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De Bruijn–Erdős-type theorems for graphs and posets

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ABSTRACT

A classical theorem of De Bruijn and Erdős asserts that any noncollinear set of n points in the plane determines at least n distinct lines. We prove that an analogue of this theorem holds for posets, where lines are defined using the natural betweenness relation in posets. More precisely, we obtain a bound on the number of lines depending on the height of the poset. The extremal configurations are also determined. Finally, we introduce a new notion of lines in graphs and show that our result for posets can be extended to this setting.

1. Introduction

The starting point of this paper is a classical result of De Bruijn and Erdős in combinatorial geometry. A set of n points in the plane is called a *near-pencil* if exactly $n - 1$ of the points are collinear.

Theorem 1. *Every noncollinear set of n points in the plane determines at least n lines. Moreover, equality occurs if and only if the configuration is a near-pencil.*

Erdős [11] showed that this result is a consequence of the Sylvester–Gallai theorem which asserts that every noncollinear set of n points in the plane determines a line containing precisely two points. Coxeter [10] showed that the Sylvester–Gallai theorem holds in a more basic setting known as *ordered geometry*. Here, the notions of distance and angle are not used and, instead, a ternary relation of *betweenness* is employed. We write $[abc]$ for the statement that b lies *between* a and c and $(V, \mathcal{B})$ to denote a set of points V together with a betweenness relation $\mathcal{B}$ on it. Now, one can naturally define lines using this betweenness relation as follows. For any two distinct points a and b in V, the line $\overline{ab}$ defined by a and b is:

$$\overline{ab} = \{a, b\} \cup \{x : [xab] \text{ or } [axb] \text{ or } [abx] \in \mathcal{B}\}. \tag{1}$$

In the rest of the paper we will define betweenness relations in various combinatorial objects. In each such case we will use (1) to define lines. In this framework, a natural object to study is a metric space. Indeed, a metric space (V, d) defines a betweenness relation on V in the following way:

$$[axb] \Leftrightarrow d(a, x) + d(x, b) = d(a, b). \tag{2}$$

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Using (1), the line $\overline{ab}$ can be defined for any two distinct points a and b in V. Observe that this definition of a line generalizes the classical notion of a line in Euclidean space to any metric space. However, it is important to note that lines in this setting may have some unusual properties. For example, in an arbitrary metric space (as well as in another framework studied later in this paper), two lines might have more than one common point, and it is even possible for a line to be a proper subset of another line.

Chen and Chvátal [6] proposed the following conjecture, which, if true, would give a vast generalization of Theorem 1. In a metric space (V, d), a line is called *universal* if it contains all points in V.

Conjecture 1 (*Chen-Chvátal [6]*). *Any finite metric space on n points either induces at least n distinct lines or contains a universal line.*

Although the conjecture has been proved in several special cases [3,4,7–9,12], it is still wide open. The best known lower bound on the number of lines in a general finite metric space with no universal line is $(1/\sqrt{2}+o(1))\sqrt{n}$ [2].

Following the lead of Chen and Chvátal, in this paper we are interested in another combinatorial object that defines a natural betweenness relation: a poset. Let $P = (X, \prec)$ be a finite poset with the order relation $\prec$ defined on the set X. As in the metric space case, a poset P induces a natural betweenness relation:

$$[abc] \Leftrightarrow a \prec b \prec c \text{ or } c \prec b \prec a. \tag{3}$$

Therefore, we can define lines in posets using (1) again. As before, a line is *universal* if it contains every point from the ground set. Our main result is to prove a De Bruijn–Erdős-type theorem for posets: A poset on n points either induces at least n distinct lines or contains a universal line. In fact, we obtain a stronger bound on the minimum number of lines in a poset P as a function of its height $h(P)$, defined to be the maximum size of a chain in P.

Theorem 2. *Let P be a poset on n points with no universal line, and with $h(P) \geq 2$. Then P induces at least*

$$h(P)\binom{\lfloor n/h(P)\rfloor}{2} + \lfloor n/h(P)\rfloor(n \bmod h(P)) + h(P) \tag{4}$$

distinct lines.

Moreover, P induces exactly n distinct lines if and only if it consists of a chain of size $n-1$ and a point which is comparable to at most one point of this chain.

