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Equilateral Triangles and the Fano Plane

Philippe Caldero and Jérôme Germoni

Abstract. We formulate a definition of equilateral triangles in the complex line that makes sense over the field with seven elements. Adjacency of these abstract triangles gives rise to the Heawood graph, which is a way to encode the Fano plane. Through some reformulation, this gives a geometric construction of the Steiner systems $S(2, 3, 7)$ and $S(3, 4, 8)$. As a consequence, we embed the Heawood graph in a torus, and we derive the exceptional isomorphism $\mathrm{PSL}_2(\mathbb{F}_7) \simeq \mathrm{GL}_3(\mathbb{F}_2)$. The study of equilateral triangles over other finite fields shows that seven is very specific.

1. EQUILATERAL TRIANGLES AND CROSS-RATIO. Three points b, c, and d in $\mathbb{C}$ form an equilateral triangle if and only if the ratio $(d-b)/(c-b)$ is $-j$ or $-j^2$, where j is a primitive cubic root of unity. With a (projective) view to extend the notion to other fields, this can be written as $[\infty, b, c, d] \in \{-j, -j^2\}$, where the bracket denotes the *cross-ratio* of four distinct elements $a, b, c, d \in \mathbb{P}^1(\mathbb{C})$:

$$[a, b, c, d] = \frac{c-a}{d-a} \times \frac{d-b}{c-b}.$$

We use the usual conventions about infinity: If $\alpha \in \mathbb{C}^*$, then $\alpha/0 = \infty$, and if $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are such that $\alpha\delta - \beta\gamma \neq 0$, then $(\alpha\infty + \beta)/(\gamma\infty + \delta) = \alpha/\gamma$.

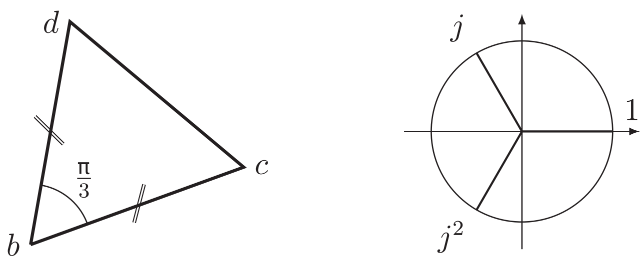

Figure 1. Equilateral triangle in the complex line

Definition. Let $\mathbb{K}$ be a field containing a primitive cubic root of unity j. Then, as in $\mathbb{C}$, there are two such roots that are the roots of $X^2 + X + 1$. An unordered triple $\{b, c, d\}$ (in short: bcd) of distinct points in $\mathbb{K}$ is said to be an *equilateral triangle* if $[\infty, b, c, d] \in \{-j, -j^2\}$.

More generally, an *equianharmonic quadrangle* is a quadruple of distinct points $\{a, b, c, d\}$ in $\mathbb{P}^1(\mathbb{K})$ such as $[a, b, c, d] \in \{-j, -j^2\}$.

Remark. Harmonic quadrangles are a projective substitute for the notion of middle in affine geometry in the following sense. A quadruple $\{a, b, c, d\}$ is harmonic, i.e.,

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MSC: Primary 05E18

$[a, b, c, d] \in \{-1, 2, 1/2\}$, if and only if, when one point is mapped to infinity by a homography, one of the other three is mapped to the middle of the last two. Similarly, a quadrangle is equianharmonic if an only if, when one point is mapped to infinity, the other three are mapped to an equilateral triangle.

These two sets of values are remarkable in the following sense. By permuting the four variables in the cross-ratio $\lambda = [a, b, c, d]$, one obtains six different values, namely λ, $(\lambda - 1)/\lambda$, $1/(1 - \lambda)$, $1/\lambda$, $1 - \lambda$, and $\lambda/(\lambda - 1)$ unless the quadrangle is equianharmonic (respectively, harmonic), in which case there are only two (respectively, three) different values. In particular, in the definition, the order of points and the choice of a cubic root j versus the other (j^2) are irrelevant.

For the sake of completeness, we prove well-known facts about $\mathrm{PGL}_2(\mathbb{K})$ and cross-ratio. Recall that $\mathrm{PGL}_2(\mathbb{K})$ (respectively, $\mathrm{PSL}_2(\mathbb{K})$) is the group of homographies $h : \mathbb{P}^1(\mathbb{K}) \to \mathbb{P}^1(\mathbb{K})$, $z \mapsto (\alpha z + \beta)/(\gamma z + \delta)$ such that $\alpha\delta - \beta\gamma$ is in $\mathbb{K}^*$ (respectively, is a square in $\mathbb{K}^*$). (If every scalar is a square, $\mathrm{PSL}_2(\mathbb{K}) = \mathrm{PGL}_2(\mathbb{K})$.)

Lemma 1. *The group* $\mathrm{PGL}_2(\mathbb{K})$ *acts simply transitively on ordered triples of distinct points of* $\mathbb{P}^1(\mathbb{K})$. *The cross-ratio is invariant under* $\mathrm{PGL}_2(\mathbb{K})$.

If -1 *is not a square in* $\mathbb{K}^*$, *the group* $\mathrm{PSL}_2(\mathbb{K})$ *acts transitively on unordered triples, but it does not act transitively on ordered triples. If* -1 *is a square in* $\mathbb{K}^*$ *and there is an element of* $\mathbb{K}$ *that is not a square, the group* $\mathrm{PSL}_2(\mathbb{K})$ *does not act transitively on unordered triples.*

Proof. Let (a, b, c) be a triple of distinct points. The homography h defined by

$$\forall z \in \mathbb{P}^1(\mathbb{K}), \quad h(z) = \frac{c - a}{c - b} \times \frac{z - b}{z - a}$$

is the unique element in $\mathrm{PGL}_2(\mathbb{K})$ that maps (a, b, c) to $(\infty, 0, 1)$ (if $\infty \in \{a, b, c\}$, simply erase the corresponding factors). Invariance of cross-ratio is easy to check.

Now, assume -1 is not a square. Then the homography s defined by $s(z) = 1 - z$ does not belong to $\mathrm{PSL}_2(\mathbb{K})$ so that either h or sh does. Since s maps $(\infty, 0, 1)$ to $(\infty, 1, 0)$, the unicity claimed above shows that $\mathrm{PSL}_2(\mathbb{K})$ is not transitive on ordered triples. On the other hand, both h and sh map $\{a, b, c\}$ to $\{\infty, 0, 1\}$ and one of them lies in $\mathrm{PSL}_2(\mathbb{K})$. Thus, $\mathrm{PSL}_2(\mathbb{K})$ is transitive on unordered triples.

