_d, 0) \mid x_i \in \mathbb{R}\}. \end{aligned}$$

Let $A = X \cap h$, $B = X \setminus A$.

Consider a $(d-1)$-plane $h_2 \subset P$ parallel to h such that X lies between h and h_2. Let

$$h_3 = h + (0, \dots, 0, r) \subset \mathbb{R}^{d+1},$$

where $r > 0$ is a sufficiently small positive number. Let $H' \subset \mathbb{R}^{d+1}$ be the hyperplane passing through h_2 and h_3. Consider sets $A_+ = A + (0, \dots, 0, 0, r)$, $A_- = A - (0, \dots, 0, 0, r)$ and let B_H be the orthogonal projection of B onto H'.

Proposition 3.2 *For a sufficiently small r and arbitrary $x, y, z \in A_+ \cup A_- \cup B_H$ such that $\{x, y, z\} \not\subset A_+ \cup A_-$, the triangle $\{x, y, z\}$ is acute.*

Proof of Proposition 3.2 The distance between $x \in X$ and any corresponding point $\tilde{x} \in A_+ \cup A_- \cup B_H$ is at most r, thus, by Proposition 2.1, for all sufficiently small r an obtuse angle can occur only in triangles $\{x, y, z\} = \{a_+, a_-, b\}$, where $a_+ = (a, 0, r)$, $a_- = (a, 0, -r)$ and $(a, 0, 0) \in A$, $b \in B_H$. Since the distance between a_+ and a_- equals $2r$, the distances between $a_\pm$ and b are bounded from below by a number not depending on r, therefore, the angle (a_+, b, a_-) is acute for small r. By the choice of H', the point b lies between the hyperplanes $P \pm (0, \dots, 0, r)$, thus the angles $(a, 0, r)$ and $(a, 0, -r)$ of the triangle $\{a_+, a_-, b\}$ are acute too. □

Proposition 3.2 shows that the constructed set $A_+ \cup A_- \cup B_H$ is almost what we need: all the angles of this set except the angles inside $A_+ \cup A_-$ are acute, the set $B_H \cup A_+$ lies in the hyperplane H' and it is obvious that A_- lies on one side of H'. To sort out the problem with the set $A_+ \cup A_-$ we perturb the hyperplane H' and the corresponding points in the following way.

For each $x \in A$ we denote by $C(v) \subset \mathbb{R}^{d+1}$ the circle of radius r with center v and orthogonal to h.

Proposition 3.3 *For any $\varepsilon > 0$ there is a hyperplane H such that:*

1. *For each point $v \in B_H$ the distance from v to H is less than ε.*
2. *For each point $(v, 0, 0) \in A$ there exists a point $\bar{v} = (v, \phi(v)) \in H \cap C(v)$ such that $\|(v, 0, r) - \bar{v}\| < \varepsilon$ and all points $\pm\phi(v)$ are distinct.*

Proof 3.3 Let $u \in \mathbb{R}^{d+1}$ be such that $H' = \{x \in \mathbb{R}^{d+1} | \langle x, u\rangle = 1\}$. Choose a vector $\alpha \in \mathbb{R}^{d+1}$ such that $\|\alpha\| < \delta$ for sufficiently small $\delta > 0$ and α is not orthogonal to any of the vectors $v_1 - v_2$ where $v_1, v_2 \in A$. We take $H = \{x \in \mathbb{R}^{d+1} \mid \langle x, u + \alpha\rangle = 1\}$.

1. For $v \in B_H$

$$\rho(v, H_2) = \frac{|\langle v, u + \alpha\rangle - 1|}{\|u + \alpha\|} \leqslant \frac{|\langle v, \alpha\rangle|}{\|u\| - \delta} \leqslant \delta \frac{\|v\|}{\|u\| - \delta} < \varepsilon$$

as δ is sufficiently small.

2. Consider the intersection l' and l of the hyperplanes H' and H with the 2-dimensional plane $\{(v, x_d, x_{d+1}) \mid x_i \in \mathbb{R}\}$, where $(v, 0, 0) \in A$. Clearly, l' intersects $C(v)$ in two points (one of them is $(v, 0, r)$), and so for small δ the line l intersects $C(x)$ in two points as well, and also one of these points tends to $(v, 0, r)$ as $\delta \to 0$. This point we denote by $(v, \phi(v))$. It is sufficient to show that all points $\pm\phi(v)$ are distinct for $(v, 0, 0) \in A$.

As $\|\phi(v) - (0, r)\| < r$ for all small enough δ, $\phi(v_1) \neq -\phi(v_2)$. Take $\bar{v}_1 = (v_1, 0, 0)$, $\bar{v}_2 = (v_2, 0, 0) \in A$. If $\phi(v_1) = \phi(v_2)$, then

$$(v_1, \phi(v_1)) - (v_1, 0, 0) = (v_2, \phi(v_2)) - (v_2, 0, 0),$$

that is

$$(v_1, \phi(v_1)) - (v_2, \phi(v_2)) = \bar{v}_1 - \bar{v}_2 = \bar{w}$$

but $(v_1, \phi(v_1))$ and $(v_2, \phi(v_2))$ lie in H, consequently $\bar{w}$ is orthogonal to $u + \alpha$ which contradicts the definition of α. Therefore all points $\pm\phi(v)$ are distinct. □

Now we take sufficiently small ε and a corresponding hyperplane H and a map ϕ. Let $\tilde{B}$ be the orthogonal projection of B_H onto H, also let

$$\tilde{A}_+ = \{(x, \phi(x)) \mid (x, 0, 0) \in A\}, \qquad \tilde{A}_- = \{(x, -\phi(x)) \mid (x, 0, 0) \in A\}.$$

Combining Propositions 2.1 and 3.3 we can claim that Proposition 3.2 is still true for corresponding sets $\tilde{A}_+$, $\tilde{A}_-$ and $\tilde{B}$. We have to check that the set $Y = \tilde{A}_+ \cup \tilde{A}_-$

is acute. But this immediately follows from Proposition 2.2. We conclude that the set $X' = \tilde{A}_+ \cup \tilde{A}_- \cup \tilde{B}$ is acute. We have $|X'| = |\tilde{A}_+| + |\tilde{A}_-| + |\tilde{B}| = |X| + |X \cap h|$, also $\tilde{A}_+ \cup \tilde{B} \subset H$ and X' lie on one side of H. Thus, the pair (X', H) has claimed in Lemma 3.1 properties. □

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