Observe that (4) is always greater than or equal to n (with equality if $h(P) \geq \lfloor n/2\rfloor$). Moreover, if $h(P) = O(n^s)$ for $0 < s \leq 1$, the number of distinct lines in P is $\Omega(n^{2-s})$.

Three points a, b, c are said to be *collinear* if $[abc]$ or $[acb]$ or $[bac]$ holds. Chen and Chvátal [6] observed that the set of lines defined by a betweenness relation depends only on collinearity. That is, when three points are collinear, it is not important which point is between which two, we just need to know that they are collinear. Let us explain this more formally.

A *hypergraph* is an ordered pair $(V, \mathcal{E})$ such that V is a set of elements called *vertices* and $\mathcal{E}$ is a family of subsets of V called *edges*. A hypergraph is *k-uniform* if each of its hyperedges consist of k vertices. Chen and Chvátal introduced the following notions. For any betweenness relation $(V, \mathcal{B})$, define the 3-uniform hypergraph $H(\mathcal{B}) = (V, \mathcal{E}(\mathcal{B}))$ with $\mathcal{E}(\mathcal{B}) := \{\{a, b, c\} : [abc] \in \mathcal{B}\}$. Then, define the line $\overline{ab}$ in a 3-uniform hypergraph as

$$\overline{ab} = \{a, b\} \cup \{x : \{a, b, x\} \in \mathcal{E}\}.$$

Points that lie in a hyperedge can be thought of as being collinear. It is then easy to see that the betweenness relation $(V, \mathcal{B})$ and the 3-uniform hypergraph $\mathcal{H}(\mathcal{B})$ determine the same set of lines.

Chen and Chvátal proved [6] that there is an infinite family of 3-uniform hypergraphs inducing only $c^{\sqrt{\log_2 n}}$ distinct lines (where n is the number of vertices and c is a constant). Therefore, a De Bruijn–Erdős-type theorem does not hold for 3-uniform hypergraphs. The best known lower bound on the number of lines in a 3-uniform hypergraph with no universal line is $(2-o(1))\log_2 n$ [1].

However, in [5], Beaudou et al., proved De Bruijn–Erdős-type theorems for some special classes of 3-uniform hypergraphs described below. Let $I \subseteq \{0, 1, 2, 3, 4\}$. Let $\mathcal{H}_I$ be the class of 3-uniform hypergraphs that satisfy the following property: any 4 vertices induce i hyperedges for an $i \in I$. For example, $H \in \mathcal{H}_{\{0,1,2\}}$ if and only if any four vertices of H induce 0, 1 or 2 hyperedges. In [5], it is proved that a hypergraph H on n vertices that belongs to $\mathcal{H}_{\{0,1,3,4\}}$ or $\mathcal{H}_{\{0,1,2,3\}}$ or $\mathcal{H}_{\{0,2,4\}}$ either has a universal line, or induces at least n distinct lines. Moreover, the family of hypergraphs discovered by Chen and Chvátal [6] containing only $c^{\sqrt{\log_2 n}}$ distinct lines belongs to $\mathcal{H}_{\{0,1,2,4\}}$. The question remains open for classes $\mathcal{H}_{\{1,2,4\}}$, $\mathcal{H}_{\{2,3,4\}}$, $\mathcal{H}_{\{1,2,3,4\}}$ and $\mathcal{H}_{\{0,2,3,4\}}$.

Our second result is a generalization from posets to graphs via comparability graphs, and lies in the framework of 3-uniform hypergraphs discussed above. For any graph $G = (V, E)$, define the 3-uniform hypergraph $H(G) = (V, \mathcal{E}(G))$ where $\{a, b, c\} \in \mathcal{E}(G)$ if and only if abc is a triangle of G. The lines of G are then defined as being the lines of $H(G)$, i.e., the line defined by two vertices a, b of G is:

$$\overline{ab} = \{a, b\} \cup \{c : abc \text{ is a triangle of } G\}. \tag{5}$$

Let us explain why this definition of lines in graphs generalizes lines in posets. With a poset $P = (V, \prec)$, we can associate a graph $G = (V, E)$ where $ab \in E$ if and only if $a \prec b$ or $b \prec a$. Such a graph is called a *comparability graph*. Hence for any three vertices a, b, x in V, abx is a triangle of G if and only if $[xab]$ or $[axb]$ or $[abx]$ hold in P (where betweenness is defined by (3)). It is then clear that a poset $(V, \prec)$ and its corresponding comparability graph define the same set of lines.