Eventually, if -1 is a square, then $s \in \mathrm{PSL}_2(\mathbb{K})$. Let S be the order-6 subgroup of $\mathrm{PSL}_2(\mathbb{K})$ generated by s and $t : z \mapsto (0z + 1)/(-z + 1)$. It stabilizes $\{\infty, 0, 1\}$ and every permutation of this set can be realized by an element of S. Let $\lambda \in \mathbb{K}$ for which there is $h \in \mathrm{PSL}_2(\mathbb{K})$ such that $h(\{\infty, 0, \lambda\}) = \{\infty, 0, 1\}$. Then λ is a square, which implies the last statement of the lemma. Indeed, if composing by an element of S if necessary, then one can assume that $h(\infty) = \infty$, $h(0) = 0$, and $h(\lambda) = 1$; hence, $h(z) = z/\lambda$, and, since $h \in \mathrm{PSL}_2(\mathbb{K})$, λ is a square. ■

2. EQUIANHARMONIC QUADRANGLES OVER $\mathbb{F}_7$. Let $\mathbb{K} = \mathbb{F}_7$ be the field with seven elements. Since $7 - 1$ is a multiple of 3, there are two primitive cubic roots of unity in $\mathbb{K}$, namely $j = 2$ and $j^2 = 4$. For example, note that the quadrangles $Q_3 = \{\infty, 0, 1, 3\}$ and $Q_5 = \{\infty, 0, 1, 5\}$ are equianharmonic.

Lemma 2. *There are* 28 *equianharmonic quadrangles in* $\mathbb{P}^1(\mathbb{F}_7)$.

Proof. Observe that the cross-ratio is a homography with respect to every variable. Hence, given three distinct points a, b, c in $\mathbb{P}^1(\mathbb{F}_7)$ and λ in $\mathbb{P}^1(\mathbb{F}_7)$, there is a unique d

such that $[a, b, c, d] = \lambda$. If $\lambda \notin \{\infty, 0, 1\}$, the point d is automatically distinct from a, b, and c. By multiplying the number of triples by the number of admissible values for the cross-ratio, one obtains $8 \times 7 \times 6 \times 2$ ordered quadruples and $8 \times 7 \times 6 \times 2/4! = 28$ equianharmonic quadrangles. ∎

Lemma 3. *Equianharmonic quadrangles form a single orbit under* $\mathrm{PGL}_2(\mathbb{F}_7)$*, and two orbits under* $\mathrm{PSL}_2(\mathbb{F}_7)$.

Proof. Let $Q = \{a, b, c, d\}$ be an equianharmonic quadrangle. Let h be the homography that maps (a, b, c) to $(\infty, 0, 1)$. Then $h(d) = [\infty, 0, 1, h(d)] = [a, b, c, d]$. Since Q is equianharmonic, h maps Q to $\{\infty, 0, 1, 3\}$ or $\{\infty, 0, 1, 5\}$. Since $s : z \mapsto 1 - z$ exchanges these two quadrangles, h or sh maps Q to $\{\infty, 0, 1, 3\}$, which proves the first claim.

Recall that the cardinality of the orbit of $Q_3 = \{\infty, 0, 1, 3\}$ under a group G acting on the set of quadrangles is $|G|/|G_{Q_3}|$, where G_{Q_3} is the stabilizer of Q_3. Besides, the order of $\mathrm{GL}_2(\mathbb{F}_7)$ is the number of bases of $\mathbb{F}_7^2$, that is $(7^2 - 1) \times (7^2 - 7)$; hence, the order of $\mathrm{PGL}_2(\mathbb{F}_7) = \mathrm{GL}_2(\mathbb{F}_7)/\mathbb{F}_7^*\mathrm{I}_2$ is 336. Since $\mathrm{SL}_2(\mathbb{F}_7)$ is the kernel of the determinant, which is surjective, its order is also $|\mathrm{GL}_2(\mathbb{F}_7)|\,/|\mathbb{F}_7^*| = 336$. Finally, since only two scalar matrices lie in $\mathrm{SL}_2(\mathbb{F}_7)$, namely $\pm\mathrm{I}_2$, the order of $\mathrm{PSL}_2(\mathbb{F}_7)$ is $336/2 = 168$.

Since $\mathrm{PGL}_2(\mathbb{F}_7)$ acts transitively on the 28 quadrangles, the stabilizer $\mathfrak{A}$ of Q_3 in $\mathrm{PGL}_2(\mathbb{F}_7)$ has cardinality $336/28 = 12$. Since the group $\mathfrak{A}$ stabilizes the set Q_3, it acts on its elements. This action is faithful. By lemma 1, if a homography fixes three points, it is the identity. Since its order is 12, it is isomorphic to the alternating group $\mathfrak{A}_4$ (the unique subgroup of index 2 in the symmetric group). In fact, using the proof of lemma 1, let us look for homographies that act on Q_3 like double transpositions. One finds that $z \mapsto 3/z$ acts on Q_3 as $(\infty 0)(13)$; that $z \mapsto (z - 3)/(z - 1)$ acts as $(\infty 1)(03)$, and $z \mapsto (3z - 3)/(z - 3)$ acts as $(\infty 3)(01)$. These are involutions, and they commute on Q_3, so they commute on $\mathbb{P}^1(\mathbb{F}_7)$.

The point is that these involutions belong to $\mathrm{PSL}_2(\mathbb{F}_7)$. Indeed, the determinants of the corresponding matrices are squares. Hence, $\mathfrak{A} \cap \mathrm{PSL}_2(\mathbb{F}_7)$ contains a subgroup $\mathfrak{K} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ of order 4; besides, it also contains the order-3 element $z \mapsto 1/(1 - z)$ that permutes $\{\infty, 0, 1\}$ cyclically and fixes 3. Hence, the group $\mathfrak{A}$ is included in $\mathrm{PSL}_2(\mathbb{F}_7)$, and $\mathfrak{A}$ is the stabilizer of Q_3 in $\mathrm{PSL}_2(\mathbb{F}_7)$. Therefore, the $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit of Q_3 has cardinality $168/12 = 14$. Since $Q_5 = \{\infty, 0, 1, 5\}$ is in the same $\mathrm{PGL}_2(\mathbb{F}_7)$-orbit as Q_3, its stabilizer is conjugated to $\mathfrak{A}$, and the orbit of Q_5 under $\mathrm{PSL}_2(\mathbb{F}_7)$ has cardinality 14 too. ∎

Lemma 4. *The complement of an equianharmonic quadrangle in* $\mathbb{P}^1(\mathbb{F}_7)$ *is equianharmonic. Moreover, both are in the same orbit under* $\mathrm{PSL}_2(\mathbb{F}_7)$.