Using this definition of lines in a graph, we prove the following De Bruijn–Erdős-type result:

Theorem 3. *If a graph G on $n \geq 4$ vertices does not contain a universal line, then it induces at least n distinct lines, and equality occurs only if G consists of a clique of size $n-1$ and a vertex that has at most one neighbour in the clique.*

It may be easily seen that the theorem also holds when $n = 3$, but we have an additional extremal example in this case: a graph where all pairs of vertices are non-adjacent.

Let us now compare our result with the results on 3-uniform hypergraphs mentioned above. Let $G = (V, E)$ be a graph and define the 3-uniform hypergraph $H(G)$ as explained above. Since four vertices of a graph cannot induce exactly three triangles, four vertices of $H(G)$ induce 0, 1, 2 or 4 hyperedges. Hence our result is concerned with a subclass of $\mathcal{H}_{\{0,1,2,4\}}$ and is thus independent from the results described above. Actually, as observed by an anonymous referee, the class of 3-uniform hypergraphs that we study are the hypergraphs $H = (V, \mathcal{E})$ that satisfy the following property: For all $a, b, c \in V$, if there exist $e_1, e_2, e_3 \in \mathcal{E}$ such that $\{a, b\} \subset e_1$, $\{a, c\} \subset e_2$, $\{b, c\} \subset e_3$, then $\{a, b, c\} \in \mathcal{E}$.

We also want to point out to the reader that there is another natural way to define lines in graphs. Indeed, since graphs naturally induce a metric space, and thus a betweenness relation, one can also use (1) to define lines in graphs and see it as a special case of Conjecture 1. We note that even this special case is still open and seems quite challenging already. The best known lower bound for the number of lines in a graph containing no universal line is $(1/2 - o(1))n^{4/7}$ [2]. From now on, when we speak about lines in a graph, we will always use (5).

The paper is organized as follows. In Section 2, we prove Theorem 2 by providing an algorithm for finding lines. In Section 3, we prove Theorem 3 by induction.

2. Lines in posets

We begin by introducing some notation that will be useful in the proof of Theorem 2. For any pair of elements a, b in a poset $P = (X, \prec)$, we write $a \nsim b$ to indicate that the points a and b are not comparable (that is, neither $a \prec b$ nor $b \prec a$ hold). Let $Y \subseteq X$. We denote by $P \setminus Y$ the poset on the set of points $X \setminus Y$ together with $\prec$ restricted to $X \setminus Y$.

In this section, we prove a lower bound on the number of lines in a poset as a function of its height. Before we proceed with the proof, we need a simple lemma.

Lemma 1. *If $a_1, \ldots, a_r$ are nonnegative integers such that $\sum_{i=1}^{r} a_i = n$, then $\sum_{i=1}^{r} \binom{a_i}{2} \geq r\binom{\lfloor n/r \rfloor}{2} + \lfloor n/r \rfloor (n \bmod r)$.*

Proof. First suppose that $|a_i - a_j| \leq 1$ for all i and j. Then every a_i is either $\lfloor n/r \rfloor$ or $\lfloor n/r \rfloor + 1$ and there are exactly $(n \bmod r)$ values of i such that $a_i = \lfloor n/r \rfloor + 1$. Using that $\binom{c+1}{2} = \binom{c}{2} + c$ for any nonnegative integer c, it follows that the bound in our lemma holds with equality. Thus, it suffices to prove that if $a_i - a_j > 1$ then simultaneously decreasing a_i and increasing a_j by 1 do not increase $\sum_{i=1}^{r} \binom{a_i}{2}$. Indeed, we have:

$$\binom{a_i - 1}{2} + \binom{a_j + 1}{2} = \binom{a_i}{2} - (a_i - 1) + \binom{a_j}{2} + a_j \leq \binom{a_i}{2} + \binom{a_j}{2}$$

and the lemma follows. □

We point out now that the fact that a poset on n points induces exactly n distinct lines if and only if it consists of a chain of size $n-1$ and a point comparable to at most one point of this chain is a trivial consequence of the equality case of Theorem 3.

Proof of Theorem 2. Observe first that, by the definition of lines in posets, if a is incomparable to b, then the line $\overline{ab} = \{a, b\}$ and, if a is comparable to b, then

$$\overline{ab} = \{a, b\} \cup \{x : x \text{ is comparable to both } a \text{ and } b\}.$$

This simple observation is used implicitly throughout the proof.

Let $\mathcal{A}$ be a maximal partition of P into antichains, and let $C \subseteq P$ be a maximal chain in P. By Mirsky's theorem [13], we know that $|\mathcal{A}| = |C| = h(P)$. For notational convenience, from now on, let $h(P)$ be denoted by H. Denote the elements of $\mathcal{A}$ and C, respectively, as $\mathcal{A} = \{a_1, \ldots, a_H\}$ and $C = \{c_1, \ldots, c_H\}$ with $c_1 \prec \cdots \prec c_H$. Assume, without loss of generality, that $c_i \in a_i$ for $i = 1, \ldots, H$.

Set

$$\mathcal{L}_0 := \bigcup_{i=1}^{H} \{\overline{ab} : a, b \in a_i, a \neq b\}.$$

Note that all of the lines in $\mathcal{L}_0$ are induced by incomparable points and are, thus, pairwise distinct. By Lemma 1, we have

$$|\mathcal{L}_0| = \sum_{i=1}^{H} \binom{|a_i|}{2} \geq H\binom{\lfloor n/H \rfloor}{2} + \lfloor n/H \rfloor (n \bmod H).$$

Next we use the chain C to find H further lines, distinct from those in $\mathcal{L}_0$. We do so via the following iterative process: Set $b_1 = 1$, $t_1 = H$ and $\mathcal{L}_1 = \emptyset$. For $k = 1, 2, \ldots$, apply the following steps until a STOP condition is met.

Step 1 If $b_k = t_k$, set $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_1 c_H}\}$ and **STOP**.
Otherwise $b_k < t_k$ and, since $\overline{c_{b_k} c_{t_k}}$ is not a universal line, there exists $s_k \in P \setminus \overline{c_{b_k} c_{t_k}}$. If s_k is incomparable with both c_{b_k} and c_{t_k}, go to **Step 2a**. If s_k is incomparable with c_{b_k} and comparable with c_{t_k}, go to **Step 2b**. Finally, if s_k is incomparable with c_{t_k} and comparable with c_{b_k}, go to **Step 2c**.

Step 2a Set $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_i s_k} : b_k \leq i \leq t_k\} \cup \{\overline{c_1 c_H}\}$ and **STOP**.

Step 2b Set $b_{k+1} = \max_{b_k \leq i < t_k}\{i : c_i \nsim s_k\} + 1$,
$t_{k+1} = t_k$, $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_i s_k} : b_k \leq i \leq b_{k+1}\}$. Go to **Step 1**.
Observe that, in this case, we have that $b_k < b_{k+1} \leq t_k$, $s_k \prec c_{b_{k+1}}$ and, for $j = b_k, \ldots, b_{k+1} - 1$, we have $s_k \nsim c_j$.

Step 2c Set $t_{k+1} = \min_{b_k < i \leq t_k}\{i : c_i \nsim s_k\} - 1$,
$b_{k+1} = b_k$, $\mathcal{L}_k := \mathcal{L}_{k-1} \cup \{\overline{c_i s_k} : t_{k+1} \leq i \leq t_k\}$. Go to **Step 1**.
Observe that, in this case, we have that $b_k \leq t_{k+1} < t_k$, $c_{t_{k+1}} \prec s_k$ and, for $j = t_{k+1} + 1, \ldots, t_k$, $s_k \nsim c_j$.

Assume that the process stops after K iterations.