Proof. We start with an example: $[\infty, 0, 1, 3] = 3 = [4, 2, 5, 6]$. The homography defined by $h(z) = 5(z - 2)/(z - 4) = (6z + 2)/(4z + 5)$ maps $(4, 2, 5, 6)$ to $(\infty, 0, 1, 3)$, and h belongs to $\mathrm{PSL}_2(\mathbb{F}_7)$. This proves the claim for $\{\infty, 0, 1, 3\}$.

Now, let $Q = \{a, b, c, d\}$ be an equianharmonic quadrangle. By lemma 3, there is a homography $g \in \mathrm{PGL}_2(\mathbb{F}_7)$ such that $g(Q) = \{\infty, 0, 1, 3\}$. Then the bijection g maps the complement of Q to $\{2, 4, 5, 6\}$, which is equianharmonic. Moreover, $g^{-1}hg$ maps the complement of Q to Q. This proves the claim for a general Q. ∎

3. EQUILATERAL TRIANGLES OVER $\mathbb{F}_7$.

Corollary 5. *There are* 14 *equilateral triangles over* $\mathbb{F}_7$. *They are the following ones:*

$$\begin{tikzcd}
013 \arrow[r, "z+1"] \arrow[d, bend right, "\frac{1}{z}"'] & 124 \arrow[r, "z+1"] & 235 \arrow[r, "z+1"] & 346 \arrow[r, "z+1"] & 450 \arrow[r, "z+1"] & 561 \arrow[r, "z+1"] & 602 \\
015 \arrow[r, "z+1"'] & 126 \arrow[r, "z+1"'] & 230 \arrow[r, "z+1"'] & 341 \arrow[r, "z+1"'] & 452 \arrow[r, "z+1"'] & 563 \arrow[r, "z+1"'] & 604.
\end{tikzcd}$$

Proof. By lemma 4, one can arrange the 28 equianharmonic quadrangles in 14 complementary pairs. In a given pair, exactly one quadrangle contains ∞. Adding or withdrawing ∞ gives a one-to-one correspondence between equilateral triangles and equianharmonic quadrangles containing ∞, hence between equilateral triangles and pairs of complementary equianharmonic quadrangles (e.g., 013 corresponds to the pair $\big\{\{\infty, 0, 1, 3\}, \{2, 4, 5, 6\}\big\}$). Hence, the first statement holds.

To write a list, one starts with 013 and 015. Using invariance of ∞ under affine transformations, one builds twelve new triangles out of the first two with the translation $z \mapsto z + 1$. These are the rows of the list in the corollary. ∎

Remark. The vertical arrow in the statement of corollary 5 has the following meaning. The triangle 013 corresponds to the pair $\big\{\{\infty, 0, 1, 3\}, \{2, 4, 5, 6\}\big\}$. The homography $z \mapsto 1/z$ maps this pair to $\big\{\{\infty, 0, 1, 5\}, \{2, 3, 4, 6\}\big\}$, which corresponds to 015. This extends to an action of $\mathrm{PGL}_2(\mathbb{F}_7)$ on triangles.

Lemma 6. *There is a canonical action of* $\mathrm{PGL}_2(\mathbb{F}_7)$ *on equilateral triangles. All triangles are in the same* $\mathrm{PGL}_2(\mathbb{F}_7)$*-orbit, but there are two* $\mathrm{PSL}_2(\mathbb{F}_7)$*-orbits described by the lines of corollary 5.*

Proof. By invariance of cross-ratio, a homography in $\mathrm{PGL}_2(\mathbb{F}_7)$ maps an equianharmonic quadrangle to another one. Since this action on parts of $\mathbb{P}^1(\mathbb{F}_7)$ commutes with taking the complement, it maps a pair of complementary equianharmonic quadrangles. Hence, the group $\mathrm{PGL}_2(\mathbb{F}_7)$ acts on the set of pairs of complementary equianharmonic quadrangles. But there is a one-to-one correspondence between such pairs and equilateral triangles; one inherits an action of $\mathrm{PGL}_2(\mathbb{F}_7)$ on triangles. If abc is a triangle and $g \in \mathrm{PGL}_2(\mathbb{F}_7)$, one defines $g \cdot abc$ as the triangle corresponding to the pair $g \cdot \big\{\{\infty, a, b, c\}, \mathbb{P}^1 \setminus \{\infty, a, b, c\}\big\}$.

Two triangles abc and $a'b'c'$ are in the same $\mathrm{PGL}_2(\mathbb{F}_7)$-orbit by lemma 1. The homography that maps (a, b, c) to (a', b', c') also maps $\big\{\{\infty, a, b, c\}, \mathbb{P}^1 \setminus \{a, b, c\}\big\}$ to $\big\{\{\infty, a', b', c'\}, \mathbb{P}^1 \setminus \{a', b', c'\}\big\}$.

As for $\mathrm{PSL}_2(\mathbb{F}_7)$-orbits, lemma 4 implies that the pairs of the form $\{Q, \mathbb{P}^1 \setminus Q\}$, where Q runs over a $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit of quadrangles, are orbits of pairs. Hence, by lemma 3, there are two orbits of pairs, corresponding to two orbits of triangles. Since triangles in the same line in corollary 5 are in the same orbit (simply apply $z \mapsto z + 1$), the two lines are exactly the two orbits. ∎

Example. Let $h(z) = 1/z$. Then $h\big(\{\infty, 0, 1, 3\}\big) = \{0, \infty, 1, 5\}$, so h maps the triangle 013 to 015; moreover, $h\big(\{\infty, 1, 2, 4\}\big) = \{0, 1, 2, 4\} = \mathbb{P}^1(\mathbb{F}_7) \setminus \{\infty, 3, 5, 6\}$ so that h maps the triangle 124 to 356.

Equilateral triangles over $\mathbb{F}_7$ as equilateral triangles on a torus. Say that two (equilateral) triangles (over $\mathbb{F}_7$) are *adjacent* if they have two vertices in common. It is easy to check that a triangle in either line of corollary 5 is adjacent to exactly three triangles, and all three lie on the other line. The upshot is that one can arrange triangles over $\mathbb{F}_7$ as equilateral triangles in the real plane in a periodic manner. By gluing the sides of a fundamental parallelogram, one tiles a torus by 14 triangles (Figure 2).