For any $k < K$, in the kth iteration, exactly one line added to $\mathcal{L}_k$ is induced by two comparable points. We call this line l_k. Thus, there are $K - 1$ such lines, namely $l_1, \ldots, l_{K-1}$. Notice that l_k is either $\overline{c_{b_{k+1}} s_k}$ or $\overline{c_{t_{k+1}} s_k}$. If $l_k = \overline{c_{b_{k+1}} s_k}$, since $s_k \prec c_{b_{k+1}}$, we have $\{c_{b_{k+1}}, c_{b_{k+2}}, \ldots, c_{b_K}, c_{t_K}, c_{t_{K-1}}, \ldots, c_{t_1}\} \subseteq \overline{c_{b_{k+1}} s_k}$, and since $c_{b_k} \nsim s_k$, we have $c_{b_k} \notin \overline{c_{b_{k+1}} s_k}$. Similarly, if $l_k = \overline{c_{t_{k+1}} s_k}$ we have $\{c_{b_1}, \ldots, c_{b_K}, c_{t_K}, \ldots, c_{t_{k+1}}\} \subseteq \overline{c_{t_{k+1}} s_k}$ and $c_{t_k} \notin \overline{c_{t_{k+1}} s_k}$. Observe now that the line $\overline{c_1 c_H}$, that is added at the Kth iteration, contains all points in C. This implies that the lines $l_1, \ldots, l_{K-1}, \overline{c_1 c_H}$ are pairwise distinct. Thus, the process finds K pairwise distinct lines which are induced by comparable points. Moreover, since all the lines in $\mathcal{L}_0$ are induced by incomparable points, none of these K lines belong to $\mathcal{L}_0$.

The rest of the lines found by the process are induced by incomparable points. Hence, it remains to prove that $H - K$ of them are pairwise distinct and do not belong to $\mathcal{L}_0$.

Let $k < K$. We claim that $\mathcal{L}_k$ contains at least $b_{k+1} - b_k + t_k - t_{k+1} - 1$ (new) lines that are not in $\mathcal{L}_{k-1}$. Assume first that, in the kth iteration, lines are added at Step 2b (so $t_k - t_{k+1} = 0$). So $b_{k+1} - b_k$ lines induced by two incomparable points are added, namely $\overline{c_{b_k} s_k}, \ldots, \overline{c_{b_{k+1}-1} s_k}$. At most one of these lines belongs to $\mathcal{L}_0$ and none of them belongs to $\mathcal{L}_{k-1} \setminus \mathcal{L}_0$ because lines induced by incomparable points that are added in previous iterations of the process involve points of C either strictly below b_k or strictly above $t_k \geq b_{k+1}$. Hence, at least $b_{k+1} - b_k - 1$ new lines induced by incomparable points are added at Step 2b. A symmetric argument proves that, in the case where the lines are added at Step 2c (so $b_{k+1} - b_k = 0$), we have added $t_k - t_{k+1} - 1$ new lines induced by incomparable points.

So, after $K - 1$ iterations, the number of lines induced by incomparable points, in $\mathcal{L}_{K-1} \setminus \mathcal{L}_0$ is

$$\sum_{k=1}^{K-1} (b_{k+1} - b_k + t_k - t_{k+1} - 1) = t_1 - b_1 - (t_K - b_K) - (K - 1) = H - K - (t_K - b_K).$$

Hence, it remains to show that $t_K - b_K$ new distinct lines induced by incomparable points are added at the Kth iteration. In the case where $b_K = t_K$ we are done, so we may assume that $b_K < t_K$ and the process terminates at Step 2a. So, the lines $\overline{c_{b_K} s_k}, \overline{c_{b_K+1} s_k}, \ldots, \overline{c_{t_K} s_k}$ are added. At most one of these lines belong to $\mathcal{L}_0$ and none of them belong to $\mathcal{L}_{K-1} \setminus \mathcal{L}_0$ since lines induced by incomparable points added at iterations $1, \ldots, K - 1$ involve points of C either strictly below b_K or strictly above t_K. It follows that $t_K - b_K$ new lines are added.

3. Lines in graphs

We first need two easy observations about lines in a graph $G = (V, E)$. A vertex x in a graph is *universal* if it is adjacent to all vertices in $V \setminus x$.

1. If $ab \notin E$, then $\overline{ab} = \{a, b\}$.
2. A line $\overline{ab}$ is universal if and only if both a and b are universal.

We are now ready to prove our generalization to the graph case.