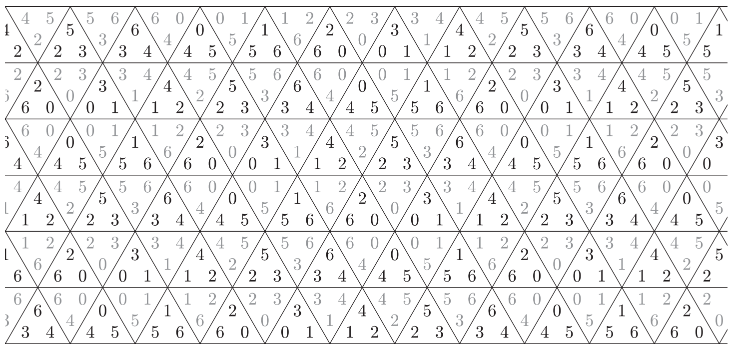

Figure 2. Triangles over $\mathbb{F}_7$ as triangles on a torus

The picture is more appealing when one draws the actual adjacency graph Γ_7: Its vertices are triangles; there is an edge between two triangles if they are adjacent. In the torus, the meeting points of triangles become seven hexagons that define a tiling (Figure 3). Since each hexagonal face touches all the others, the chromatic number of the torus is at least 7. A polyhedral version of this tiling was constructed by L. Szilassi [**7**].

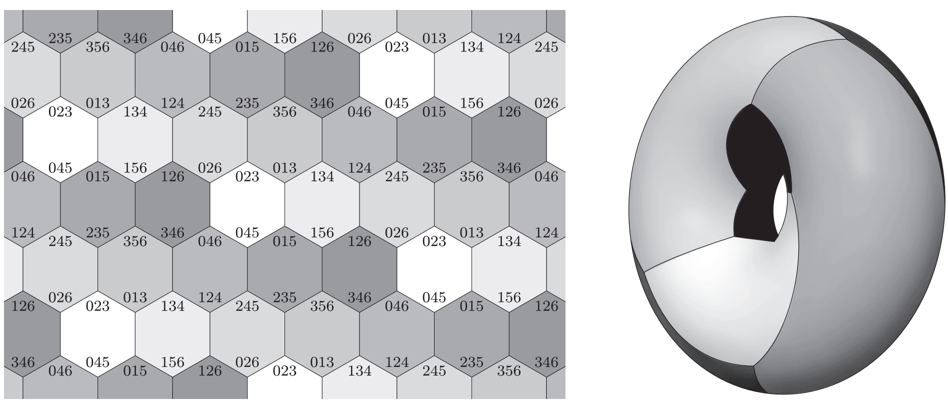

Figure 3. Seven hexagons tiling a torus

Equilateral triangles and the Fano plane. Forgetting the tilings, let us consider the graph Γ_7 with 14 vertices labeled by equilateral triangles over $\mathbb{F}_7$, where two vertices are connected if the corresponding triangles are adjacent. This gives rise to a graph known as the Heawood graph (see [**5**]). It is bipartite because a triangle on a line of corollary 5 is adjacent to triangles on the other line.

The Heawood graph is the incidence graph of the Fano plane: One can label vertices and lines of the Fano plane by triangles so that adjacency of triangles corresponds to incidence (Figure 4).

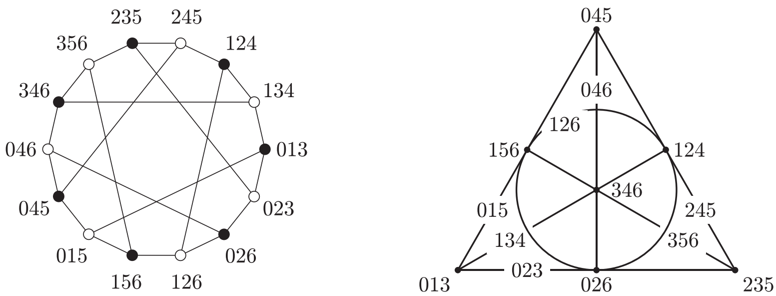

Figure 4. Incidence of triangles: Heawood graph Γ_7 and Fano plane

Application: Two exceptional isomorphisms.

Theorem 7. *One has:* $\mathrm{PSL}_2(\mathbb{F}_7) \simeq \mathrm{GL}_3(\mathbb{F}_2)$.

Proof. One can label vertices of the Fano plane by nonzero vectors in $\mathbb{F}_2^3$ so that the third vertex on the line containing v and v' is labelled by $v + v'$ (put, for instance, the canonical basis vectors on the vertices of the triangle and use this rule to complete the labelling). Hence, a permutation f of vertices and lines of the Fano plane, once completed into a map $\mathbb{F}_2^3 \to \mathbb{F}_2^3$ by setting $f(0) = 0$, is additive. However, on a prime field, additivity is equivalent to linearity so that automorphisms of the Fano plane are linear automorphisms of $\mathbb{F}_2^3$. This is in fact a special case of the "fundamental theorem of projective geometry" ([**1**, Theorem 2.26]) by which any incidence preserving map is a projective map, i.e., an element in $\mathrm{PGL}_3(\mathbb{F}_2) \simeq \mathrm{GL}_3(\mathbb{F}_2)$. ∎

Remark. The action of $\mathrm{PGL}_2(\mathbb{F}_7)$ on $\mathrm{PSL}_2(\mathbb{F}_7)$ by conjugation embeds the former group in $\mathrm{Aut}(\mathrm{PSL}_2(\mathbb{F}_7))$. On the other hand, the automorphism group $\mathrm{Aut}(\mathrm{GL}_3(\mathbb{F}_2))$ is the semidirect product of $\mathrm{GL}_3(\mathbb{F}_2)$ and $\mathbb{Z}/2\mathbb{Z}$ acting by $g \mapsto (g^T)^{-1}$. Hence, $\mathrm{PGL}_2(\mathbb{F}_7) \simeq \mathrm{Aut}\,\mathrm{GL}_3(\mathbb{F}_2)$.

One can identify $\mathrm{PGL}_2(\mathbb{F}_7)$ to its image in the automorphism group of the Heawood graph Γ_7.

Corollary 8. $\mathrm{Aut}(\Gamma_7) = \mathrm{PGL}_2(\mathbb{F}_7)$.

Proof. The Heawood graph Γ_7 is bipartite; its vertices are colored in black and white as in Figure 4. Since a given vertex can be characterized as the unique vertex adjacent to three suitable vertices of the other color, an automorphism that fixes all vertices of a given color is the identity.

Let $\varphi \in \mathrm{Aut}(\Gamma_7)$. Fix a vertex, say 013. If $\varphi(013)$ is not the same color as 013, one replaces φ by $h\varphi$, where $h : z \mapsto 1/z$. Theorem 7 means that the group of color-preserving automorphisms of Γ_7 is $\mathrm{PSL}_2(\mathbb{F}_7)$. So one can find $g \in \mathrm{PSL}_2(\mathbb{F}_7)$ such that $g\varphi$ (or $gh\varphi$) fixes all vertices the same color as 013. Then $g\varphi$ (or $gh\varphi$) is the identity so that $\varphi \in \mathrm{PGL}_2(\mathbb{F}_7)$. ∎

4. STEINER SYSTEMS. Recall a *Steiner system* with parameters (t, k, n), written $S(t, k, n)$, is a set of cardinality n and a collection of k-sets called *blocks* such that every t-set is contained in a unique block.