Proof of Theorem 3. We will use induction on n on the full statement of the theorem. First, we show that the theorem holds when $n = 4$. If there is no triangle in our graph, then every pair of vertices induces a distinct line, giving us 6 lines. If there are two different triangles in our graph, then there exist 2 vertices p, q, that belong to both triangles, and the line $\overline{pq}$ is universal, a contradiction. Therefore, we have exactly one triangle, and it is easy to see that in this case we have exactly 4 lines in our graph and the extremal graphs are exactly as desired.

Let $G = (V, E)$ be a graph on $n \geq 5$ vertices having no universal lines, and assume the statement holds for smaller n.

Let $V' \subseteq V$ be the set of points $x \in V$ such that $G \setminus \{x\}$ has no universal line. Assume first that $V' = \emptyset$. Thus, for any $x \in V$, $V \setminus \{x\}$ induces a universal line. Since G has no universal lines, $V \setminus \{x\}$ is a line of G for any $x \in V$. Thus, G induces n distinct lines of size $n-1$. Moreover, since it has no universal lines, G has at least two non-adjacent vertices, providing a line of size two. Thus, if $V' = \emptyset$, we have that G induces at least $n+1$ lines.

So we may assume from now on that $V' \neq \emptyset$. We will distinguish between two cases:

Case 1. There exists a point x in V' that is not universal.

Let y be a vertex non-adjacent to x. Since $G \setminus \{x\}$ has no universal lines, by induction, $G \setminus \{x\}$ induces at least $n-1$ distinct lines. If ℓ is a line of $G \setminus \{x\}$, then either ℓ or $\ell \cup \{x\}$ is a line of G. It follows that these lines are all distinct in G. Moreover, if they contain x, then they have at least 3 vertices and so they are all distinct from $\overline{xy} = \{x, y\}$. Hence, G has at least n distinct lines.

Now, assume that G induces exactly n distinct lines. Then, $G \setminus \{x\}$ must contain exactly $n-1$ lines and so by induction, $G \setminus \{x\}$ consists of a clique K on $n-2$ vertices, $x_1, x_2, \ldots, x_{n-2}$, and a vertex z which has at most one neighbour in K. Notice that the set of lines of $G \setminus \{x\}$ is $\mathcal{L}_{G\setminus\{x\}} := \{\{x_1, \ldots, x_{n-2}\}, \{z, x_1\}, \ldots, \{z, x_{n-2}\}\}$, giving us $n-1$ distinct lines of G, namely, $\mathcal{L} := \{\ell \text{ or } \ell \cup \{x\} : \ell \in \mathcal{L}_{G\setminus\{x\}}\}$.

We claim that x is adjacent to all vertices of $V \setminus \{x, y\}$. Suppose for contradiction that x is non-adjacent to a vertex $y' \in V \setminus \{x, y\}$. Then, $\overline{xy'} = \{x, y'\}$ is a line of G. Since $\overline{xy} = \{x, y\}$ is also a line of G, and we have $\overline{xy'}, \overline{xy} \notin \mathcal{L}$, there are a total of $n - 1 + 2 = n + 1$ distinct lines in G, a contradiction to our assumption that G has exactly n distinct lines.

Assume that $y \in K$, and let z' be the unique neighbour of z in K. Consider a vertex w in $K \setminus \{y, z'\}$ (such a vertex exists because $n \geq 5$). Since $y \notin \overline{xw}$ and $z \notin \overline{xw}$, we have $\overline{xw} \notin \mathcal{L}$. Of course, $\overline{xy} \notin \mathcal{L}$ is a line of G like before. Thus, G induces at least $n+1$ distinct lines again. Therefore, $y = z$, and $K \cup \{x\}$ is a clique of G as desired.

Case 2. All points of V' are universal.

Since G has no universal line, it follows that V' contains exactly one vertex, say x. So, for any $u \in V \setminus \{x\}$, $V \setminus \{u\}$, is a line of G. This yields $n-1$ lines of size $n-1$. Moreover, it is easy to see that since G has no universal lines, it must contain at least two pairs of non-adjacent vertices (as $n \geq 4$), providing us with two more distinct lines of size two. Hence, G has at least $n+1$ distinct lines.

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