Proposition 9. *An orbit under* $\mathrm{PSL}_2(\mathbb{F}_7)$ *of equianharmonic quadrangles forms an* $S(3,4,8)$. *An orbit under* $\mathrm{PSL}_2(\mathbb{F}_7)$ *of equilateral triangles forms an* $S(2,3,7)$ *(here, the action is the one defined in lemma 6).*

To be more explicit, recall that a $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit of triangles is but a line in corollary 5. Starting from such an orbit, one rebuilds quadrangles by adding ∞ to every triangle and by adding to the collection the complements of these quadrangles. For instance, starting from the orbit containing 013, one builds the following $S(3,4,8)$:

$$\begin{array}{ccccccc} \infty013 & \infty124 & \infty235 & \infty346 & \infty045 & \infty156 & \infty026 \\ 2456 & 0356 & 0146 & 0125 & 1236 & 0234 & 1345. \end{array}$$

Proof. The underlying set of the $S(3,4,8)$ is $\mathbb{P}^1(\mathbb{F}_7)$; blocks are quadrangles in a given $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit. To fix notations, let us consider that of $\{\infty,0,1,3\}$. Let $\{a,b,c\}$ be a 3-set. By lemma 1, there exists $h \in \mathrm{PSL}_2(\mathbb{F}_7)$ that maps $\{a,b,c\}$ to $\{\infty,0,1\}$. Then $\{a,b,c,h^{-1}(3)\}$ is an equianharmonic quadrangle in the same $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit as $\{\infty,0,1,3\}$. If $\{a,b,c\}$ is included in some block $\{a,b,c,d\}$, then $h(d) = [\infty,0,1,h(d)] = [a,b,c,d] \in \{3,5\}$. Since $\{\infty,0,1,5\}$ is not a block (013 and 015 are not in the same line in corollary 5; see lemma 6), one has $h(d) = 3$, which proves the unicity of a block containing $\{a,b,c\}$.

Now, the underlying set of the $S(2,3,7)$ is $\mathbb{F}_7$; blocks are triangles in a fixed $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit, say, that of 013. A block is in particular a triple $\{a,b,c\}$ such that $\{\infty,a,b,c\}$ is equianharmonic. Hence, given a pair $\{a,b\}$ in $\mathbb{F}_7$, the unique block containing $\{a,b\}$ is obtained by erasing ∞ from the unique equianharmonic triangle in the $\mathrm{PSL}_2(\mathbb{F}_7)$-orbit of $\{\infty,0,1,3\}$ that contains $\{\infty,a,b\}$. ■

Remark. The passage from quadrangles to triangles follows the classical construction of an $S(t-1,k-1,n-1)$ out of an $S(t,k,n)$ by selecting blocks containing a fixed point (here, ∞), then erasing it.

Remark. There are other ways to relate the Fano plane and the projective line over $\mathbb{F}_7$. For instance, one can replace equilateral triangles by self-polar triangles with respect to a conic in the projective plane or elementary 2-subgroups of $\mathrm{PSL}_2(\mathbb{F}_7)$, i.e., isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. See, e.g., [**3**].

5. EQUILATERAL TRIANGLES OVER OTHER FIELDS. In this section, we study the graph of equilateral triangles over other finite fields. Characteristic 7 turns out to be the only one where a projective group governs an apparently affine situation.

Let $\mathbb{K} = \mathbb{F}_q$ be a finite field of characteristic $p \neq 3$ and cardinality $q = p^e$. There are cubic roots of unity j and j^2 either in $\mathbb{F}_p$ or in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$, depending on the Legendre symbol $\left(\frac{-3}{p}\right)$. Let us assume that $\mathbb{K}$ contains them, i.e., that $q \equiv 1 \pmod 3$. As above, one defines a graph Γ_q: Its vertices are equilateral triangles; two triangles are related by an edge if they have a common side. Let Γ_q^0 be the connected component of Γ_q that contains the triangle $01d$, where $d = -j^2$.

Example. For $q = 4$, write $\mathbb{F}_4 = \{0,1,j,j^2\}$. There are four triangles, all of them are equilateral, and Γ_4 is the complete graph with 4 vertices.

Example. For $q = 13$, it is obvious that Γ_{13} has at least two connected components. Indeed, let bcd be an equilateral triangle. Since $-1 = 5^2$ is a square in $\mathbb{F}_{13}$ so that

$-j^2$ is one too, $b - c$ is a square if and only if $c - b$ is a square if and only if $d - b = -j^2(c - b)$ is a square. If one dare say so, this means that a side of a triangle is a square if and only if its three sides are squares. Hence, there are two types of triangles: triangles the sides of which are squares and the other ones (Figure 5).

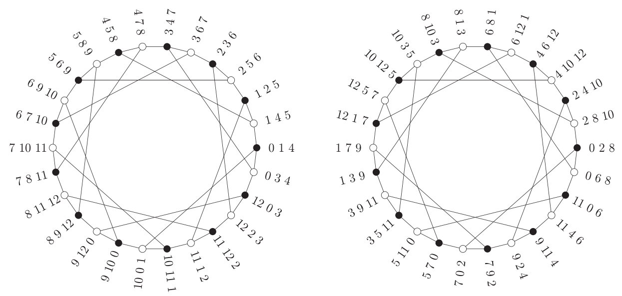

Figure 5. Equilateral triangles over $\mathbb{F}_{13}$

Connected components of Γ_q.

Proposition 10. *There are $q(q-1)/3$ equilateral triangles over $\mathbb{F}_q$. The affine group over $\mathbb{F}_q$ acts edge-transitively on Γ_q. In any connected component of Γ_q, there are:*

- *$2p$ vertices if p is odd and $j \in \mathbb{F}_p$;*
- *$2p^2$ vertices if p is odd and $j \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$;*
- *4 vertices if $p = 2$.*

Proof. Given distinct points b and c in $\mathbb{F}_q$, there are two points d such that bcd is equilateral, namely $d_1 = b - j(c - b)$ and $d_2 = b - j^2(c - b)$. Since every triangle has three sides and since there are $\binom{q}{2}$ pairs $bc = \{b, c\}$, there are $2\binom{q}{2}/3 = q(q-1)/3$ equilateral triangles.

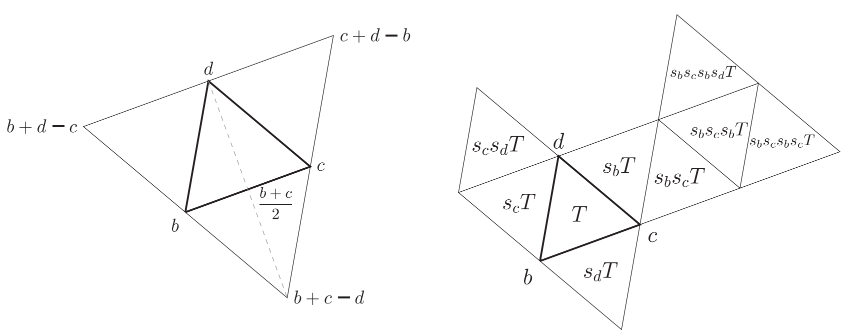

Figure 6. Adjacent equilateral triangles

The adjacent triangles bcd_1 and bcd_2 with a common side bc are mapped to each other by the affine involution $s_d : z \mapsto b + c - z$. (If p is odd, this is the symmetry with respect to the middle of bc; see Figure 6.) Indeed, s_d interchanges b and c and, with notation as above, $d_1 + d_2 = 2b - (j + j^2)(c - b) = 2b + (c - b) = b + c$.

Let $(T_1, T_2) = (bcd_1, bcd_2)$ and $(T_1', T_2') = (b'c'd_1', b'c'd_2')$ be two pairs of adjacent equilateral triangles. The affine map $f : z \mapsto \frac{c'-b'}{c-b}(z-b)+b'$ sends b to b', c to c', and d_1 to d_1' or d_2' so that f maps (T_1, T_2) to (T_1', T_2') or (T_2', T_1'), and f or $s_{d_1'} f$ maps (T_1, T_2) to (T_1', T_2').

We claim that the connected component of a triangle $T = bcd$ is its orbit under the subgroup H generated by the involutions $s_d : z \mapsto b + c - z$, $s_b : z \mapsto c + d - z$, and $s_c : z \mapsto d + b - z$. Indeed, the three neighbors of T are $s_b T$, $s_c T$, and $s_d T$. More generally, since H preserves adjacency, given $h \in H$, the three neighbors of hT are $hs_b T$, $hs_c T$, and $hs_d T$ (see Figure 6). This implies the claim.

Besides, the stabilizer of T in H is trivial. Indeed, elements of H are all of the form $s : z \mapsto \alpha z + \beta$ for $\alpha \in \{-1, 1\}$ and $\beta \in \mathbb{F}_q$. If p is odd, an involution $z \mapsto -z + \beta$ has only one fixed point: $\beta/2$; if it stabilized $T = bcd$, as it can act neither trivially nor as a 3-cycle on $\{b, c, d\}$, it would permute two vertices, say b and c, and fix the third one, say d. This would imply $(b + c)/2 = d$, which is absurd since $\{\infty, b, c, d\}$ cannot be both harmonic and equianharmonic. Assume a translation $z \mapsto z + \beta$ fixes a triangle bcd, with $\beta \neq 0$: Since it has no fixed point, it acts on $\{b, c, d\}$ as a 3-cycle; this forces $3\beta = 0$, thus $3 = 0$, which contradicts the standing assumption $p \neq 3$.

Therefore, the cardinality of the connected component of T is the order of H. Thus, it only remains to compute the order of H. First assume p is odd. Then H contains an index 2 subgroup H_0 generated by the translations $s_c s_b : z \mapsto z + b - c$, $s_d s_c : z \mapsto z + c - d$, and $s_b s_d : z \mapsto z + d - b$. Since $b - c + c - d + d - b = 0$, H_0 is generated by $s_c s_b$ and $s_d s_c$. Recall that $d - b = -j(c - b)$ or $d - b = -j^2(c - b)$. If $j \in \mathbb{F}_p$, there is an integer k such that $d - b = k(c - b)$ so that $s_d s_c = (s_c s_b)^{-k}$, but the order of $s_c s_b$ is p so that the order of H_0 is p and the order of H is $2p$. If $j \notin \mathbb{F}_p$, then $c - b$ and $d - b$ span an $\mathbb{F}_p$-vector space of dimension 2 so that the order of H_0 is p^2 and the order of H is $2p^2$. Finally, if $p = 2$, then s_b, s_c, and s_d are translations of order 2 and $s_d = s_b s_c$ so that the order of H is 4. ∎

Remark. If p is odd, then Γ_q is bipartite. Indeed, with notations as above, triangles in the connected component of T split into $\{hT,\ h \in H_0\}$ and $\{hT,\ h \in H \setminus H_0\}$; an edge relates one vertex in each of these sets.

Corollary 11. *All connected components of Γ_q are isomorphic to Γ_q^0 and, in fact, either to Γ_p^0 if $j \in \mathbb{F}_p$ or to $\Gamma_{p^2}^0$ if $j \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$.*

Cycles of minimal length. From now on, p is assumed to be odd. We proceed to show why $p = 7$ is exceptional.

Lemma 12. *If $p \neq 7$, then $j \notin \{\pm 2, \pm 1/2\}$. If $p = 7$, then $\{j, j^2\} = \{2, 1/2\}$.*

Proof. Recall $p \neq 2, 3$. The cases $p = 5$ and $p = 7$ are straightforward. For $p > 7$, the integers $(\pm 2)^2 + (\pm 2) + 1$ are odd, and their absolute values are strictly less than p so that $j \neq \pm 2$ in $\mathbb{F}_q$. This implies that $j \neq \pm 1/2$ as well. ∎

Lemma 13. *Assume p is odd. Then Γ_q has girth 6. The number of hexagons (cycles of length 6) in any connected component of Γ_q is:*

- *p if $p \neq 7$ and $j \in \mathbb{F}_p$;*
- *p^2 if $p \neq 7$ and $j \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$;*
- *28 if $p = 7$.*

Proof. Let $T = bcd$ be a triangle in Γ_q. Thanks to the affine group, one can assume without loss of generality that $b = 0$, $c = 1$, and $d = -j^2$. By proposition 10 and its proof, a path of length ℓ starting at T is a sequence $(T, s_{a_1}T, \dots, s_{a_1}\cdots s_{a_\ell}T)$, where $(a_1, \dots, a_\ell) \in \{0, 1, d\}^\ell$ is such that $a_{i+1} \neq a_i$ if $1 \le i < \ell$. Such a path is a cycle if and only if $s_{a_1}\cdots s_{a_\ell}T = T$, i.e., if $s_{a_1}\cdots s_{a_\ell} = \mathrm{id}$. Reversed sequences $(a_1, \dots, a_\ell)$ and $(a_\ell, \dots, a_1)$ describe the same cycle traversed in the opposite direction.

Since p is odd, the relation $s_{a_1}\cdots s_{a_\ell} = \mathrm{id}$ implies that ℓ is even and that

$$a_2 - a_1 + a_4 - a_3 + \cdots + a_\ell - a_{\ell-1} = 0 \tag{\S}$$

(recall that $s_{a_1}s_{a_2}$ is the translation by $a_2 - a_1$). The six differences $a - a'$ (where $a, a' \in \{0, 1, d\}$, $a \neq a'$) are

$$\begin{array}{ccc} 0 - d = j^2 & d - 1 = j & 1 - 0 = 1, \\ d - 0 = -j^2 & 1 - d = -j, & 0 - 1 = -1. \end{array}$$

They are all distinct. This forces $\ell > 4$, since $a_2 - a_1 + a_4 - a_3 = 0$ would imply $a_3 = a_2$ (and $a_4 = a_1$), which is excluded.

Since $j^2 + j + 1 = 0$, one has $s_d s_0 s_1 s_d s_0 s_1 = \mathrm{id}$. This gives a hexagon, namely $(T, s_dT, s_ds_0T, s_ds_0s_1T, s_ds_0s_1s_dT, s_ds_0s_1s_ds_0T, T)$; hence, Γ_q has girth 6.

Now assume $p \neq 7$. Let us count the hexagons. By lemma 12, the only zero sums of three elements of $\{\pm 1, \pm j, \pm j^2\}$ are $j^2 + j + 1 = 0$ and $-j^2 - j - 1 = 0$ (this is wrong if $p = 7$). Hence, if (§) holds with $\ell = 6$, then $\{a_2 - a_1, a_4 - a_3, a_6 - a_5\}$ is $\{j^2, j, 1\}$ or $\{-j^2, -j, -1\}$. This means one must compose s_ds_0, s_1s_d, and s_0s_1 in the first case (resp. s_0s_d, s_ds_1, and s_1s_0 in the second case) so that two consecutive s_a's are different. This gives three relations for each case:

$$\begin{cases} s_ds_0s_1s_ds_0s_1 = s_0s_1s_ds_0s_1s_d = s_1s_ds_0s_1s_ds_0 = \mathrm{id} \\ s_1s_0s_ds_1s_0s_d = s_ds_1s_0s_ds_1s_0 = s_0s_ds_1s_0s_ds_1 = \mathrm{id}\,. \end{cases}$$

Recall that two relations read in reverse order, e.g., $s_ds_0s_1s_ds_0s_1$ and $s_1s_0s_ds_1s_0s_d$, correspond to the same cycle. In other terms, every triangle lies in three hexagons. Each connected component contains $2p$ triangles (respectively, $2p^2$) if $j \in \mathbb{F}_p$ (respectively, $j \notin \mathbb{F}_p$), so there are $3 \times 2p/6 = p$ (respectively, p^2) hexagons.

For $p = 7$, we interpret the Heawood graph Γ_7 as lines and planes in $\mathbb{F}_2^3$. Up to cyclic permutation and reversal, a hexagon is a sequence $D_1 \subset P_{12} \supset D_2 \subset P_{23} \supset D_3 \subset P_{31} \supset D_1$ where the D_i's are distinct lines and the P_{ij}'s are distinct planes. Such a hexagon is determined by the three noncoplanar lines D_i's (by the rule $P_{ij} = D_i + D_j$). There are seven lines in $\mathbb{F}_2^3$, hence $\binom{7}{3}$ triplets of lines. There are seven planes, each one contains three lines, hence seven triplets of coplanar lines. There remain $\binom{7}{3} - 7 = 28$ hexagons. ■

Corollary 14. *If $p \neq 2, 7$, every edge is contained in exactly two hexagons.*

Proof. Since the affine group acts edge-transitively on Γ_q, all edges are contained in the same number of hexagons. Assume that $j \in \mathbb{F}_p$. There are $2p$ triangles, and each of them belongs to three edges, so there are $3p$ edges. On the other side, there are p hexagons, each of which contains six edges, so each edge belongs to $(6p)/(3p) = 2$ hexagons. If $j \notin \mathbb{F}_p$, simply replace p by p^2. ■

Proposition 15. *Assume $p \neq 2, 7$. All triangles in a given hexagon of Γ_q contain a unique common point in $\mathbb{F}_q$. This induces a bijection between hexagons in the connected component Γ_q^0 of the triangle $01d$ (where $d = -j^2$) and the subgroup K spanned by 1 and $-j^2$ in $\mathbb{F}_q$.*

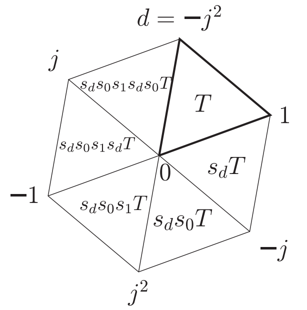

Figure 7. The relation $s_ds_0s_1s_ds_0s_1 = \mathrm{id}$ and the point 0 are associated to the same hexagon.

Proof. The scalar 0 is the unique one that belongs to every triangle in the hexagon $C = (T, s_dT, s_ds_0T, \ldots, T)$ (Figure 7). Recall the subgroup H_0 generated by s_0s_1 and s_0s_d from the proof of proposition 10. It is isomorphic to $\mathbb{F}_p$ if $j \in \mathbb{F}_p$ (respectively, $\mathbb{F}_{p^2}$ if $j \notin \mathbb{F}_p$). For $h \in H_0$, the scalar $h(0)$ is the unique one in every triangle of the hexagon $hC = (hT, hs_dT, hs_ds_0T, \ldots, hT)$; moreover, H_0 acts by translations on scalars. So all hexagons hC are different. Since the number of hexagons is the order of H_0, the action of H_0 on cycles is simply transitive, hence the proposition. ■

Remark. Let $p \neq 2, 7$. Points of K, sides of equilateral triangles, and triangles in Γ_q^0 form a combinatorial cell-complex C of dimension 2. Since two triangles share at most one edge, this is orientable. Thus, C can be embedded into a surface, the Euler characteristic of which is 0 ($= 2p - 3p + p$ or $2p^2 - 3p^2 + p^2$). Then, Γ_q^0 is the dual graph of the 1-skeleton of C. This extends the torus tilings in section 3 (Figures 8 and 9).

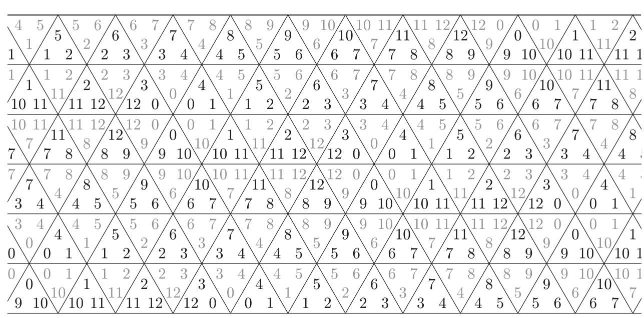

Figure 8. Equilateral triangles in Γ_{13}^0 pave a torus ($\{-j^2, -j\} = \{4, 10\}$).

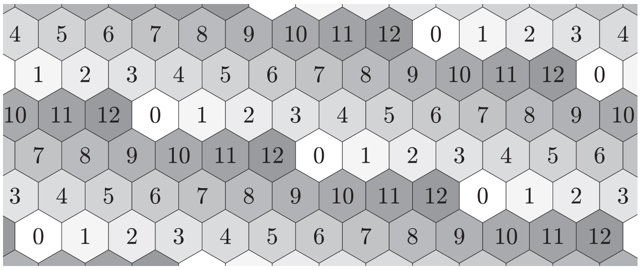

Figure 9. The graph Γ_{13}^0 and its hexagons tile a torus ($\{-j^2, -j\} = \{4, 10\}$).

Automorphisms of a connected component of Γ_q. From now on, we assume that $p \neq 2, 7$. We introduce two "geometric" groups that act on the connected component Γ_q^0, then we identify its full automorphism group.

Let G be the group generated by $s_0 : z \mapsto 1 - j^2 - z$, $s_1 : z \mapsto -j^2 - z$, $s_d : z \mapsto 1 - z$ and $t : z \mapsto j(z - e) + e$, where $d = -j^2$ and $e = (0 + 1 + d)/3$. The map t has order 3; it stabilizes the subgroup K of $\mathbb{F}_q$ spanned by 1 and $d = -j^2$ (note that $K \simeq H_0$). Elements of G are the maps $z \mapsto \alpha(z - e) + \beta + e$, where $\alpha \in \{\pm 1, \pm j, \pm j^2\}$ (a sixth root of unity) and $\beta \in K$. Hence, G has order $6p$ if $j \in \mathbb{F}_p$ (respectively, $6p^2$ if $j \notin \mathbb{F}_p$).

Being affine, t preserves adjacency, so t acts on Γ_q; it fixes the triangle $T = 01d$, so it acts on Γ_q^0. The group G acts faithfully on Γ_q^0. Indeed, the kernel of the action of G is contained in the stabilizer S of $T = 01d$. But S contains the order-3 element t, and S intersects H trivially, so S embeds into $G/H \simeq \mathbb{Z}/3\mathbb{Z}$ (recall G is generated by H and t); in other terms, S is generated by t. But t permutes cyclically the triangles incident to T, so the kernel is trivial.

Assume moreover that $j \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$. Let $\sigma : x \mapsto x^p$ be the unique (Frobenius) involution in the Galois group of $\mathbb{F}_{p^2}/\mathbb{F}_p$. Let $\widehat{G}$ be the group generated by σ and G (in the symmetric group of $\mathbb{F}_{p^2}$). Then $\widehat{G}$ has order $12p^2$. Since $\sigma(j) = j^2$, σ preserves equilateral triangles, so $\widehat{G}$ acts on Γ_q. Since σs_d fixes $T = 01d$, the group $\widehat{G}$ acts on Γ_q^0. The action is faithful by the same kind of arguments as we used for G.

Lemma 16. *Let T be a triangle in Γ_q^0. The only automorphism of Γ_q^0 that fixes T, an edge $\{T, T'\}$, and that stabilizes a hexagon containing the edge $\{T, T'\}$ is the identity.*

Proof. Assume an automorphism φ fixes T and T' adjacent to T and stabilizes a hexagon containing T and T'. The only automorphism of a hexagon with two consecutive fixed vertices is the identity so that φ fixes all vertices of the hexagon. Since $p \neq 7$, there is exactly one other hexagon containing T and T', which φ necessarily fixes (vertex-wise). In particular, φ fixes the other two vertices that are adjacent to T too. One can thus extend the assumption from one vertex to its adjacent vertices. Hence, φ fixes vertex-wise the connected component of T. ■

Lemma 17. *Assume $p \neq 7$ and $j \in \mathbb{F}_p$. There is no automorphism of Γ_q that fixes two adjacent triangles and permutes the two hexagons that contain them.*

Proof. One can assume the fixed triangles are $T = 01d$ and $T' = 01d'$, with $d = -j^2$ and $d' = -j$. Since $j \in \mathbb{F}_p$, hexagons are numbered by $\mathbb{F}_p$ according to which scalar belongs to all triangles in the hexagon. Recall from proposition 15 that the $(k+1)^{\text{th}}$ hexagon is the image of the k^{th} one by $s_0 s_1 : z \mapsto z + 1$.

Assume an automorphism φ fixes T and T' and permuted the hexagons 0 and 1. Consider the edge opposite to $\{T, T'\}$ in hexagon 0. It belongs to hexagons 0 and -1, and it is mapped to the edge opposite to $\{T, T'\}$ in hexagon 1, which belongs to hexagon 2. Hence, φ maps hexagon -1 to hexagon 2 and, more generally, hexagon k to hexagon $1-k$ for all $k \in \mathbb{F}_p$ (see Figure 9). However, φ necessarily fixes hexagon d because it is the only one that shares a triangle with hexagons 0 and 1. Hence, one has $d = 1 - d$, which contradicts lemma 12. ■

Theorem 18. *Assume $p \neq 2, 7$. Then the automorphism group of Γ_q^0 is either the subgroup G of the affine group if $j \in \mathbb{F}_p$ or its twofold extension $\widehat{G}$ if $j \in \mathbb{F}_{p^2} \backslash \mathbb{F}_p$.*

Proof. Let φ be an automorphism of Γ_q^0. Since H is transitive on vertices of Γ_q^0 and t permutes cyclically the vertices of T and, therefore, the triangles adjacent to T, the group G is transitive on oriented edges of Γ_q^0. By composing by some $\psi \in G$, one can thus assume that $\psi\varphi$ fixes T and $T' = s_d T$.

Assume $j \in \mathbb{F}_p$. By lemma 17, $\psi\varphi$ fixes both hexagons containing T and $s_d T$ too. By lemma 16, $\psi\varphi$ is the identity, so $\varphi \in G$.

Assume $j \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$. If $\psi\varphi$ fixes a hexagon containing T and $s_d T$, we can conclude as above. Otherwise, σs_d fixes T and $s_d T$ and permutes both hexagons containing this edge so that $\sigma s_d \psi\varphi$ fixes T, $s_d T$ and a hexagon. By lemma 16, $\varphi \in G$. ■

Conclusion. We have seen how the geometric idea of equilateral triangles over $\mathbb{F}_7$ gives rise to natural constructions of combinatorial structures related to the Fano plane–the Heawood graph, a tiling of a torus by seven hexagons, the Steiner systems $S(2, 3, 7)$ and $S(3, 4, 8)$. Moreover, characteristic 7 is exceptional: Here, this notion falls under projective geometry, while for other finite fields, it is essentially affine.

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